AN "ISOMORPHIC" VERSION OF DVORETZKY'S THEOREM, II

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Abstract - A different proof is given to the result announced in [MS2]: For each $1 \le k < n$ we give an upper bound on the minimal distance of a k-dimensional subspace of an arbitrary n-dimensional normed space to the Hilbert space of dimension k. The result is best possible up to a multiplicative universal constant.

Our main result is the following extension of Dvoretzky's theorem (from the range $1 < k < c \log n$ to $c \log n \le k < n$) first announced in [MS2, Theorem2]. As is remarked in [MS2], except for the absolute constant involved the result is best possible.

Theorem. *There exists a $K > 0$ such that, for every n and every $\log n \le k < n$, any n-dimensional normed space, X, contains a k-dimensional subspace, Y, satisfying $d(Y, \ell_2^k) \le K\sqrt{\frac{k}{\log(1+n/k)}}$. In particular, if $\log n \le k \le n^{1-K\varepsilon^2}$, there exists a k-dimensional subspace Y (of an arbitrary n-dimensional normed space X) with $d(Y, \ell_2^k) \le \frac{K}{\varepsilon}\sqrt{\frac{k}{\log n}}$.*

Jesus Bastero pointed out to us that the proof of the theorem in [MS2] works only in the range $k \le cn/\log n$. Here we give a different proof which corrects this oversight. The main addition is a computation due to E. Gluskin (see the proof of the Theorem in [Gl1] and the remark following the proof of Theorem 2 in [Gl2]). In the next lemma we single out what we need from Gluskin's argument and sketch Gluskin's proof.

Gluskin's Lemma. *Let $1 \le k \le n/2$ and let $\nu_{n,k}$ denote the normalized Haar measure on the Grasmannian of k-dimensional subspaces of $\mathbb{R}^n$. Then, for some absolute positive constant c,*

$$\nu_{n,k}\left(\left\{E \; ; \; \exists x \in E \text{ with } \|x\|_\infty < c\sqrt{\frac{\log(1+n/k)}{n}}\|x\|_2\right\}\right) < 1/2.$$

Proof. Let $g_{i,j}$, $i = 1, \ldots, k$, $j = 1, \ldots, n$, be independent standard Gaussian variables. The invariance of the Gaussian measure under the orthogonal group implies that the conclusion of the lemma is equivalent to

$$\text{Prob}\left(\inf_{x \in S^{k-1}} \max_{1 \le j \le n} \frac{|\sum_{i=1}^k x_i g_{i,j}|}{(\sum_{l=1}^n (\sum_{i=1}^k x_i g_{i,l})^2)^{1/2}} < c\sqrt{\frac{\log(1+n/k)}{n}}\right) < 1/2.$$

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As is well known the variable $(\sum_{j=1}^n(\sum_{i=1}^k x_i g_{i,j})^2)^{1/2}$ is well concentrated near the constant $\|x\|_2 = (\sum_{i=1}^k x_i^2)^{1/2}\sqrt{n}$. In particular,

$$\text{Prob}\left(\exists x \in S^{k-1} \text{ with } (\sum_{j=1}^{n}(\sum_{i=1}^{k} x_i g_{i,j})^2)^{1/2} > 2\sqrt{n}\right) < 1/4$$

if n is large enough. It is thus enough to prove that

$$\text{Prob}\left(\inf_{x\in S^{k-1}} \max_{1\le j\le n} |\sum_{i=1}^{k} x_i g_{i,j}| < 2c\sqrt{\log(1+n/k)}\right) < 1/4.$$

The left hand side here is clearly dominated by

$$\text{Prob}\left(\inf_{x\in S^{k-1}} (\sum_{j=1}^{2k} |\sum_{i=1}^{k} x_i g_{i,j}|^{*2})^{1/2} < 2c\sqrt{2k\log(1+n/k)}\right)$$

where $\{a_i^*\}$ denotes the decreasing rearrangement of the sequence $\{|a_i|\}$. To estimate the last probability we use the usual deviation inequalities for Lipschitz functions of Gaussian vectors (see e.g. [MS1] or [Pi]). The only two facts one should notice are that the norm $\|a\| = (\sum_{j=1}^{2k} a_j^{*2})^{1/2}$ on $\mathbb{R}^n$ is dominated by the Euclidean norm, i.e. that the Lipschitz constant of the function $\|a\| = (\sum_{j=1}^{2k} a_j^{*2})^{1/2}$ on $\mathbb{R}^n$ is at most one and that the expectation of $\|(g_1,\dots,g_n)\|$ is larger than $c_1(2k)^{1/2}\sqrt{\log(1+n/k)}$ for some absolute constant c_1. Then, for an appropriate c,

$$\text{Prob}\left(\inf_{x\in S^{k-1}} (\sum_{j=1}^{2k} |\sum_{i=1}^{k} x_i g_{i,j}|^{*2})^{1/2} < 2c\sqrt{2k\log(1+n/k)}\right) < \exp(-k\log(1+n/k)) < 1/4$$

if n is large enough. ■

Proof of the theorem. The proof follows in parts the proof in [MS2]. For completeness we shall repeat these parts. In what follows $0 < \eta, K < \infty$ denote absolute constants, not necessarily the same in each instance. By a result of Bourgain and Szarek (Theorem 2 of [BS] but refer to remark 4 in [MS2] for an explanation why we need only a much simpler form of their result), we may assume without loss of generality that there exists a subspace, $Z \subseteq X$, with $m = dim Z > n/2$ and $\alpha\|x\|_{\ell_\infty^m} \le \|x\|_Z \le \|x\|_{\ell_2^m}$ for all $x \in Z$ for some absolute constant $\alpha > 0$.

Let M denote the median of $\|x\|_Z$ over $S^{m-1} = S^{n-1} \cap Z$. Fix k as in the statement of the theorem. If $M > K\sqrt{\frac{k}{n}}$ then, by [Mi] (see also [FLM] or [MS1], Theorem 4.2), Z and thus X contains a k-dimensional subspace, Y, satisfying $d(Y, \ell_2^k) \le 2$. Also, for each k, with probability $> 1 - e^{-\eta k}$

$$(*) \qquad \|x\| \le 2\left(M + K\sqrt{\frac{k}{n}}\right)\|x\|_2.$$

This again follows by the usual deviation inequalities. Let us refresh the reader's memory: Let $\mathcal{M}$ be a $\frac{1}{2}$-net in the sphere of a fixed k-dimensional subspace, Y_0, of Z with $|\mathcal{M}| \le 6^k$ (see [MS1], Lemma 2.6). Denoting by ν the Haar measure on the orthogonal group $O(m)$ we get,

$$\nu\left(\left\{U \;;\; \|Ux\|_Z \ge M + K\sqrt{\frac{k}{n}} \text{ for some } x \in \mathcal{M}\right\}\right) \le \exp(k\log 6 - \eta K^2 k).$$

Thus, a successive approximation argument gives that, with probability larger than $1 - e^{-\eta k}$, a k-dimensional subspace E of Z satisfies

$$\|x\| \le 2\left(M + K\sqrt{\frac{k}{n}}\right)\|x\|_2, \quad \text{for all } x \in E.$$

By Gluskin's Lemma, for $k < n/4$,

$$\begin{aligned}&\nu_{m,k}\left(\left\{E \;;\; \exists x \in E \text{ with } \|x\| < c\alpha\sqrt{\frac{\log(1+n/k)}{n}}\|x\|_2\right\}\right)\\ &\le \nu_{m,k}\left(\left\{E \;;\; \exists x \in E \text{ with } \|x\|_\infty < c\sqrt{\frac{\log(1+m/k)}{m}}\|x\|_2\right\}\right) < 1/2.\end{aligned}$$

That is, with probability larger than $1/2$ a k-dimensional subspace E satisfies

$$\|x\| \ge c\alpha\sqrt{\frac{\log(1+n/k)}{n}}\|x\|_2, \quad \text{for all } x \in E.$$

Combining this with $(*)$, we get that, if $M \le K\sqrt{\frac{k}{n}}$, there exists a k-dimensional subspace whose distance to Euclidean space is smaller than

$$4K\sqrt{\frac{k}{n}}/c\alpha\sqrt{\frac{\log(1+n/k)}{n}} = K'\sqrt{k/\log(1+n/k)}.$$

■

Remark. Can one show that the conclusion of the Theorem holds for a random subspace Y? The only obstacle in the proof here and also in [MS2] is the use of the result from [BS]. Michael Schmuckenschläger showed us how to overcome this obstacle in the proof of [MS2]: Instead of using [BS] one can use Proposition 4.11 in [ScSc] which says that any multiple of the ℓ_∞^n unit ball has larger or equal Gaussian measure than the same multiple of the unit ball of any other norm on $\mathbb{R}^n$ whose ellipsoid of maximal volume is S^{n-1}. This can replace the first inequality on the last line of p. 542 in [MS2] (with $m = n$ and $\alpha = 1$. The change from the Gaussian measure to the spherical measure is standard.)
It follows that at least for $k \le cn/\log n$ the answer to the question above is positive.

What is the "isomorphic' version of Dvoretzky's Theorem for spaces with non-trivial cotype? It is known that in this case one has a version of Dvoretzky's theorem with a much better dependence of the dimension of the Euclidean section on the dimension of the space ([FLM] or see [MS1], 9.6). We do not know if one can extend this theorem in a similar "isomorphic" way as the theorem above. The proposition below gives such an extension under the additional assumption that the space also has non-trivial type. Recall that it is a major open problem whether an n-dimensional normed space with non-trivial cotype has a subspace of dimension $[n/2]$ which is of type 2 (with the type 2 constant depending on the cotype and the cotype constant only) or at least of some non-trivial type. If this open problem has a positive solution, the next proposition would imply the desired "isomorphic" cotype case of the theorem. The proof we sketch here (as well as the statement of the proposition) uses quite a lot of background material (which can be found in [MS1]) and is intended for experts.

Proposition. *For every n and every $n^{2/q} \le k < n/2$, any n-dimensional normed space, X, contains a k-dimensional subspace, Y, satisfying $d(Y, \ell_2^k) \le K k^{1/2}/n^{1/q}$. Here K depends on $q < \infty$, the cotype q constant of X and the norm of the Rademacher projection in $L_2(X)$ only. Up to the exact value of the constant involved the result is best possible and is attained for $X = \ell_q^n$.*

Sketch of proof. We use the notations of [MS1]. We first find an operator $T : \ell_2^n \to X$ for which

$$\ell(T)\ell(T^{-1*}) \le Kn.$$

where K depends on the norm of the Rademacher projection in $L_2(X)$ only (see [MS1], 15.4.1).

Next we use the "lower bound theorem" of the first named author ([MS1], 4.8) to find an $3n/4$ dimensional subspace $E \subseteq \ell_2^n$ for which

$$\|(T_{|E})^{-1}\| \le C\ell(T^{-1*})/\sqrt{n},$$

for an absolute constant C.

By a theorem of Figiel and Tomczak ([FT] or [MS1], 15.6), there exists a further subspace $F \subseteq E$ of dimension larger than $n/2$ for which

$$\|T_{|F}\| \le Kn^{-1/2} n^{1/2-1/q} \ell(T_{|E}) \le Kn^{-1/q}\ell(T),$$

for K depending on q and the cotype q constant of X only.

This reduces the problem to the following: Given a norm $\|\cdot\|$ on $\mathbb{R}^{n/2}$ for which

$$C^{-1}\|x\|_2 \le \|x\| \ \le Kn^{1/2-1/q}\|x\|_2$$

for all x, for constants C, K depending only on p, q and the type p and cotype q constants of X, and for which $M = \int_{S^{n/2-1}} \|x\| = 1$, find a subspace Y of dimension k as required in the statement of the proposition. Since we have to take care of the upper bound only, this can be accomplished by the usual "concentration" method as described in the first few chapters of [MS1].

The fact that, for some absolute constant η and for all k, ℓ_q^n does not have k-dimensional subspaces of distance smaller than $\eta q^{-1/2}k^{1/2}/n^{1/q}$ to ℓ_2^k follows from the method developed in [BDGJN] (or see [MS1], 5.6). One just need to replace the constant 2 in 5.6 of [MS1] by a general constant d and follow the proof to get a lower bound on d. ■

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