

BANACH SPACES DETERMINED BY THEIR UNIFORM STRUCTURES

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Dedicated to the memory of E. Gorelik

Abstract

Following results of Bourgain and Gorelik we show that the spaces ℓ_p , $1 < p < \infty$, as well as some related spaces have the following uniqueness property: If X is a Banach space uniformly homeomorphic to one of these spaces then it is linearly isomorphic to the same space. We also prove that if a $C(K)$ space is uniformly homeomorphic to c_0 , then it is isomorphic to c_0 . We show also that there are Banach spaces which are uniformly homeomorphic to exactly 2 isomorphically distinct spaces.

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0. Introduction

The first result in the subject we study is the Mazur-Ulam theorem which says that an isometry from one Banach space onto another which takes the origin to the origin must be linear. This result, which is nontrivial only when the Banach spaces are not strictly convex, means that the structure of a Banach space as a metric space determines the linear structure up to translation. On the other hand, the structure of an infinite dimensional Banach space as a topological space gives no information about the linear structure of the space [Kad], [Tor]. In this paper we are concerned with equivalence relations of Banach spaces which lie between isometry and homeomorphism, mainly Lipschitz equivalence and uniform homeomorphism. There exists a considerable literature on this topic (see [Ben2] for a nice survey to about 1983). Nevertheless, the subject is still in its infancy, as many fundamental, basic questions remain unanswered. What we find fascinating is that the subject combines topological arguments and constructions with deep facts from the linear structure theory of Banach spaces. The present paper is also largely concerned with this interplay.

Early work in this subject, especially that of Ribe [Rib1], showed that if two Banach spaces are uniformly homeomorphic, then each is finitely crudely representable in the other. (Recall that Z is *finitely crudely representable* in X provided there is a constant λ so that each finite dimensional subspace E of Z is λ -isomorphic to a subspace of X). In a short but imprecise way, this means that the two spaces have the same finite dimensional subspaces. Since the spaces ℓ_p and $L_p(0,1)$ have the same finite dimensional subspaces, a natural question was whether they are uniformly homeomorphic for $1 \leq p < \infty$, $p \neq 2$. This question was answered in the negative first for $p = 1$ by Enflo [Ben2], then for $1 < p < 2$ by Bourgain [Bou], and, finally, for $2 < p < \infty$ by Gorelik [Gor]. The main new point in Gorelik's proof is a nice topological argument using the Schauder fixed point theorem. In section 1 we formulate what Gorelik's approach yields as "The Gorelik Principle". This principle is most conveniently applicable for getting information about spaces which are uniformly homeomorphic to a space which has an unconditional basis with a certain convexity or concavity property (see e.g. Corollary 1.7).

In section 2 we combine Bourgain's result [Bou] and the Gorelik Principle with structural results from the linear theory to conclude that the linear structure of ℓ_p , $1 < p < \infty$, is determined by its uniform structure; that is, the only Banach spaces which are uniformly homeomorphic to ℓ_p are those which are isomorphic to ℓ_p . The case $p = 2$ was done twenty-five years ago by Enflo [Enf]. The main part of section 2 is devoted to proving that some other spaces are determined by their uniform structure and also to investigating the possible number of linear structures on spaces which are "close" to ℓ_p in an appropriate local sense.

In section 3 we use the Gorelik Principle to study the uniqueness question for c_0 , answering in the process a question Aharoni asked in 1974 [Aha].

Very roughly, the passage from a uniform homeomorphism U between Banach spaces to a linear isomorphism involves two steps:

1. Passage from a uniformly continuous U to a Lipschitz map F via the "formula" $F(x) = \lim_{n \rightarrow \infty} n^{-1}U(nx)$.
2. Passage from a Lipschitz map F to its derivative.

Of course, both steps lead to difficulties. In step one a major problem is that the limit does not exist in general and thus one is forced to use ultrafilters and ultraproducts. In step two the problem is again the existence of derivatives. Derivatives in the sense of Gateaux of Lipschitz functions exist under rather mild conditions, but they usually do not suffice; on the other hand, derivatives in the sense of Fréchet, which are much more useful, exist (or at least are known to exist) only in special cases.

It is therefore natural that in section 2 ultraproducts are used as well as a recent result [LP] on differentiation which ensures the existence of a derivative in sense which is between those of Gateaux and Fréchet.

It is evident that for step one of the procedure above it is important that U be defined on the entire Banach space. In fact, it often happens that the unit balls of spaces X and Y are uniformly homeomorphic while X and Y are not. The simplest example of this phenomenon goes back to Mazur [Maz] who noted that the map from $L_p(\mu)$, $1 < p < \infty$, to $L_1(\mu)$ defined by $f \mapsto |f|^p \text{sign } f$ is a uniform homeomorphism between the unit balls of these spaces, while in [Lin1] and [Enf] it is proved that the spaces themselves are not uniformly homeomorphic (this of course also follows from Ribe's result mentioned above). The Mazur map was extended recently to more general situations by Odell and Schlumprecht [OS]. In section 4 we obtain estimates on the modulus of continuity of these generalized Mazur maps. The proofs are based on complex interpolation.

In the rather technical section 5 we combine the results of sections 2 and 4 with known constructions to produce examples, for each $k = 1, 2, 3, \dots$, of Banach spaces which admit exactly 2^k linear structures. The most easily described examples are certain direct sums of convexifications of Tsirelson's space (see [Tsi], [CS]).

Although our interest is mainly in the separable setting, we present in section 6 some results for nonseparable spaces.

As pointed out by Ribe [Rib1], Enflo's result on ℓ_2 [Enf] mentioned earlier follows from the fact that ℓ_2 is determined by its finite dimensional subspaces. Section 7, which is formally independent of the rest of the paper, is motivated by the problem of characterizing those Banach spaces which are determined by their finite dimensional subspaces. We conjecture that only ℓ_2 has this property. We show that any space which is determined by its finite dimensional subspaces must be close to ℓ_2 , and if we replace "determined by its finite dimensional subspaces" by a natural somewhat stronger property then, besides ℓ_2 , there are other spaces which enjoy this property.

The paper ends with a section which mentions a few of the many open problems connected to the results of sections 1–5.

We use standard Banach space theory language and notation, as may be found in [LT1,2] and [T-J].

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As is clear from this introduction, our work on this paper was motivated by Gorelik's paper [Gor]. A few days after he submitted the final version of [Gor], Gorelik was fatally injured by a car while he was jogging. We all had the privilege of knowing Gorelik, unfortunately for only a very short time.

1. The Gorelik Principle

A careful reading of Gorelik's paper [Gor] leads to the formulation of the following principle:

The Gorelik Principle. *A uniform homeomorphism between Banach spaces cannot take a large ball of a finite codimensional subspace into a small neighborhood of a subspace of infinite codimension.*

The precise formulation of the Gorelik Principle in the form we use is:

Theorem 1.1. *Let U be a homeomorphism from a Banach space X onto a Banach space Y with uniformly continuous inverse V . Suppose that d, b are such that there exist a finite codimensional subspace X_0 of X and an infinite codimensional subspace Y_0 of Y for which*

$$U[d\text{Ball}(X_0)] \subset Y_0 + b\text{Ball}(Y).$$

Then $\omega(V, 2b) \geq d/4$, where $\omega(V, t) \equiv \sup\{\|Vy_1 - Vy_2\| : \|y_1 - y_2\| \leq t\}$ is the modulus of uniform continuity of V .

The proof of the Gorelik Principle is based on two lemmas which are minor variations of Lemmas 5 and 6 in Gorelik's paper [Gor]. The first lemma is obvious, while the second is a simple consequence of Brouwer's fixed point theorem.

Lemma 1.2. *let Y_0 be an infinite codimensional subspace of the Banach space Y and B a compact subset of Y . Then for every $\tau > 0$ there is a y in Y with $\|y\| < \tau$, so that $d(B + y, Y_0) \geq \tau/2$.*

Lemma 1.3. *Let X_0 be a finite codimensional subspace of the Banach space X . For every $\tau > 0$ there is a compact subset A of $\tau\text{Ball}(X)$ so that whenever ϕ is a continuous map from A into X for which $\|\phi(x) - x\| < \tau/2$ for all x in A , then $\phi(A) \cap X_0 \neq \emptyset$.*

Proof of Lemma 1.3. By the Bartle-Graves theorem or Michael's selection theorem [Mic], there is a continuous selection $f : \frac{3}{4}\tau\text{Ball}(X/X_0) \rightarrow \tau\text{Ball}(X)$ of the inverse of the quotient mapping Q from X to X/X_0 . Set $A = f[\frac{3}{4}\tau\text{Ball}(X/X_0)]$, and apply Brouwer's theorem to the mapping $x \mapsto x - Q\phi f(x)$ from $\frac{3}{4}\tau\text{Ball}(X/X_0)$ to itself. ■

Proof of Theorem 1.1. Get the set A from Lemma 1.3 for the value $\tau = d/2$. Applying Lemma 1.2 to the compact set $U[A]$, we get y in Y with $\|y\| < 2b$ so that $d(U[A] + y, Y_0) \geq b$. The mapping from A into X defined by $a \mapsto V(Ua + y)$ moves each point of A a distance of at most $\omega(V, \|y\|) \leq \omega(V, 2b)$; so if this is less than $d/4 = \tau/2$, we have from Lemma 1.3 that there must be a point a_0 in A for which $V(Ua_0 + y)$ is in X_0 and hence in $d\text{Ball}(X_0)$. But then $Ua_0 + y$ would be in $Y_0 + b\text{Ball}(Y)$, which contradicts the choice of y . Therefore $\omega(V, 2b) \geq d/4$, as desired. ■

Recall that an unconditional basis is said to have an upper p -estimate (respectively, lower p -estimate) provided that there is a constant $0 < C < \infty$ so that for every finite sequence $\{x_k\}_{k=1}^n$ of vectors which are disjointly supported with respect to the uncondi-

tional basis, the quantity $\left\| \sum_{k=1}^n x_k \right\|^p$ is less than or equal to (respectively, greater than or

equal to) $C^p \sum_{k=1}^n \|x_k\|^p$. To characterize the uniform homeomorphisms of ℓ_p and the Tsirelson spaces, we need the following implementation of the Gurelik Principle.

Theorem 1.4. *Suppose that X has an unconditional basis which has an upper p -estimate and X is uniformly homeomorphic to Y . Then no quotient of Y can have an unconditional basis which has a lower r -estimate with $r < p$.*

For the proof of Theorem 1.4, we need to recall the concept of *approximate metric midpoint*, which plays a role also in the proofs of Enflo [Enf] and Bourgain [Bou]. Given points x and y in a Banach space X and $\delta \geq 0$, let

$$\text{Mid}(x, y, \delta) = \left\{ z \in X : \|x - z\| \vee \|z - y\| \leq (1 + \delta) \frac{\|x - y\|}{2} \right\}.$$

We also need a quantitative way of expressing the well-known fact that a uniformly continuous mapping from a Banach space is “Lipschitz for large distances”. If U is a uniformly continuous mapping from a Banach space X into a Banach space Y , set for each $t > 0$

$$u_t = \sup \left\{ \frac{\|Ux_1 - Ux_2\|}{t \vee \|x_1 - x_2\|} : x_1, x_2 \in X \right\}.$$

The statement that U is *Lipschitz for large distances* is just that u_t is finite for each $t > 0$. Obviously u_t is a decreasing function of t ; denote its limit by u_∞ .

Lemma 1.5. *Let U be a uniformly continuous mapping from a Banach space X and let $d > 0$. Suppose that x, y in X satisfy for certain $\epsilon \geq 0, \delta \geq 0$*

$$(i) \quad \|x - y\| \geq \frac{2d}{1+\delta}$$

$$(ii) \quad \|Ux - Uy\| \geq \frac{u_d}{(1+\epsilon)} \|x - y\|.$$

Then $U[\text{Mid}(x, y, \delta)] \subset \text{Mid}(Ux, Uy, \epsilon + \delta + \epsilon\delta)$.

Proof. Let z be in $\text{Mid}(x, y, \delta)$. Then

$$\begin{aligned} \|Ux - Uz\| \vee \|Uz - Uy\| &\leq u_d \left(d \vee (1 + \delta) \frac{\|x - y\|}{2} \right) = u_d(1 + \delta) \frac{\|x - y\|}{2} \\ &\leq (1 + \epsilon)(1 + \delta) \frac{\|Ux - Uy\|}{2}. \end{aligned} \quad \blacksquare$$

In a space which has an unconditional basis which has an upper or lower p -estimate, the set of approximate metric midpoints between two points has some obvious structure. We state the next lemma only for points symmetric around 0; by translation one obtains a similar statement for general sets $\text{Mid}(x, y, \delta)$.

Lemma 1.6. *Suppose that x is in X and X has a basis $\{x_n\}_{n=1}^\infty$ whose unconditional constant is one.*

(i) If $\{x_n\}_{n=1}^\infty$ has an upper p -estimate with constant one, then for each $\delta > 0$ there is a finite codimensional subspace X_0 of X so that $\text{Mid}(x, -x, \delta \|x\|) \supset \delta^{\frac{1}{p}} \|x\| \text{Ball}(X_0)$.

(ii) If $\{x_n\}_{n=1}^\infty$ has a lower r -estimate with constant one, then for each $\delta > 0$ there is a finite dimensional subspace X_1 of X so that $\text{Mid}(x, -x, \delta \|x\|) \subset X_1 + (r\delta)^{\frac{1}{r}} \|x\| \text{Ball}(X)$.

Proof. We assume $p > 1$ and $x \neq 0$ since otherwise the conclusion is trivial. To prove (i), suppose that y is in X , $\|y\| \leq \delta^{\frac{1}{p}} \|x\|$, and y is disjoint from x relative to the basis $\{x_n\}_{n=1}^\infty$. Then $\|y \pm x\| \leq (1 + \delta)^{\frac{1}{p}} \|x\| < (1 + \delta) \|x\|$. So when x is finitely supported relative to $\{x_n\}_{n=1}^\infty$, we can take for X_0 the subspace of all vectors in X which vanish on the support of x . The general case follows by approximating x by a vector with finite support; the degree of approximation depending on δ .

The proof of (ii) is similar. If x is in $\text{span}\{x_k\}_{k=1}^n$, then $X_1 = \text{span}\{x_k\}_{k=1}^n$ “works” independently of δ ; the general case follows by approximation. ■

Proof of Theorem 1.4 If the conclusion is false, we may assume, after renorming X and Y , that the basis $\{x_n\}_{n=1}^\infty$ for X has unconditional constant one and also the upper p -estimate constant for $\{x_n\}_{n=1}^\infty$ is one, and that there is a quotient mapping Q from Y onto a space Z having a basis $\{z_n\}_{n=1}^\infty$ with unconditional constant one and with lower r -estimate constant one. Let U be a uniform homeomorphism from X onto Y . Since it is easy to check that the “Lipschitz for large distance” constants s_t of $S = QU$ are bounded away from 0, we can assume without loss of generality that $s_\infty = 1$.

Fix $\delta > 0$; later we shall see how small δ need be to yield a contradiction. Since $s_t \downarrow 1$ as $t \uparrow \infty$, we can find a pair x, y of vectors in X with $\|x - y\|$ as large as we please (for one thing we want $\delta^{\frac{1}{r}} \|x - y\| > 2$), so that $\frac{\|Sx - Sy\|}{\|x - y\|}$ is as close to one as we please. From Lemma 1.5 we then get, as long as $\|x - y\|$ is sufficiently large, that

$$(1.1) \quad S[\text{Mid}(x, y, \delta \|x - y\|)] \subset \text{Mid}(Sx, Sy, 2\delta \|x - y\|).$$

By making translations, we only need to consider the case when $y = -x$ and $U(-x) = -Ux$; that is, we can assume that there is x in X with $\|x\|$ as large as we like and

$$(1.2) \quad S[\text{Mid}(x, -x, 2\delta \|x\|)] \subset \text{Mid}(Sx, -Sx, 4\delta \|x\|).$$

From Lemma 1.6 we get a finite codimensional subspace X_0 of X and a finite dimensional subspace Z_0 of Z so that

$$(1.3) \quad S\left[(2\delta)^{\frac{1}{p}} \|x\| \text{Ball}(X_0)\right] \subset Z_0 + (4r\delta)^{\frac{1}{r}} \|x\| \text{Ball}(Z).$$

Set $Y_0 = Q^{-1}Z_0$. Then Y_0 has infinite codimension in Y and

$$(1.4) \quad U\left[(2\delta)^{\frac{1}{p}} \|x\| \text{Ball}(X_0)\right] \subset Y_0 + (4r\delta)^{\frac{1}{r}} \|x\| \text{Ball}(Y).$$

The Gorelik Principle tells us that

$$(1.5) \quad \omega(V, 2(4r\delta)^{\frac{1}{r}} \|x\|) \geq \frac{(2\delta)^{\frac{1}{p}} \|x\|}{4},$$

where $V = U^{-1}$.

However, keeping in mind that $\delta^{\frac{1}{r}} 2 \|x\| > 2$, we have

$$(1.6) \quad \omega(V, 2(4r\delta)^{\frac{1}{r}} \|x\|) \leq v_1 2(4r\delta)^{\frac{1}{r}} \|x\|.$$

Putting (1.5) and (1.6) together gives

$$\frac{(2\delta)^{\frac{1}{p}}}{4} \leq v_1 2(4r\delta)^{\frac{1}{r}},$$

which is a contradiction for small enough δ . ■

In Section 2 we need the next corollary.

Corollary 1.7. *Suppose that X has an unconditional basis and X is uniformly homeomorphic to Y . Assume that either the unconditional basis for X has a lower r -estimate for some $r < 2$, or that X is superreflexive and the unconditional basis for X has an upper p -estimate for some $p > 2$. Then ℓ_2 is not isomorphic to a subspace of Y .*

Proof. Bourgain [Bou] proved that for $r < 2$ there is no homeomorphism U from ℓ_2 into ℓ_r for which both U and U^{-1} are “Lipschitz for large distances”; a nonessential modification yields the same result for any space X which has an unconditional basis which has a lower r -estimate for some $r < 2$. Consequently, ℓ_2 is not isomorphic to a subspace of any space which is uniformly homeomorphic to such an X . So assume that X is superreflexive and the unconditional basis for X has an upper p -estimate for some $p > 2$. This implies that X is of type 2 in the sense of Maurey-Pisier (see [LT2, 1.f]), and hence so is Y , since Y is finitely crudely representable in X by Ribe’s theorem [Rib1] (or see [Ben2]). Thus by Maurey’s theorem [Mau3], any isomorphic copy of ℓ_2 in Y is complemented in Y , so Theorem 1.4 implies that ℓ_2 is not isomorphic to a subspace of Y . ■

2. Uniform Homeomorphs of ℓ_p , \mathcal{T}^p , and related spaces

The first result in this section is an immediate consequence of Corollary 1.7 and previously known results.

Theorem 2.1. *If X is a Banach space which is uniformly homeomorphic to ℓ_p , $1 < p < \infty$, then X is isomorphic to ℓ_p .*

Proof. By Ribe’s theorem [Rib2], X is isomorphic to a complemented subspace of L_p . But X does not contain an isomorphic copy of ℓ_2 by Corollary 1.7 if $p \neq 2$, so by the results of [JO], X is isomorphic to ℓ_p . ■

Remark. The case $p = 2$ in Theorem 2.1 was proved by Enflo [Enf]. A simple proof using ultraproducts was provided by Heinrich and Mankiewicz [HM], [Ben2]. They also used ultraproducts to give a simple proof of Ribe’s theorem [Rib2]. Since we need the type of reasoning introduced in [HM] to prove the remaining theorems, we recall some more-or-less standard facts about ultraproducts of Banach spaces. Much more information, as well as references to the original sources, can be found in [Hei].

Given a sequence $\{X_n\}_{n=1}^\infty$ of Banach spaces and a free ultrafilter \mathcal{U} on the natural numbers, denote by $(X_n)_\mathcal{U}$ (or just $X_\mathcal{U}$ if all the X_n 's are the same) the Banach space ultraproduct of $\{X_n\}_{n=1}^\infty$, defined as the collection of all bounded sequences $\{x_n\}_{n=1}^\infty$ with $x_n \in X_n$ under the norm $\lim_{\mathcal{U}} \|x_n\|$. Here we identify two sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ as being the same if $\lim_{\mathcal{U}} \|x_n - y_n\| = 0$. The space $X_\mathcal{U}$ is finitely representable in X . A less commonly used fact which is important for us is that finite dimensional complemented subspaces of $X_\mathcal{U}$ pull down to X . Precisely, if E is a finite dimensional subspaces of $X_\mathcal{U}$ which is λ -complemented in $X_\mathcal{U}$, then for each $\epsilon > 0$, E is $(1 + \epsilon)$ -isomorphic to a subspace of X which is $(\lambda + \epsilon)$ -complemented in X . This follows from the *ultrapower version of local duality* ([Hei], p. 90)), which says that every finite dimensional subspace of $(X_\mathcal{U})^*$ $(1 + \epsilon)$ -embeds into $(X^*)_\mathcal{U}$ in such a way that the action on any fixed finite subset of $X_\mathcal{U}$ is preserved. (The space $(X^*)_\mathcal{U}$ is always a subspace of $(X_\mathcal{U})^*$, but these spaces coincide only when X is superreflexive.)

The space X is naturally embedded into $X_\mathcal{U}$ as the diagonal. If X is reflexive, it is norm one complemented in $X_\mathcal{U}$ (map $\{x_n\}_{n=1}^\infty$ in $X_\mathcal{U}$ to the \mathcal{U} -weak limit in X of $\{x_n\}_{n=1}^\infty$); the kernel $X_{\mathcal{U},0}$ of this projection consists of those bounded sequences in X which tend weakly to zero along the ultrafilter \mathcal{U} .

An ultrapower of a Banach lattice X is again a Banach lattice; moreover, if X is a L_p -space, then the norm in X is p -additive for disjoint vectors, and then so is the norm in the ultrapower. Thus (by the generalized Kakutani representation theorem) the ultrapower is also a L_p -space if $1 \leq p < \infty$.

Heinrich and Mankiewicz [HM] (see also [Ben2]) used ultraproducts to give simple proofs of a number of previously known theorems concerning uniformly homeomorphic Banach spaces and to answer a number of open problems. Here we just recall the basic approach in a situation which is general enough to meet our needs.

Suppose that X and Y are separable, uniformly homeomorphic Banach spaces. Using the fact that a uniformly continuous mapping from a Banach space is Lipschitz for large distances, one sees easily that X and Y have Lipschitz equivalent ultrapowers $X_\mathcal{U}$ and $Y_\mathcal{U}$. Suppose now that X , hence also $X_\mathcal{U}$, is superreflexive. Using a back-and-forth procedure and a classical weak compactness argument [Lin2], one gets a separable, norm one complemented subspace X_1 of $X_\mathcal{U}$ which contains X and is Lipschitz equivalent to a subspace Y_1 of $Y_\mathcal{U}$ which contains Y . A differentiation argument (combined with a technique from [Lin1]) now yields that Y_1 isomorphically embeds into X_1 as a complemented subspace. This shows that Y is also superreflexive, and so X_1 embeds into Y_1 as a complemented subspace. Moreover, now that we know that $Y_\mathcal{U}$ is reflexive, we can make the earlier construction produce a Y_1 which is norm one complemented in $Y_\mathcal{U}$.

One of several consequences of this construction which we use later is that uniformly homeomorphic spaces X and Y have the same finite dimensional complemented subspaces if X is separable and superreflexive (in fact, without any restriction on X , but this requires more work).

Given the necessary background on ultrapowers, analysis only slightly more involved than that of Theorem 2.1 yields:

Theorem 2.2. *Let $1 < p_1 < p_2 < \dots < p_n < 2$ or $2 < p_1 < p_2 < \dots < p_n < \infty$ and set*

$X = \sum_{k=1}^n \ell_{p_k}$. If X is uniformly homeomorphic to a Banach space Y , then X is isomorphic to Y .

Proof. For simplicity of notation, we treat the case $n = 2$. By one of the results of Heinrich-Mankiewicz [HM] (or see [Ben2]), X, Y have Lipschitz equivalent ultrapowers $X_{\mathcal{U}}, Y_{\mathcal{U}}$, respectively; and $X_{\mathcal{U}}$ splits as the direct sum of an L_{p_1} space with an L_{p_2} space: $X_{\mathcal{U}} = L_{p_1}(\tilde{\mu}) \oplus L_{p_2}(\tilde{\mu})$. (We do not need that the measures for p_1 and p_2 are the same, but they are.) Ribe proved that Y is finitely crudely representable in X ([Rib1], [HM], [Ben2]), so Y is superreflexive and $Y_{\mathcal{U}}$ splits as the direct sum of Y and some space Z (which is, incidentally, embeddable as a subspace of $L_{p_1}(\nu) \oplus L_{p_2}(\nu)$ for some measure ν). Since Y is separable, there exists a separable subspace Z of \tilde{Z} so that the image of $Y + Z$ under the Lipschitz isomorphism from $Y_{\mathcal{U}}$ onto $X_{\mathcal{U}}$ is of the form $L_{p_1}(\mu) \oplus L_{p_2}(\mu)$, where μ is the restriction of $\tilde{\mu}$ to a separable sigma subalgebra. Hence by [HM], Y is isomorphic to a complemented subspace of $L_{p_1}(\mu) \oplus L_{p_2}(\mu)$, whence of $L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$.

Suppose now that p_1 and p_2 are larger than two. Let J denote the embedding of Y onto a complemented subspace of $L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$ and let S_1, S_2 be the natural projections from $L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$ onto $L_{p_1}(0, 1)$ and $L_{p_2}(0, 1)$, respectively. We know from Corollary 1.7 that ℓ_2 does not embed into Y , so by the generalization in [Joh1] of the theorem from [JO] used earlier, for $k = 1, 2$, the operator $S_k J$ factors through ℓ_{p_k} . Hence J factors through $\ell_{p_1} \oplus \ell_{p_2}$, whence Y embeds into $\ell_{p_1} \oplus \ell_{p_2}$ as a complemented subspace. However, by a result of Edelstein-Wojtaszczyk [Ede], [Woj], [EW] (or see [LT1, p. 82]), every complemented subspace of $\ell_{p_1} \oplus \ell_{p_2}$ is isomorphic to ℓ_{p_1}, ℓ_{p_2} , or $\ell_{p_1} \oplus \ell_{p_2}$. Theorem 2.1 eliminates the first two possibilities.

When p_1 and p_2 are smaller than two, we pass to the duals: Y^* is isomorphic to a complemented subspace of $L_{q_1}(0, 1) \oplus L_{q_2}(0, 1)$, where q_k is the conjugate index to p_k . Y^* also does not contain an isomorph of ℓ_2 (since Y^* has type 2, every copy of ℓ_2 in Y^* is complemented). The reasoning in the last paragraph shows that Y^* is isomorphic to $\ell_{q_1} \oplus \ell_{q_2}$. ■

In sections 4 and 5 we prove that for $1 \leq p < \infty$, the p -convexified version \mathcal{T}^p of Tsirelson's space is uniformly homeomorphic to $\mathcal{T}^p \oplus \ell_p$. Here we show that for $1 < p < \infty$, \mathcal{T}^p is uniformly homeomorphic to at most two isomorphically distinct spaces. Since it is desirable to have general conditions which limit the isomorphism class of spaces uniformly homeomorphic to a given space, it seems worthwhile to prove some results in this direction which might be used elsewhere. The reader who is interested only in the example can skip to Proposition 2.7 and substitute \mathcal{T}^{p_k} for X_k in the statement.

We begin with a definition.

Definition 2.3. A Banach space X is said to be as. \mathcal{L}_p , $1 \leq p \leq \infty$, provided there exists λ so that for every n there is a finite codimensional subspace Y so that every n dimensional subspace of Y is contained in a subspace of X which is λ -isomorphic to $L_p(\mu)$ for some μ .

A space is as. \mathcal{L}_2 if and only if it is asymptotically Hilbertian in the sense of Pisier [Pis]. We avoided the term asymptotically \mathcal{L}_p because when $p \neq 2$ as. \mathcal{L}_p may not be the "right" definition for asymptotically \mathcal{L}_p . The definition we give is also not the definition

one gets by specializing Pisier's [Pis, p. 221] *as. property* (P) to $P = \mathcal{L}_p$ -structure; however, Pisier's definition seems "right" only for hereditary properties.

In section 5 we review some of the properties of the Tsirelson spaces. Here we just mention that \mathcal{T}^p has an unconditional basis $\{e_n\}_{n=1}^\infty$ for which there is a constant λ such that for every n , there exists an m , so that every n -tuple of disjointly supported unit vectors in $\text{span}\{e_k\}_{k=m}^\infty$ is λ -equivalent to the unit vector basis in ℓ_p^n . It is easily seen that a space X with this property is *as. \mathcal{L}_p* and also that $X_{\mathcal{U},0}$ is isomorphic to $L_p(\mu)$ for some measure μ . This property of such spaces can be generalized to the class of *as. \mathcal{L}_p* spaces.

Proposition 2.4.a. *Suppose that X is *as. \mathcal{L}_p* , $1 < p < \infty$. Then for every ultrafilter \mathcal{U} on the natural numbers, the space $X_{\mathcal{U},0}$ is a \mathcal{L}_p space.*

Proof. First use the James' characterization of reflexivity/superreflexivity to see that X is superreflexive (see, for example, the argument in [Pis, pp. 220&222] for the similar case of asymptotically Hilbertian spaces). Therefore $X_{\mathcal{U}}$ splits as the direct sum of X and $X_{\mathcal{U},0}$. Now if \tilde{E} is a finite; say, n ; dimensional subspace of $X_{\mathcal{U},0}$, then it is not hard to see that there exists a sequence $\{E_k\}_{k=1}^\infty$ of n -dimensional subspaces of X which converges weakly to zero along \mathcal{U} (in the sense that every bounded sequence $\{x_k\}_{k=1}^\infty$ with x_k in E_k converges weakly to 0 along \mathcal{U}) so that $(E_k)_{\mathcal{U}} = \tilde{E}$. Since $\{E_k\}_{k=1}^\infty$ converges weakly to zero along \mathcal{U} and have bounded dimension, given any finite codimensional subspace Y of X , a standard small perturbation argument shows that there are space automorphisms T_k on X with $\lim_{\mathcal{U}} \|I - T_k\| = 0$ and $T_k E_k \subset Y$ for all k . Consequently, since X is *as. \mathcal{L}_p* (with constant λ , say), there are superspaces F_k in X of E_k with F_k λ_k -isomorphic to $L_p(\mu_k)$ and $\lim_{\mathcal{U}} \lambda_k \leq \lambda$. In particular, the inclusion mappings from the E_k 's into X uniformly factor through L_p -spaces for a \mathcal{U} -large set of k 's. Passing to ultraproducts and using the fact that an ultraproduct of L_p -spaces is again an L_p -space, we conclude that the inclusion mapping from \tilde{E} into $X_{\mathcal{U}}$ factors through an L_p -space with the norms of the factoring maps independent of k . The same can be said about the inclusion mapping from \tilde{E} into $X_{\mathcal{U},0}$ since $X_{\mathcal{U},0}$ is complemented in $X_{\mathcal{U}}$. Finally, we conclude from [LR] that $X_{\mathcal{U},0}$ is either a \mathcal{L}_p or isomorphic to a Hilbert space. But the latter is impossible for $p \neq 2$. Indeed, ℓ_p is finitely crudely representable in X and hence also in every finite codimensional subspace of X , and this implies that ℓ_p embeds into $X_{\mathcal{U},0}$. ■

Proposition 2.4.b. *Let X be a reflexive space for which there exists an ultrafilter \mathcal{U} on the natural numbers so that $X_{\mathcal{U},0}$ is \mathcal{L}_p , $1 < p < \infty$. Then X is *as. \mathcal{L}_p* .*

Proof. Assume that the \mathcal{L}_p constant of $X_{\mathcal{U},0}$ is smaller than λ , but that $X_{\mathcal{U},0}$ is not *as. \mathcal{L}_p* with constant λ . Let n be such that every finite codimensional subspace of X contains an n -dimensional subspace no superspace of which is λ -isomorphic to an L_p space. Using a standard basic sequence construction, we can construct a finite dimensional decomposition $\{E_k\}_{k=1}^\infty$ of n -dimensional subspaces of X so that no superspace of any E_k is λ -isomorphic to an L_p space. But since X is reflexive, the sequence $\{E_k\}_{k=1}^\infty$ converges weakly to zero, and hence the ultraproduct $E = (E_k)_{\mathcal{U}}$ is an n -dimensional subspace of $X_{\mathcal{U},0}$. Then there is a finite (say, m ,) dimensional subspace of $X_{\mathcal{U},0}$ which contains E and whose Banach-Mazur distance from ℓ_p^m is less than λ . It follows that F can be represented as $(F_k)_{\mathcal{U}}$,

where for each k F_k is an m -dimensional subspace of X which contains E_k . Necessarily the set of k 's for which F_k is λ -isomorphic to ℓ_p^m is in \mathcal{U} , hence is nonempty. ■

The next lemma is an as. version of the result [LR] that a complemented subspace of a \mathcal{L}_p space is either a \mathcal{L}_p space or a \mathcal{L}_2 space.

Lemma 2.5. *If X is as. \mathcal{L}_p , $1 < p < \infty$, and Y is a complemented subspace of X , then Y is as. \mathcal{L}_p or as. \mathcal{L}_2 . Y is as. \mathcal{L}_p iff ℓ_p is finitely crudely representable in Y iff Y contains uniformly complemented ℓ_p^n 's for all n .*

Proof. The ultraproduct of a projection from X onto Y defines a projection from $X_{\mathcal{U}}$ onto $Y_{\mathcal{U}}$ which projects $X_{\mathcal{U},0}$ onto $Y_{\mathcal{U},0}$. Thus the first conclusion follows from Proposition 2.4 and the classical theory of \mathcal{L}_p -spaces [LPe], [LR]. The last sentence follows from the fact [KP] that a complemented subspace of an L_p space which is not isomorphic to a Hilbert space contains a complemented subspace isomorphic to ℓ_p and the fact, mentioned earlier, that one can pull down finite dimensional well-complemented subspaces of an ultrapower to the base space. ■

We also need the as. version of Ribe's [Rib2] theorem that a uniform homeomorph of a \mathcal{L}_p space, $1 < p < \infty$, is again a \mathcal{L}_p space.

Lemma 2.6. *If X is separable as. \mathcal{L}_p , $1 < p < \infty$, and Y is uniformly homeomorphic to X , then Y is as. \mathcal{L}_p .*

Proof. Y is finitely crudely representable in X by [Rib1] (or see [HM]), so Y is superreflexive. Reasoning as in the proof of Theorem 2.2, we get Lipschitz equivalent ultraproducts $X_{\mathcal{U}} = X \oplus X_{\mathcal{U},0}$ and $Y_{\mathcal{U}} = Y \oplus Y_{\mathcal{U},0}$ of X and Y , respectively; and, by Proposition 2.4, $X_{\mathcal{U},0}$ is \mathcal{L}_p . Continuing as in the earlier proof, we get separable subspaces X_1 of $X_{\mathcal{U},0}$ and Y_1 of $Y_{\mathcal{U},0}$ with X_1 a \mathcal{L}_p space so that $X \oplus X_1$ is Lipschitz equivalent to $Y \oplus Y_1$. By another result from [HM], Y is isomorphic to a complemented subspace of $X \oplus X_1$, which of course is as. \mathcal{L}_p . Hence by Lemma 2.5, Y is as. \mathcal{L}_p or as. \mathcal{L}_2 . But X is finitely representable in Y , so the latter is impossible. ■

Remark. The separability assumption is not needed in Lemma 2.6. One way to see this is to check that if X is as. \mathcal{L}_p , $1 < p < \infty$, then $X = X_1 \oplus X_2$ with X_1 separable and X_2 \mathcal{L}_p . Casazza and Shura [CS, p. 150] proved this for $p = 2$ and the general case can be done similarly.

The proof of Lemma 2.6 yields that if the separable as. \mathcal{L}_2 -space X is uniformly homeomorphic to Y , then $X \oplus \ell_2$ and $Y \oplus \ell_2$ are Lipschitz equivalent and hence by [HM] each is isomorphic to a complemented subspace of the other. Proposition 2.7 gives a version of this for certain direct sums of as. \mathcal{L}_p spaces. For its proof, we need a consequence of the preceding. Suppose that $X \oplus Y$ contains uniformly complemented copies of ℓ_r^n for all n , X is as. \mathcal{L}_p , $1 < p < \infty$, and $2 \neq r \neq p$. Then Y contains uniformly complemented copies of ℓ_r^n for all n . Indeed, one observes that ℓ_r embeds complementably into $(X \oplus Y)_{\mathcal{U}} = X_{\mathcal{U}} \oplus Y_{\mathcal{U}}$, and hence complementably into either $X_{\mathcal{U}}$ or $Y_{\mathcal{U}}$. But $X_{\mathcal{U}}$ is as. \mathcal{L}_p (for example, by Proposition 2.4.a), so if ℓ_r embeds complementably into $X_{\mathcal{U}}$, then ℓ_r is as. \mathcal{L}_p or as. \mathcal{L}_2 by Lemma 2.5. The latter is clearly impossible, and so is the former by the last statement in Lemma 2.5 and well-known properties of ℓ_r . Hence ℓ_r embeds complementably into $Y_{\mathcal{U}}$, whence $Y_{\mathcal{U}}$, *a fortiori* Y itself, contains uniformly complemented copies of ℓ_r^n for all n .

Proposition 2.7. *Let $1 < p_1 < p_2 < \dots < p_n < 2$ or $2 < p_1 < p_2 < \dots < p_n < \infty$ and for $1 \leq k \leq n$, assume that X_k is a separable, as. \mathcal{L}_{p_k} space with an unconditional basis which has an lower r -estimate for some $r < 2$ or an upper p -estimate for some $p > 2$. Set $X = \sum_{k=1}^n X_k$. If X is uniformly homeomorphic to a Banach space Y , then $X \oplus \sum_{k=1}^n \ell_{p_k}$ and $Y \oplus \sum_{k=1}^n \ell_{p_k}$ are isomorphic to complemented subspaces of each other.*

Proof. As in the proof of Theorem 2.2, we assume, for simplicity of notation, that $n = 2$. Reasoning as in the proof of Lemma 2.6, we get Lipschitz equivalent ultraproducts $X_{\mathcal{U}}$ and $Y_{\mathcal{U}}$ of X and Y , respectively. $X_{\mathcal{U}}$ is $X \oplus \tilde{X}_1 \oplus \tilde{X}_2$, where \tilde{X}_k is \mathcal{L}_{p_k} for $k = 1, 2$. Continuing as in the proof of Lemma 2.6, we conclude that $X \oplus Z_1 \oplus Z_2$ is Lipschitz equivalent to $Y \oplus W$ for some separable \mathcal{L}_{p_k} spaces Z_k and some W . By enlarging the Z_k 's and W , if necessary, we can assume that $Z_k = L_{p_k}(0, 1)$. So Y embeds into $(X_1 \oplus L_{p_1}) \oplus (X_2 \oplus L_{p_2})$ as a complemented subspace. However, we also know from Corollary 1.7 that ℓ_2 is not isomorphic to a subspace of Y , so, reasoning as in the proof of Theorem 2.2, we conclude that Y is isomorphic to a complemented subspace of $(X_1 \oplus \ell_{p_1}) \oplus (X_2 \oplus \ell_{p_2})$. The two spaces in parentheses are totally incomparable, so by the theorem of Edelstein-Wojtaszczyk [EW] (or [LT1, p. 80]), $Y = Y_1 \oplus Y_2$ with Y_k isomorphic to a complemented subspace of $X_k \oplus \ell_{p_k}$. Lemma 2.5 now tells us that Y_k is as. \mathcal{L}_{p_k} or as. \mathcal{L}_2 . But Y is uniformly homeomorphic to X and X contains uniformly complemented $\ell_{p_k}^n$'s for all n , hence so does Y by [HM] (see the earlier discussion on ultraproducts). But Y_1 cannot contain uniformly complemented $\ell_{p_2}^n$'s for all n , and so Y_2 must. Thus Y_2 is as. \mathcal{L}_{p_2} and, similarly, Y_1 is as. \mathcal{L}_{p_1} . This shows that the situation with X and Y is symmetrical, and we can conclude that X is isomorphic to a complemented subspace of $Y \oplus \ell_{p_1} \oplus \ell_{p_2}$. ■

Remark. In Proposition 2.7, even when $n = 1$ the hypothesis on X_1 cannot be weakened to “ X is \mathcal{L}_{p_1} , X has an unconditional basis, and no subspace of X is isomorphic to ℓ_2 ”. Indeed, $\mathcal{T}^{p_1} \oplus \mathcal{T}^2$ is as. \mathcal{L}_{p_1} for $1 < p_1 < \infty$ and thus is a counterexample by Proposition 5.7.

The conclusion in Proposition 2.7 is not completely satisfactory, since it is open whether a complemented subspace of \mathcal{T}^p which contains a complemented copy of \mathcal{T}^p must be isomorphic to \mathcal{T}^p . Fortunately, it is possible to improve Proposition 2.7 for spaces of Tsirelson type without solving this problem about \mathcal{T}^p . However, this requires rather more work, so we treat first the simpler case of \mathcal{T}^2 and related spaces, which already provides examples of spaces that are isomorphic to exactly two isomorphically distinct spaces. Basic to this argument is a recent result concerning differentiation [LP]:

Assume that X and Y are separable superreflexive Banach spaces. Let F be a Lipschitz map from X into Y and G a Lipschitz map from X into a finite dimensional normed space. Then for every $\epsilon > 0$, there is a point x_0 in X so that F and G are Gateaux differentiable at x_0 and moreover there exists $\delta > 0$ so that for $\|u\| \leq \delta$,

$$\|G(x_0 + u) - G(x_0) - Su\| \leq \epsilon \|u\|,$$

where S is the Gateaux derivative of G at x_0 .

In other words, G is “almost” (up to ϵ) even Fréchet differentiable at x_0 . An easy consequence of this result and the reasoning used in [HM] is the following:

Proposition 2.8. *Suppose that F is a mapping from the separable, superreflexive space X onto a Banach space Y , the Lipschitz constant of F is one and the Lipschitz constant of F^{-1} is $C < \infty$. Then for every finite dimensional subspace E of Y^* and each $1 > \epsilon' > 0$, there is a norm one operator T from X into Y so that $\|T^{-1}\| \leq C$, TX is C -complemented in Y , and $\|T|_E^*{}^{-1}\| \leq (1 - \epsilon')^{-1}C$.*

Proof. Note that the space Y is also superreflexive (e.g., by [Rib1]). We shall see that for T one may take the Gateaux derivative of F at a suitable point x_0 in X ; the point depends on the subspace E of Y^* . In view of (the proof of) Proposition 2.1 of [HM] and the fact that T is the Gateaux derivative of F at some point, the conclusions that $\|T^{-1}\| \leq C$ and TX is C -complemented in Y are automatic. To find the point x_0 , apply the result from [LP] to the mapping F and an auxiliary mapping G , which we now define: Let G be the map F followed by the evaluation (quotient) mapping Q from Y onto E^* , so that for x in X and y^* in E , $Gx(y^*) = y^*(Fx)$. Given $\epsilon' > 0$, set $\epsilon = \frac{\epsilon'}{C}$ and get x_0 , $\delta > 0$, and S from the [LP] result stated above. For notational simplicity, assume that $x_0 = 0$ and $F(0) = 0$ (this is “without loss of generality” because it amounts to making one translation in X and another in Y). Of course, $S = QT$, where T is the Gateaux derivative of F at 0. Only the last conclusion, that $\|T|_E^*{}^{-1}\| \leq (1 - \epsilon')^{-1}C$, needs to be checked. Since Q^* is the inclusion mapping from E into Y^* , this is the same as checking that $\|S^{*-1}\| \leq (1 - \epsilon')^{-1}C$. So suppose that y^* is a norm one vector in E and choose a vector y in Y of norm $\frac{\delta}{C}$ so that $y^*(y)$ is equal to $\frac{\delta}{C}$ (an approximation would be OK but is not needed because Y is reflexive). Set $x = F^{-1}y$, so that $\|x\| \leq \delta$. Then

$$\begin{aligned} |S^*y^*(x) - \frac{\delta}{C}| &= |S^*y^*(x) - y^*(Fx)| = |y^*(Sx) - Gx(y^*)| \\ &\leq \|y^*\| \|Sx - Gx\| \leq \frac{\epsilon'}{C} \|x\| \leq \frac{\epsilon'}{C} \delta. \end{aligned}$$

Since $\|x\| \leq \epsilon$, this gives $\|S^*y^*\| \geq \frac{1-\epsilon'}{C}$, so that $\|S^{*-1}\| \leq (1 - \epsilon')^{-1}C$, as desired. \blacksquare

Proposition 2.9. *Suppose that X is separable, as \mathcal{L}_2 , does not contain a complemented subspace isomorphic to ℓ_2 , and is isomorphic to its hyperplanes. If X is uniformly homeomorphic to Y , then Y is isomorphic either to X or to $X \oplus \ell_2$.*

Proof. The main part of the proof is devoted to proving the following

Claim. *$X \oplus \ell_2$ is isomorphic to $Y \oplus \ell_2$.*

Assuming the Claim, we complete the proof of Proposition 2.9 as follows: If Y contains a complemented subspace isomorphic to ℓ_2 , then Y is isomorphic to $Y \oplus \ell_2$ and so we are done, so assume otherwise. By a slight abuse of notation, write $X \oplus \ell_2 = Y \oplus W$ with W isomorphic to ℓ_2 . Since X does not contain a complemented subspace isomorphic to

ℓ_2 , every operator from X into ℓ_2 is strictly singular; that is, not an isomorphism on any infinite dimensional subspace of Y . Hence by the result of Edelstein-Wojtaszczyk [EW], by applying a space automorphism to $X \oplus \ell_2$, we can assume, without loss of generality, that $Y = Y_X \oplus Y_2$, $W = W_X \oplus W_2$ with Y_X, W_X subspaces of X and Y_2, W_2 subspaces of ℓ_2 . But Y_2 must be finite dimensional since Y does not contain a complemented subspace isomorphic to ℓ_2 . Similarly, W_X is finite dimensional since X does not contain a complemented subspace isomorphic to ℓ_2 . Thus Y is isomorphic to X plus or minus a finite dimensional space, which is isomorphic to X .

We turn to the proof of the Claim.

As mentioned earlier, the proof of Lemma 2.6 shows that $Y' \equiv Y \oplus \ell_2$ is Lipschitz equivalent to $X' \equiv X \oplus \ell_2$. Let $\{E_n\}_{n=1}^\infty$ be an increasing sequence of finite dimensional subspaces of X^* whose union is dense in X^* and, using Proposition 2.8, get for each n a norm one isomorphism T_n from Y' onto a C -complemented subspace of X' with $\|T_n^{-1}\| \leq C$, $\|T_n^*|_{E_n}^{-1}\| \leq C$. The ultraproduct T of the isomorphisms T_n is an isomorphism from the ultrapower $Y'_\mathcal{U}$ onto a complemented subspace of the ultrapower $X'_\mathcal{U}$: $T(y_1, y_2, \dots) = (T_1 y_1, T_2 y_2, \dots)$. Since X' is superreflexive, $(X'_\mathcal{U})^* = X'^*\mathcal{U}$ and the adjoint T^* is defined for $\{x_n\}_{n=1}^\infty$ a bounded sequence in X'^* by $T^*(x_1^*, x_2^*, \dots) = (T_1^* x_1^*, T_2^* x_2^*, \dots)$. Identify X^* with $\{(x^*, x^*, \dots) : x^* \in X^*\}$; evidently $T|_{X^*}$ is an isomorphism. By Proposition 2.4.a, $X'^*\mathcal{U}$ is isomorphically just X^* direct summed with a nonseparable Hilbert space. Thus by coming down to appropriate separable subspaces of the ultrapowers, we get that there is a projection P from $X \oplus \ell_2$ onto a subspace isomorphic to $Y \oplus \ell_2$ for which $P|_{X^*}^*$ is an isomorphism. This implies that the projection from $X^* \oplus \ell_2$ onto ℓ_2 is an isomorphism on the subspace $\mathcal{R}(P)^\perp$ of $X^* \oplus \ell_2$; in particular, $\mathcal{R}(P)^\perp$ is isomorphic to a Hilbert space. But $\mathcal{R}(P)^\perp$ is isomorphic to the dual of the null space of P , so the null space of P is also isomorphic to a Hilbert space. \blacksquare

Remark. The last two hypotheses on X were not used in the proof of the Claim.

Remark. The proof of Proposition 2.9 yields that if X is a separable as. \mathcal{L}_2 space and every infinite dimensional subspace of X contains a complemented subspace which is isomorphic to ℓ_2 , then X is determined by its uniform structure. The simplest examples of such spaces which are different from ℓ_2 are ℓ_2 -sums of $\ell_{p_n}^{k_n}$ with $p_n \rightarrow 2$ and $k_n \rightarrow \infty$ sufficiently quickly.

Proposition 2.10. *Let $1 < p_1 < p_2 < \dots < p_n < 2$ or $2 < p_1 < p_2 < \dots < p_n < \infty$ and for $1 \leq k \leq n$, assume that X_k is a separable, as. \mathcal{L}_{p_k} , has an unconditional basis which has an upper p -estimate for some $p > 2$ or a lower r -estimate for some $r < 2$, is isomorphic to its hyperplanes, and X_k^* does not contain a subspace isomorphic to ℓ_s for any s . Set $X = \sum_{k=1}^n X_k$. If X is uniformly homeomorphic to a Banach space Y , then Y is isomorphic to $X \oplus \sum_{k \in F} \ell_{p_k}$ for some subset F of $\{1, 2, \dots, n\}$.*

Proof. As usual, assume $n = 2$. From the proof of Proposition 2.7, we know that $Y = Y_1 \oplus Y_2$ with Y_k as. \mathcal{L}_{p_k} and that $X \oplus L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$ is Lipschitz equivalent to $Y \oplus L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$. The main part of the proof is devoted to proving the following

Claim. *There exists a quotient W of $L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$ so that $Y \oplus W$ is isomorphic to $X \oplus L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$.*

Assuming the Claim, we complete the proof as follows: Every operator from X^* to $L_{q_1}(0, 1) \oplus L_{q_2}(0, 1)$ (where q_k is the conjugate index to p_k) is strictly singular by [KP], [Ald], [KM], so the Edelstein-Wojtaszczyk theorem [EW], [LT1, p. 80] says that by applying a space automorphism to $X \oplus L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$, we may assume, without loss of generality, that $Y = Y_X \oplus Y_L$, $W = W_X \oplus W_L$ with Y_X, W_X subspaces of X and Y_L, W_L subspaces of $L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$. As in the proof of Theorem 2.2, Theorem 1.7 and old results from the linear theory imply that Y_L is isomorphic to $\ell_{p_1}, \ell_{p_2}, \ell_{p_1} \oplus \ell_{p_2}$, or is finite dimensional (in which case we can assume the dimension is zero because X is isomorphic to its hyperplanes). So the proof is complete once we observe that W_X must be finite dimensional. But W_X^* embeds into $L_{q_1}(0, 1) \oplus L_{q_2}(0, 1)$ and hence by [KP], [Ald], [KM] would contain a subspace isomorphic to ℓ_s for some s if it were infinite dimensional. This would contradict the hypotheses on X^* since W_X^* embeds into X^* .

We turn to the proof of the claim, which is only slightly more complicated than the proof of the Claim in Proposition 2.9.

Since $X' \equiv X \oplus L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$ is Lipschitz equivalent to $Y' \equiv Y \oplus L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$, we use Lemma 2.6 just as in the proof of Proposition 2.9 to get good isomorphisms T_n from Y' onto well-complemented subspaces of X' so that T_n^* is a good isomorphism on a subspace E_n of Y^* , where $\{E_n\}_{n=1}^\infty$ is an increasing sequence of finite dimensional subspaces of Y^* whose union is dense in Y^* . Taking ultrapowers, we get an isomorphism T from an ultrapower $Y'_{\mathcal{U}}$ of Y' onto a complemented subspace of $X'_{\mathcal{U}}$ so that T^* is an isomorphism on X^* . But we know that $Y'_{\mathcal{U}}$; respectively, $X'_{\mathcal{U}}$, is just Y ; respectively X , direct summed with the direct sum of a \mathcal{L}_{p_1} -space with a \mathcal{L}_{p_2} -space. Therefore by coming down to appropriate separable subspaces of the ultrapowers and, if necessary, enlarging the \mathcal{L}_{p_k} -spaces, we conclude that there is a projection P from $X \oplus L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$ onto a subspace isomorphic to $Y \oplus L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$ for which $P|_{X^*}$ is an isomorphism. This implies that the natural projection from $X^* \oplus L_{q_1}(0, 1) \oplus L_{q_2}(0, 1)$ onto $L_{q_1}(0, 1) \oplus L_{q_2}(0, 1)$ is an isomorphism on the subspace $\mathcal{R}(P)^\perp$; in particular, $\mathcal{R}(P)^\perp$ is isomorphic to a subspace of $L_{q_1}(0, 1) \oplus L_{q_2}(0, 1)$. But $\mathcal{R}(P)^\perp$ is isomorphic to the dual of the null space of P , so the null space of P is isomorphic to a quotient of $L_{p_1}(0, 1) \oplus L_{p_2}(0, 1)$. ■

3. Uniform Homeomorphs of c_0

In this section we prove:

Theorem 3.1. *Let X be a Banach space which has $C(\omega^\omega)$ as a quotient space. Then X is not uniformly homeomorphic to c_0 .*

In particular, c_0 is not uniformly homeomorphic to $C[0, 1]$ or any other $C(K)$ space except those which are isomorphic to c_0 , so Theorem 3.1 solves the problem mentioned in [Aha] whether c_0 and $C[0, 1]$ are Lipschitz equivalent. In fact, an immediate application of Theorem 3.1 and results of Alspach, Zippin, and Benyamini is:

Corollary 3.2. *Let X be a complemented subspace of a $C(K)$ space. If X is uniformly homeomorphic to c_0 , then X is isomorphic to c_0 .*

Proof. By [Als] (or see [AB]), since X does not have a quotient isomorphic to $C(\omega^\omega)$, the ϵ -Szlenk index of X is finite for each $\epsilon > 0$. But in [Ben1, proof of Theorem 3] it was shown (using, among other things, a small variation on a lemma in [Zip]) that then X must be isomorphic to a quotient of c_0 , hence X is isomorphic to c_0 by [JZ].

Although it is known [HM] that a uniform homeomorph of c_0 is a \mathcal{L}_∞ space, it is open whether it must be isomorphic to a complemented subspace of a $C(K)$ space.

Proof of Theorem 3.1. Denote by K the space ω^ω and let $K^{(1)}, K^{(2)}, \dots$ be the derived sets of K . Let R_n be the restriction mapping from $C(K)$ onto $C(K^{(n)})$.

Assume that there is a uniform homeomorphism U from c_0 onto X with inverse V and that there is a quotient mapping Q from X onto $C(K)$. Without loss of generality we assume that Q maps the open unit ball of X onto the open unit ball of $C(K)$.

For each n , let $s_{n,\infty}$ be the limiting Lipschitz constant of the map $S_n \equiv R_n Q U$. So $s_{n,\infty}$ is the smallest constant so that for each $\epsilon > 0$, there is a $d = d(\epsilon, n)$ so that if y, z are in c_0 with $\|y - z\| \geq d$, then $\|S_n y - S_n z\| \leq (s_{n,\infty} + \epsilon) \|y - z\|$. The sequence $\{s_{n,\infty}\}_{n=1}^\infty$ is decreasing and tends to a limit $s_{\infty,\infty}$ which is easily seen to be positive, so we can assume without loss of generality that $s_{\infty,\infty} = 1$.

Fix $\epsilon > 0$ small and take n_0 so that $s_{n_0,\infty} < 1 + \epsilon$. We can find a pair x, y of vectors in c_0 with $\|x - y\|$ as large as we please so that $(1 - \epsilon) \|x - y\| < \|S_{n_0+1} x - S_{n_0+1} y\|$; and, since $\|x - y\|$ is large, of necessity $\|S_{n_0} x - S_{n_0} y\| < (1 + \epsilon) \|x - y\|$. By making translations, we can assume that $y = -x$ and $U(-x) = -Ux$, so we have:

$$(3.1) \quad (1 - \epsilon) \|x\| < \|S_{n_0+1} x\| \leq \|S_{n_0} x\| < (1 + \epsilon) \|x\|.$$

Moreover, since $\|x\|$ can be taken arbitrarily large, we can assume, in view of Lemma 1.5 that

$$(3.2) \quad S_{n_0} [\text{Mid}(x, -x, 0)] \subset \text{Mid}(S_{n_0} x, -S_{n_0} x, \epsilon).$$

Choose N so that the magnitude of the k -th coordinate of x is less than ϵ for $k > N$ and let Y_0 be the finite codimensional subspace of c_0 consisting of those vectors whose first N coordinates are zero, so that

$$(3.3) \quad (\|x\| - \epsilon) \text{Ball}(Y_0) \subset \text{Mid}(x, -x, 0).$$

Now take t_0 in $K^{(n_0+1)}$ with $|S_{n_0+1} x(t_0)| = \|S_{n_0+1} x\| > (1 - \epsilon) \|x\|$. The point t_0 is a limit point of $K^{(n_0)}$, so $|S_{n_0} x(t)| > (1 - \epsilon) \|x\|$ for infinitely many points t in $K^{(n_0)}$; say, for $t \in B$.

Let W be the subspace of $C(K^{(n_0)})$ of functions which vanish on B . Recalling that $\|S_{n_0} x\| < (1 + \epsilon) \|x\|$, we observe that

$$(3.4) \quad \text{Mid}(S_{n_0} x, -S_{n_0} x, \epsilon) \subset W + 2\epsilon \|x\| \text{Ball} \left(C(K^{(n_0)}) \right).$$

Setting $X_0 = (R_{n_0} Q)^{-1} W$, we have from (3.2), (3.3), and (3.4) that

$$U[(\|x\| - \epsilon)\text{Ball}(Y_0)] \subset X_0 + 2\epsilon \|x\| \text{Ball}(X),$$

which contradicts the Gorelik Principle since $\|x\|$ is arbitrarily large and ϵ is arbitrarily small. \blacksquare

4. Uniform homeomorphisms between balls in “close” spaces

S. Mazur [Maz] proved that the unit balls of any two ℓ_p , $1 \leq p < \infty$, spaces are uniformly homeomorphic. This fact was extended by E. Odell and T. Schlumprecht [OS], as a tool in solving the distortion problem, to any two spaces with unconditional bases and nontrivial cotype. It follows that any two balls in such spaces are uniformly homeomorphic. However, the modulus of continuity of the uniform homeomorphism or its inverse generally gets worse as the radius of the balls increases. In Corollary 4.7 we show that if the two spaces are close to a common ℓ_p space then “large” balls are uniformly homeomorphic with good modulus (where “large” depends on p and how “close” the two spaces are to ℓ_p .)

We begin by recalling the definition of the generalized Mazur map of Odell and Schlumprecht. Let $b = \{b_i\}_{i=1}^n \in \mathbb{R}^+{}^n$ with $\|b\|_1 = \sum b_i = 1$ and let X be a Banach space with a 1-unconditional basis $\{e_i\}_{i=1}^\infty$. For $x = \sum_{i=1}^\infty x_i e_i \in X$ with $x_i \geq 0$, let $G_X(b, x) = \prod_{i=1}^n x_i^{b_i}$, $G_X(b) = \sup_{\|x\|=1} G_X(b, x)$ and $F_X(b) = F_{\ell_1, X}(b)$ the unique x with the same support as b for which the sup in the definition of $G_X(b)$ is attained.

Recall that, for $x = \sum_{i=1}^\infty x_i e_i$ and $0 < p < \infty$, we denote $|x|^p = \sum_{i=1}^\infty |x_i|^p e_i$ and that the unconditional basis $\{e_i\}_{i=1}^\infty$ is said to be p -convex (respectively, q -concave), with constant C , if $\|(\sum_{n=1}^N |x^n|^p)^{1/p}\| \leq C(\sum_{n=1}^N \|x^n\|^p)^{1/p}$ (respectively, $\|(\sum_{n=1}^N |x^n|^q)^{1/q}\| \geq C^{-1}(\sum_{n=1}^N \|x^n\|^q)^{1/q}$) for all finite sequences $\{x^n\}_{n=1}^N$ of vectors in X . (See [LT2, section 1.d] for a discussion of p -convexity and q -concavity).

Odell and Schlumprecht proved that, if X is q -concave with constant one for some $q < \infty$, then F_X extends naturally (i.e., homogeneously to the positive quadrant in ℓ_1 , then extending to the other quadrants so that F_X will commute with changes of signs, and finally by continuity from ℓ_1^n to ℓ_1) to a uniform homeomorphism of the unit balls of ℓ_1 and X . Moreover, the modulus of continuity of F_X and F_Y depend only on q . If X and Y both have 1-unconditional basis and nontrivial cotype we shall denote $F_{X,Y} = F_{\ell_1, Y} \circ F_{\ell_1, X}^{-1}$. The following lemma follows from a proof of Lozanovskii’s theorem [Loz], (see, for example, [T-J, Lemma 39.3], but note that misleading notation should be corrected).

Lemma 4.1. *Let X be a Banach space with a 1-unconditional basis $\{e_i\}_{i=1}^\infty$, let b be a finitely supported vector in the unit sphere of ℓ_1 with nonnegative coordinates, and set $x = F_X(b)$. Then $f = \sum_{i=1}^\infty \frac{b_i}{x_i} e_i^*$ is a norming functional of x . Here $\{e_i^*\}_{i=1}^\infty$ denotes the biorthogonal basis to $\{e_i\}_{i=1}^\infty$ and $\frac{0}{0}$ is interpreted as 0.*

If Y and Z are two spaces with 1-unconditional bases (or, more generally, Banach lattices of functions) and $0 < \theta < 1$ we denote by $X = Y^\theta Z^{1-\theta}$ the space of all sequences x for which $|x|$ can be written as $|x| = y^\theta z^{1-\theta}$ with $y \in Y$, $z \in Z$ (the operations are defined pointwise) with the norm

$$\|x\| = \inf\{\|y\|^\theta \|z\|^{1-\theta} : |x| = y^\theta z^{1-\theta}\}.$$

By a theorem of Calderon [Cal], $Y^\theta Z^{1-\theta}$ is the interpolation space, in the complex method, between X and Y with parameter $1-\theta$ commonly denoted by $(Y, Z)^{1-\theta}$ (see, for example, [BL]). The following lemma relates the functions F_X , F_Y , and F_Z .

Lemma 4.2. *Let $b \geq 0$ have finite support with $\|b\|_1 = 1$. Then for all spaces Y, Z with 1-unconditional bases,*

$$F_{Y^\theta Z^{1-\theta}}(b) = F_Y^\theta(b) F_Z^{1-\theta}(b).$$

Proof.

$$\begin{aligned} G_{Y^\theta Z^{1-\theta}}(b) &= \sup_{\|y\|^\theta \|z\|^{1-\theta}=1} \prod y_i^{\theta b_i} \prod z_i^{(1-\theta)b_i} \\ &= \sup_{y \in Y} \left(\prod y_i^{\theta b_i} \sup_{\|z\|_Z = \|y\|_Y^{-\theta/(1-\theta)}} \left(\prod z_i^{b_i} \right)^{1-\theta} \right) \\ &= \sup_{y \in Y} \|y\|_Y^{-\theta} \prod y_i^{\theta b_i} G_Z^{1-\theta}(b) \\ &= G_Y^\theta(b) G_Z^{1-\theta}(b). \end{aligned}$$

Let $y = F_Y(b)$, $z = F_Z(b)$, and $x = y^\theta z^{1-\theta}$. Then $\|x\| \leq 1$ and

$$G_{Y^\theta Z^{1-\theta}}(b, x) = \prod y_i^{\theta b_i} \prod z_i^{(1-\theta)b_i} = G_Y^\theta(b) G_Z^{1-\theta}(b) = G_{Y^\theta Z^{1-\theta}}(b).$$

Consequently, $\|x\| = 1$ and $F_{Y^\theta Z^{1-\theta}}(b) = x$. ■

Recall that the p -convexification of a space X with a 1-unconditional basis (or a general Banach lattice of functions) is the space

$$X^{(p)} = \{x = \sum x_i e_i : \|x\|_{X^{(p)}} = \| |x|^p \|^{1/p} < \infty\}.$$

The expression $\|\cdot\|_{X^{(p)}}$ is a norm whenever $p \geq 1$. If $p < 1$, $X^{(p)}$ is sometimes called the $1/p$ -concavification of X . It is still a Banach space if the original X is $1/p$ -convex with constant one. (See [LT2, section 1.d] for more information on these operations). Note that $X^{(1/\theta)} = X^\theta \ell_\infty^{1-\theta}$.

Corollary 4.3. *If b is as in Lemma 4.2, then $F_{X^{(1/\theta)}}(b) = F_X^\theta(b)$ for all spaces X with 1-unconditional basis.*

Proof. Since b has finite support, it is enough to prove the corollary when X is finite dimensional. This case follows from Lemma 4.2 and the observation that for b a strictly positive vector in ℓ_1^n , $F_{\ell_\infty^n}(b) = (1, 1, \dots, 1)$. ■

For $1 \leq p \leq \infty$, p' denotes the conjugate index to p .

Lemma 4.4. *If X is p -convex and $r \leq 2p$ concave, both with constant one, then $X = \ell_p^\theta Y^{1-\theta}$ with Y p -convex and $2p$ -concave, both with constant one, and $\theta = 1 - \frac{2}{(r/p)'} = \frac{2p}{r} - 1$.*

Proof. Even if we just assume that X is a space of functions on the natural numbers, the convexity and concavity assumptions guarantee that the unit vector basis is a

1-unconditional basis for X . This implies that the general case follows from the case where X is finite dimensional, which we assume in the sequel to avoid irrelevant topological and duality problems that arise in the infinite dimensional setting.

Assume first that $p = 1$. Since for any space W with 1-unconditional basis, $W^{(p)} = W^{1/p} \ell_\infty^{1-1/p}$, $X^* = (X^{*(\frac{2}{r})})^{\frac{2}{r}} \ell_\infty^{1-\frac{2}{r}}$ so $X = (X^{*(\frac{2}{r})})^{1-\theta} \ell_1^\theta$ and $X^{*(\frac{2}{r})}$ is 2-concave with constant one.

For a general p , $X = (X^{(\frac{1}{p})})^{\frac{1}{p}} \ell_\infty^{1-\frac{1}{p}} = (Z^{1-\tau} \ell_1^\tau)^{\frac{1}{p}} \ell_\infty^{1-\frac{1}{p}}$ with $\tau = 1 - \frac{2}{(r/p)'} and Z 2-concave with constant one. This can be rewritten as$

$$X = (\ell_1^{\frac{1}{p}} \ell_\infty^{1-\frac{1}{p}})^\tau (Z^{\frac{1}{p}} \ell_\infty^{1-\frac{1}{p}})^{1-\tau} = \ell_p^\tau (Z^{(p)})^{1-\tau}.$$

■

We are now ready to investigate the modulus of continuity of $F_{\ell_p, X}$ for spaces X close to ℓ_p .

Proposition 4.5. *Let X be p -convex and r -concave, both with constant one, and with $r \leq 2p$. Then the modulus of uniform continuity $\varphi(\epsilon)$ of $F_{\ell_p, X}$ on the unit ball of ℓ_p is bounded by $2(\epsilon^{\frac{2p}{r}-1} + (1 - \frac{p}{r})\varphi_0(\epsilon))$ for some function φ_0 depending only on p .*

Proof. Let u, v be two nonnegative vectors of norm one in ℓ_p . It follows from Lemma 4.2 that, with the notation of Lemma 4.4,

$$\begin{aligned} \|F_{\ell_p, X}(u) - F_{\ell_p, X}(v)\|_X &= \|u^\theta F_{\ell_p, Y}^{1-\theta}(u) - v^\theta F_{\ell_p, Y}^{1-\theta}(v)\|_X \\ (4.1) \quad &\leq \left\| w^\theta \left| F_{\ell_p, Y}^{1-\theta}(u) - F_{\ell_p, Y}^{1-\theta}(v) \right| \right\|_X + \left\| |u^\theta - v^\theta| F_{\ell_p, Y}^{1-\theta}(u) \vee F_{\ell_p, Y}^{1-\theta}(v) \right\|_X, \end{aligned}$$

where $w = \{w_i\}$ is defined by $w_i = u_i$ whenever the i -th component of $F_{\ell_p, Y}(u)$ is smaller than that of $F_{\ell_p, Y}(v)$ and $w_i = v_i$ otherwise.

Now, the second term in 4.1 is dominated by

$$(4.2) \quad \left\| |u - v|^\theta F_{\ell_p, Y}^{1-\theta}(u) \vee F_{\ell_p, Y}^{1-\theta}(v) \right\|_X \leq \|u - v\|_p^\theta \|F_{\ell_p, Y}(u) \vee F_{\ell_p, Y}(v)\|_Y^{1-\theta} \leq 2^{1-\theta} \|u - v\|_p^\theta,$$

while the first term is dominated by

$$\begin{aligned} &(1 - \theta) \left\| \left(\frac{w}{F_{\ell_p, Y}(u) \wedge F_{\ell_p, Y}(v)} \right)^\theta |F_{\ell_p, Y}(u) - F_{\ell_p, Y}(v)| \right\|_X \\ &\leq (1 - \theta) \left\| \left(\frac{w}{F_{\ell_p, Y}(u) \wedge F_{\ell_p, Y}(v)} \right) |F_{\ell_p, Y}(u) - F_{\ell_p, Y}(v)| \right\|_p^\theta \|F_{\ell_p, Y}(u) - F_{\ell_p, Y}(v)\|_Y^{1-\theta} \\ &= (1 - \theta) \left\| \left(\frac{w}{F_{\ell_p, Y}(u) \wedge F_{\ell_p, Y}(v)} \right)^p |F_{\ell_p, Y}(u) - F_{\ell_p, Y}(v)|^p \right\|_1^{\theta/p} \|F_{\ell_p, Y}(u) - F_{\ell_p, Y}(v)\|_Y^{1-\theta}. \end{aligned}$$

By Corollary 4.3, $F_{\ell_p, Y}(u) = F_{\ell_1, Y}(u^p) = F_{\ell_1, Y^{(1/p)}}(u^p)$. Thus, by Lemma 4.1, $\left(\frac{u}{F_{\ell_p, Y}(u)} \right)^p$ and $\left(\frac{v}{F_{\ell_p, Y}(v)} \right)^p$ are norm one functionals on $Y^{(1/p)}$ and the first term in (4.1) is dominated

by

$$\begin{aligned}
(1-\theta) & \left\| \left(\frac{u}{F_{\ell_p, Y}(u)} \vee \frac{v}{F_{\ell_p, Y}(v)} \right)^p |F_{\ell_p, Y}(u) - F_{\ell_p, Y}(v)|^p \right\|_1^{\theta/p} \|F_{\ell_p, Y}(u) - F_{\ell_p, Y}(v)\|_Y^{1-\theta} \\
& \leq (1-\theta) 2^{\theta/p} \| |F_{\ell_p, Y}(u) - F_{\ell_p, Y}(v)|^p \|_{Y^{(1/p)}}^{\theta/p} \|F_{\ell_p, Y}(u) - F_{\ell_p, Y}(v)\|_Y^{1-\theta} \\
& = (1-\theta) 2^{\theta/p} \|F_{\ell_p, Y}(u) - F_{\ell_p, Y}(v)\|_Y.
\end{aligned}$$

Combining this with (4.1) and (4.2) and using the Odell and Schlumprecht result we get that

$$\begin{aligned}
\|F_{\ell_p, X}(u) - F_{\ell_p, X}(v)\|_X & \leq 2^{1-\theta} \|u - v\|_p^\theta + (1-\theta) \varphi_0(\|u - v\|_p) \\
& \leq 2(\|u - v\|_p^{\frac{2p}{r}-1} + (1 - \frac{p}{r}) \varphi_0(\|u - v\|_p))
\end{aligned}$$

for some function φ_0 depending only on p . ■

Next we investigate the modulus of uniform continuity of F_{X, ℓ_p} for spaces close to ℓ_p .

Proposition 4.6. *Let X be p -convex and r -concave, both with constant one, and with $r \leq 2p$. Then the modulus of uniform continuity $\varphi(\epsilon)$ of $F_{\ell_p, X}^{-1} = F_{X, \ell_p}$ on the unit ball of X is bounded by $2(\epsilon + (1 - \frac{p}{r})^{1/p} \varphi_0(\epsilon))$ for some function φ_0 depending only on p .*

Proof. Write $X^{(1/p)} = \ell_1^\theta Y^{1-\theta}$ with Y 2-concave and $\theta = \frac{2p}{r} - 1$. Note that for $x \in X$ with $\|x\| = 1$ and $x \geq 0$, $F_{\ell_p, X}^{-1}(x) = F_{X, \ell_p}(x) = (x \cdot f_x)^{1/p}$ where f_x is the norming functional of x . Let x, x' be two norm one vectors in the positive cone of X . Writing $x^p = y \cdot f_y^\theta$, $\frac{f_x}{x^{p-1}} = f_y^{1-\theta}$, we get from Lemma 4.2 that y has norm one in Y and f_y norms it. Similarly let y' be the norm one vector in Y corresponding to x' . Let z, z' be defined by

$$z(i) = \begin{cases} x(i) & \text{if } f_y(i) \leq f_{y'}(i) \\ x'(i) & \text{otherwise} \end{cases}$$

and

$$z'(i) = \begin{cases} x'(i) & \text{if } z(i) = x(i) \\ x(i) & \text{otherwise} \end{cases}.$$

Then

$$\begin{aligned}
\|(x \cdot f_x)^{1/p} - (x' \cdot f_{x'})^{1/p}\|_p & = \|x \cdot (\frac{f_x}{x^{p-1}})^{1/p} - x' \cdot (\frac{f_{x'}}{x'^{p-1}})^{1/p}\|_1^{1/p} \\
& \leq 2^{(p-1)/p} [\|x - x'\|_p^{1/p} \cdot \|\frac{f_{z'}}{z'^{p-1}}\|_1^{1/p} + \|z^p \cdot (\frac{f_x}{x^{p-1}})^{1/p} - (\frac{f_{x'}}{x'^{p-1}})^{1/p}\|_1^{1/p}] \\
& \equiv 2^{(p-1)/p} [I + II].
\end{aligned}$$

Now,

$$I \leq \|x - x'\|_p^{1/p} \max\left\{ \frac{f_x}{x^{p-1}}, \frac{f_{x'}}{x'^{p-1}} \right\} \|x - x'\|_X^{1/p} \leq 2^{1/p} \|x - x'\|_X$$

by Lemma 4.1, and

$$\begin{aligned}
II &\leq \|z^p \cdot |\frac{f_x}{x^{p-1}} - \frac{f_{x'}}{x'^{(p-1)}}|\|_1^{1/p} \\
&\leq \|z^p \cdot |f_y^{1-\theta} - f_{y'}^{1-\theta}|\|_1^{1/p} \\
&\leq (1-\theta)^{1/p} \|z^p \cdot f_y^{-\theta} \cdot |f_y - f_{y'}|\|_1^{1/p} \\
&\leq (1-\theta)^{1/p} \|\max\{y, y'\} \cdot |f_y - f_{y'}|\|_1^{1/p} \\
&\leq (1-\theta)^{1/p} [\|yf_y - y'f_{y'}\|_1 + \|y - y'\| \max\{f_y, f_{y'}\}\|_1]^{1/p} \\
&\leq (1-\theta)^{1/p} [\phi(\|y - y'\|) + 2\|y - y'\|]^{1/p}
\end{aligned}$$

where ϕ is the uniformity function of F_{Y, ℓ_1} on the unit ball of Y . Finally $\|y - y'\| \leq \psi(\|x - x'\|)$ where ψ is the uniformity function of $F_{X, Y}$. Note that ϕ and ψ can be bounded by functions depending only on p . ■

Corollary 4.7. *Let $1 \leq p < \infty$. Then there is a function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for all $1 \leq R < \infty$ there exists an $\varepsilon > 0$ such that, if X is p -convex and $(p + \varepsilon)$ -concave, both with constant one, then there is a map ψ from the ball of radius R in ℓ_p onto the ball of radius R in X with the modulus of continuity of ψ and ψ^{-1} bounded by φ .*

Proof. Let ψ be the natural (homogeneous) extension of $F_{\ell_p, X}$ and use Propositions 4.5 and 4.6. ■

5. Uniform homeomorphisms between X and $\ell_p \oplus X$

In Corollary 5.3 we give a useful sufficient condition for a space X to be uniformly homeomorphic to $X \oplus \ell_p$. This is used in Corollary 5.7 to prove that \mathcal{T}^p is uniformly homeomorphic to $\mathcal{T}^p \oplus \ell_p$.

Theorem 5.1. *Let $1 \leq p < \infty$. Let $X = \sum_{n=0}^{\infty} X_n$ be a 1-unconditional sum of spaces with 1-unconditional bases such that there are norm preserving, homogeneous homeomorphisms $f_n : \ell_p \xrightarrow{\text{onto}} X_n$ with the property that, for some sequence $R_n \rightarrow \infty$, the sequence $f_n|_{B(R_n)}, f_n^{-1}|_{B(R_n)}, n = 0, 1, \dots$, is equi-uniformly continuous. ($B_Y(R)$, or just $B(R)$ if there is no ambiguity, denotes the closed ball of radius R in Y .) Then $\ell_p \oplus X$ is uniformly homeomorphic to X .*

For $X = (\sum \oplus \ell_{p_n})_q$ and $p_n \rightarrow p$, this theorem was proved by M. Ribe [Rib3] for $p = 1$ and extended by I. Aharoni and J. Lindenstrauss [AL2] for $p > 1$. The proof we sketch closely follows a simplification of the proof in [AL2] given by Y. Benyamini in his nice exposition [Ben2].

In what follows we shall frequently use the spaces $\ell_p \oplus X_n, \ell_p \oplus X_n \oplus X_{n+1}$ and other spaces of similar nature. Since the isometric nature of the spaces we deal with is important in the proof, we emphasize that by, e.g., $\ell_p \oplus X_n \oplus X_{n+1}$ we mean the space of all triples (u, x_n, x_{n+1}) , $u \in \ell_p, x_i \in X_i, i = n, n+1$, with norm

$$\|(u, x_n, x_{n+1})\| = (\|u\|^p + \|(0, \dots, 0, x_n, x_{n+1}, 0, \dots)\|_X^p)^{1/p}.$$

By coupling the X_n 's, we may assume that for each n , $X_n = Y_n \oplus Z_n$ and there are norm preserving homogeneous homeomorphisms $g_n : \ell_p \xrightarrow{\text{onto}} Y_n$ and $h_n : X_n \xrightarrow{\text{onto}} Z_n$ such that the sequence $g_{n|B(R_n)}, g_{n|B(R_n)}^{-1}, h_{n|B(R_n)}, h_{n|B(R_n)}^{-1}$, $n = 0, 1, \dots$, is equi-uniformly continuous.

Define now, $I_n : \ell_p \oplus X_n \xrightarrow{\text{onto}} X_n$ by

$$I_n(u, x) = I_n(u, (y, z)) = \frac{\|(u, x)\|}{\|(g_n(u), h_n(x))\|} (g_n(u), h_n(x)).$$

Clearly, the I_n 's are norm preserving, homogeneous, and $I_{n|B(S_n)}, I_{n|B(S_n)}^{-1}$, $n = 0, 1, \dots$, form an equi-uniformly continuous family for some sequence $S_n \rightarrow \infty$. Since, in order to prove the theorem we may pass to a subsequence of the X_n 's, we may assume that $S_n > 2^{n+1}$.

The following proposition imitates Proposition 6.2 in [Ben2].

Proposition 5.2. *There is a family of homeomorphisms, $\{F_t ; 0 \leq t < \infty\}$, so that*

- (i) *For $n = 0, 1, \dots$ and $2^{n-1} \leq t < 2^n$, ($0 \leq t < 1$ for $n = 0$), F_t maps $\ell_p \oplus X_n \oplus X_{n+1}$ onto $X_n \oplus X_{n+1}$.*
- (ii) *F_t is homogeneous and norm preserving, $0 \leq t < \infty$.*
- (iii) *$F_{2^{n-1}}(x, y, z) = (I_n(x, y), z)$, $n = 1, 2, \dots$*
- (iv) *$F_{2^n}(x, y, z) = (y, I_{n+1}(x, z))$, $n = 0, 1, \dots$*
- (v) *Denote, for $2^{n-1} \leq t < 2^n$, $G_t = F_{t|B_{\ell_p \oplus X_n \oplus X_{n+1}}(t)}$. Then the families $\{G_t(a)\}$, $\{G_t^{-1}(a)\}$ are equi-uniformly continuous in both t and a .*

Before we sketch the proof of the proposition let us apply it to give the

Proof of Theorem 5.1. Define an homeomorphism $f : \ell_p \oplus X \xrightarrow{\text{onto}} X$ in the following way: For $a = (u, x_0, x_1, \dots) \in \ell_p \oplus X$ with $2^{n-1} \leq \|a\| < 2^n$ ($0 \leq \|a\| < 1$ for $n = 0$) let

$$f(a) = (x_0, x_1, \dots, x_{n-1}, F_{\|a\|}(u, x_n, x_{n+1}), x_{n+2}, \dots)$$

and

$$g(a) = f(a) / \|f(a)\|.$$

For $(x_n, x_{n+1}) \in X_n \oplus X_{n+1}$ and $2^{n-1} \leq t < 2^n$ denote $(u^t, y_n^t, y_{n+1}^t) = F_t^{-1}(x_n, x_{n+1})$. Then it is easily checked that for $x = (x_0, x_1, \dots)$

$$g^{-1}(x) = \frac{(u^{\|x\|}, x_0, x_1, \dots, x_{n-1}, y_n^{\|x\|}, y_{n+1}^{\|x\|}, x_{n+2}, \dots)}{\|(u^{\|x\|}, x_0, x_1, \dots, x_{n-1}, y_n^{\|x\|}, y_{n+1}^{\|x\|}, x_{n+2}, \dots)\|}.$$

The 1-unconditionality of the decomposition of X to $\sum \oplus X_n$ and Proposition 5.2 now show that g and g^{-1} are uniformly continuous. ■

Proof of Proposition 5.2. Consider the maps

$$\varphi_n : \ell_p \oplus X_n \oplus X_{n+1} \rightarrow \ell_p \oplus_p \ell_p \oplus_p \ell_p$$

and

$$\psi_n : X_n \oplus X_{n+1} \rightarrow \ell_p \oplus_p \ell_p$$

given by

$$\varphi_n(u, x, y) = \frac{\|(u, x, y)\|}{\|(u, f_n(x), f_{n+1}(y))\|} (u, f_n(x), f_{n+1}(y))$$

and

$$\psi_n(x, y) = \frac{\|(x, y)\|}{\|(f_n(x), f_{n+1}(y))\|} (f_n(x), f_{n+1}(y)).$$

Note that these are norm preserving homogeneous maps and the sequence

$\varphi_{n|_{B(2^{n+1})}}, \psi_{n|_{B(2^{n+1})}}, \varphi_{n|_{B(2^{n+1})}}^{-1}, \psi_{n|_{B(2^{n+1})}}^{-1}$ is equi-uniformly continuous. Assume we can prove the proposition in the special case where $X_n = \ell_p$ for all n and $\sum \oplus X_n = \sum \oplus_p X_n$ and call the resulting maps H_t instead of F_t . Then for $n = 0, 1, \dots$ and $2^{n-1} \leq t < 2^n$ define $F_t = \psi_n^{-1} \circ H_t \circ \varphi_n$. Clearly F_t satisfy the conclusions of the proposition.

The construction of H_t is given in steps II-IV of the proof of Proposition 6.2 of [Ben2]. The only additional thing to notice is that all the maps there are homogeneous. ■

Corollary 5.3. *Let $1 \leq p < \infty$ and let $X = \sum X_n$ be an unconditional sum of spaces X_n with unconditional bases which are p -convex with constant C and $(p + \varepsilon_n)$ -concave with constant C , for some positive sequence $\varepsilon_n \rightarrow 0$ and some $1 \leq C < \infty$. Then $\ell_p \oplus X$ is uniformly homeomorphic to X .*

Proof. We may assume that the unconditionality constant of the sum $\sum X_n$ is 1. By a theorem from [FJ], each X_n is isomorphic, with constant independent of n , to a space with a 1-unconditional basis which is p -convex and $(p + \varepsilon_n)$ -concave with constants one. Applying Corollary 4.7 stated after Proposition 4.6 (and the above isomorphisms) we get the maps $f_n : \ell_p \rightarrow X_n$ as in the statement of Theorem 5.1. Now apply Theorem 5.1. ■

We shall denote by \mathcal{T}^p (or just \mathcal{T} when $p = 1$) the closed span of a certain subsequence of the unit vector basis for the p -convexification of the modified Tsirelson space. In fact, it is known (see [CO], [CS]) that this space is (up to an equivalent renorming) the space which is usually denoted by \mathcal{T}^p ; that is, the p -convexification of the dual to Tsirelson's original space; but we avoid using this for the convenience of the reader.

The construction of \mathcal{T} , as well as an elementary proof that \mathcal{T} is reflexive, is contained in [Joh2]. The space \mathcal{T} is the completion of c_{00} under the norm defined implicitly by the formula:

$$(5.1) \quad \|x\| = \max \left\{ \|x\|_{c_0}, \frac{1}{2} \sup \left\{ \sum_{j=1}^{(n+1)^n} \|E_j x\| \right\} \right\},$$

where the sup is over all sequences of disjoint finite sets E_j of positive integers with $j \leq E_j$ for each $1 \leq j \leq n$. Such sequences are called *allowable*. The choice of the growth $n \mapsto (n+1)^n$ is simply one of convenience; the results of [CJT], [CO] imply that the space is the same if e. g. " n " instead of " $(n+1)^n$ " is used.

In order to show that for all $1 \leq p < \infty$, \mathcal{T}^p , the p -convexification of \mathcal{T} , is uniformly homeomorphic to $\mathcal{T}^p \oplus \ell_p$, we need that for any $q > 1$ the span of a sufficiently thin subsequence of the unit vector basis $\{e_n\}_{n=1}^\infty$ for \mathcal{T} has q -concavity constant independent of q . (In fact, the subsequence can be just a “tail” of the basis.) The proof uses an elementary lemma:

Lemma 5.4. *Suppose $q > 1$, $\epsilon > 0$, and $1 = m_1 < m_2 < \dots$ with $\sum_{k=2}^\infty (m_k - m_{k-1})^{1-q} < \epsilon$. Then for all sequences $a_1 \geq a_2 \geq \dots \geq 0$,*

$$\sup_k \sum_{j=m_k}^{m_{k+1}-1} a_j \geq (1 + \epsilon)^{-1/q} \left(\sum_j a_j^q \right)^{1/q}.$$

Proof. Assume, without loss of generality, that the left side is one. Then

$$\begin{aligned} \sum_j a_j^q &= \sum_{k=1}^\infty \sum_{j=m_k}^{m_{k+1}-1} a_j^q \leq \sum_{k=1}^\infty \left(\sum_{j=m_k}^{m_{k+1}-1} a_j \right) a_{m_k}^{q-1} \leq \\ &\left(\sup_k \sum_{j=m_k}^{m_{k+1}-1} a_j \right) \left(\sum_{k=1}^\infty a_{m_k}^{q-1} \right) \leq \left(\sup_k \sum_{j=m_k}^{m_{k+1}-1} a_j \right) \left(a_{m_1}^{q-1} + \sum_{k=2}^\infty (m_k - m_{k-1})^{1-q} \right), \end{aligned}$$

so the desired conclusion follows. ■

Corollary 5.5. *There is a constant $M < \infty$ so that for any $1 < q < \frac{3}{2}$ the span of $\{e_n\}_{n=N(q)}^\infty$ in \mathcal{T} has q -concavity constant less than M , where $N(q) = 2^{\frac{q}{q-1}}$.*

Proof: A general result of Maurey’s [Mau2] (see [LT2, proof of Theorem 1.f.7]) shows that the q -concavity constant of a Banach lattice is at most a constant multiple of the lower s constant of the lattice, where $q^* = s^* + 1$; i.e., $s = \frac{1}{2-q}$. (Trace the constants in the proof in [LT2] for the dual statement in the special case $2 < r = p + 1$.)

Thus it is enough to prove the corollary with “ q -concavity constant” replaced with “lower q constant” and “ $N(q)$ ” replaced with “ $M(q) \equiv N(s)$ ” where, as above, $s = \frac{1}{2-q}$, so that $M(q) = 2^{\frac{1}{q-1}}$.

So let $1 < q < \frac{3}{2}$ be fixed and suppose that x_1, \dots, x_n are disjoint vectors in $\mathcal{T} \cap c_{00}$ which are supported on $[M(q), \infty)$. We can assume that $\|x_1\| \geq \|x_2\| \geq \dots \geq \|x_n\|$. If, for some m , F is a set of at most $(m+1)^m$ indices so that for each j in F , x_j is supported in

$$[m, \infty), \text{ then by the definition of } \mathcal{T}, \left\| \sum_{j=1}^n x_j \right\| \geq \left\| \sum_{j \in F} x_j \right\| \geq \frac{1}{2} \sum_{j \in F} \|x_j\|.$$

Define $m_1 = 1$ and $m_k = m_{k-1} + M(q)^{k-1}$ for $k > 1$. Since for $k > 1$, $m_{k+1} - m_k \leq (m_k + 1)^{m_k}$ and since at most m_k of the x_i ’s, $1 \leq i \leq m_{k+1}$, fail to be supported on $[m_k, \infty)$, we get

$$\left\| \sum_{j=1}^n x_j \right\| \geq \frac{1}{2} \sup_k \sum_{j=m_k}^{m_{k+1}-1} \|x_j\|.$$

The sequence $\{m_k\}_{k=1}^\infty$ satisfies the condition of the lemma with $\epsilon = 1$, and thus the lemma yields that $\{e_n\}_{n=M(q)}^\infty$ has a lower q estimate with constant $2^{-1-1/q}$. ■

If X has a 1-unconditional basis and is q -concave with constant M , then its p -convexification is pq -concave with constant $M^{\frac{1}{p}}$. Thus the next proposition is an immediate consequence of Corollary 5.5.

Proposition 5.6. *There is a constant $M < \infty$ so that for any $1 \leq p < r < \frac{3p}{2}$ the span of $\{e_n\}_{n=N(p,r)}^\infty$ in \mathcal{T}^p has r -concavity constant less than $M^{1/p}$, where $N(p, r) = 2^{\frac{r}{r-p}}$.*

As an immediate consequence of Proposition 5.6 and Corollary 5.3 we get:

Proposition 5.7. *For $1 \leq p < \infty$, \mathcal{T}^p is uniformly homeomorphic to $\mathcal{T}^p \oplus \ell_p$.*

Finally, by combining Proposition 5.7 and Proposition 2.8 we get:

Theorem 5.8. *Let $1 < p_1 < p_2 < \dots < p_n < 2$ or $2 < p_1 < p_2 < \dots < p_n < \infty$. Set $X = \sum_{k=1}^n \mathcal{T}^{p_k}$. If X is uniformly homeomorphic to a Banach space Y if and only if Y is isomorphic to $X \oplus \sum_{k \in F} \ell_{p_k}$ for some subset F of $\{1, 2, \dots, n\}$. Consequently, X is uniformly homeomorphic to exactly 2^n mutually nonisomorphic Banach spaces.*

Proof. X is uniformly homeomorphic to spaces Y of the desired form by Proposition 5.7. As mentioned in section 2, the (obvious) fact that every n -tuple (even $(n+1)^n$ -tuple) of disjoint unit vectors in \mathcal{T}^p whose first n coordinates are zero is 2-equivalent to the unit vector basis of ℓ_{p_n} implies that \mathcal{T}^p is as \mathcal{L}_p . Of course, ℓ_r does not embed into \mathcal{T}^p for any r (the general case is immediate from the special case $p = 1$, proved in [Joh2]). So it remains to observe that \mathcal{T}^p is isomorphic to its hyperplanes. In fact, the right shift is an isomorphism on \mathcal{T} , hence on \mathcal{T}^p for all $1 \leq p < \infty$. This is included in the first result in [CJT] for the classical Tsirelson space. To avoid using the fact that this space is the same space (up to an equivalent renorming) that we denote by \mathcal{T} , the reader will have to check directly that the right shift is an isomorphism on \mathcal{T} . So Proposition 2.8 applies to show that any uniform homeomorph of X must be of the form for Y in the statement of Theorem 5.8. Finally, ℓ_{p_j} embeds into $X \oplus \sum_{k \in F} \ell_{p_k}$ iff and only if j is in F , so all the Y 's of the mentioned form are isomorphically distinct. ■

6. The nonseparable case

Some of the uniqueness theorems proved in earlier sections have nonseparable analogues. In particular, there is a nonseparable version of Theorem 2.1:

Theorem 6.1. *If X is a Banach space which is uniformly homeomorphic to $\ell_p(\Gamma)$, $1 < p < \infty$, then X is isomorphic to $\ell_p(\Gamma)$.*

Proof. A back-and-forth argument yields that each separable subspace of X is contained in another subspace of X which is uniformly homeomorphic to ℓ_p , hence isomorphic to ℓ_p . It is then easy to see that there is a $\lambda < \infty$ so that each separable subspace of X is contained in another subspace of X which is λ -isomorphic to ℓ_p . So X is \mathcal{L}_p and hence is isomorphic to a complemented subspace of $L_p(\mu)$ for some measure μ [LPe], and of course

ℓ_2 does not embed into X . The conclusion then follows from the next lemma, which is implicit in [JO] although not stated there.

Lemma 6.2. *Let X be an infinite dimensional complemented subspace of $L_p(\mu)$ for some measure μ , $1 < p < \infty$, and suppose that no subspace of X is isomorphic to ℓ_2 . Then X is isomorphic to $\ell_p(\Gamma)$, where Γ is the density character of X .*

Proof. Since every Hilbertian subspace of L_p is complemented for $2 < p < \infty$ [KP], by duality we can assume that $2 < p < \infty$. By Theorem 3 of [JO], X embeds into $\ell_p(\Gamma)$, and,

as noted in the proof of that theorem, X is isomorphic to a space of the form $\left(\sum_{\gamma \in \Gamma} X_\gamma \right)_{\ell_p(\Gamma)}$ with each space X_γ separable (and, without loss of generality, infinite dimensional). But then the X_γ 's are uniformly isomorphic to uniformly complemented subspaces of $\ell_p(\Gamma)$, hence by [JO] and [JZ] (or see [Joh1]) are uniformly isomorphic to ℓ_p . ■

The isomorphic classification of $C(K)$ for compact metric spaces K [BP] implies that c_0 is isomorphic to $C(K)$ if and only if the ω -th derived set $K^{(\omega)}$ of K is empty. In [DGZ] it is proved that for any compact Hausdorff space K , if $K^{(\omega)}$ is empty, then $C(K)$ is uniformly homeomorphic (even Lipschitz equivalent) to $c_0(\Gamma)$ with Γ the density character of $C(K)$. Part of the interest of this result is that, in contrast to the separable case, in the nonseparable setting this can happen with $C(K)$ not isomorphic to $c_0(\Gamma)$, [AL1]. Gilles Godefroy pointed out to us that Corollary 3.2 and previously known results yield a converse to the mentioned theorem from [DGZ] and kindly suggested that we include the result in this paper.

Theorem 6.3. *(G. Godefroy) Suppose that K is a compact Hausdorff space and $C(K)$ is uniformly homeomorphic to $c_0(\Gamma)$. Then $K^{(\omega)}$ is empty.*

Proof. The usual back-and-forth argument shows that every separable subspace of $C(K)$ is contained in a subalgebra which is uniformly homeomorphic to c_0 hence isomorphic to c_0 by Corollary 3.2. This implies that K is scattered by [PS] (or see [Sem, Theorem 8.5.4]); that is, whenever $K^{(\alpha)}$ is nonempty, $K^{(\alpha+1)}$ is a proper subset of $K^{(\alpha)}$. But if K is scattered and $K^{(\omega)}$ is nonempty, then ω^ω is a continuous image of K by a result of Baker's [Bak]. But then $C(\omega^\omega)$ is isomorphic to a subspace of $C(K)$ and hence, by the first sentence of this proof, $C(\omega^\omega)$ is isomorphic to a subspace of c_0 , which of course is false. ■

Remark. Theorem 6.3 and the mentioned result in [DGZ] yield that if $c_0(\Gamma)$ is uniformly homeomorphic to a $C(K)$ space X , then $c_0(\Gamma)$ is Lipschitz equivalent to X . We do not know whether this holds for a general space X even in the separable case.

7. Spaces determined by their finite dimensional subspaces

We begin with some notions and definitions.

A *paving* of a (separable) Banach space X is a sequence $E_1 \subset E_2 \subset \dots$ of finite dimensional subspaces of X whose union is dense in X . We say that the two Banach

spaces X and Y have a *common paving* if there exist pavings $\{E_n\}_{n=1}^\infty$ of X and $\{F_n\}_{n=1}^\infty$ of Y so that the isomorphism constants between E_n and F_n are uniformly bounded.

Example (warning). c_0 and $c_0 \oplus \ell_1$ have a common paving, since both are paved by $\ell_\infty^{2^n} \oplus_\infty \ell_1^n$.

Definition 7.1. A separable Banach space X is said to be *determined by its pavings* provided that X is isomorphic to every separable Banach space which has a common paving with X .

Definition 7.2. X and Y are said to have the *same finite dimensional subspaces* if each is finitely crudely representable in the other. A separable Banach space X is said to be *determined by its finite dimensional subspaces* provided that X is isomorphic to every separable Banach space which has the same finite dimensional subspaces as X .

Obviously, every Banach space which is determined by its finite dimensional subspaces is determined by its pavings. It is easy to see and well known that ℓ_2 is determined by its pavings. In this section we shall show, among other things, that any space determined by its pavings must be “close” to ℓ_2 , but that there are spaces not isomorphic to ℓ_2 which are determined by their pavings. On the other hand, we make the following conjecture:

Conjecture 7.3. Every (separable, infinite dimensional) Banach space which is determined by its finite dimensional subspaces is isomorphic to ℓ_2 .

We start the discussion of the concepts introduced above by proving:

Proposition 7.4. *If X is determined by its pavings, then for every $\epsilon > 0$, X has type $2 - \epsilon$ and cotype $2 + \epsilon$.*

Proof. Otherwise ℓ_p is finitely representable in X for some $1 \leq p \neq 2 \leq \infty$ by the Krivine-Maurey-Pisier theorem ([Kri], [MP]). If $\{F_n\}_{n=1}^\infty$ is a paving of any subspace of L_p , then using the fact that ℓ_p is finitely representable in every finite codimensional subspace of X , it is easy to see that X has a paving of the form $\{E_n \oplus_p F_n\}_{n=1}^\infty$ with $\{E_n\}_{n=1}^\infty$ itself a paving of X . This means that $X \oplus Y$ has a common paving with X for every subspace Y of L_p , which implies that every subspace of L_p is isomorphic to a complemented subspace of X . But in [JS] it was proved that for $2 < p < \infty$, no separable Banach space is complementably universal for subspaces of ℓ_p . Probably the same result is true when $1 \leq p < 2$, although we can get by with the weaker fact that there is no separable Banach space which is complementably universal for subspaces of L_p for p in this range. This weaker fact follows from the proof of what is called in [JS] the Basic Result and two facts which can be found in [LT1,2]: (1) ℓ_r embeds into L_p when $1 \leq p < r < 2$; and (2) for each $r \neq 2$, ℓ_r has a subspace which fails the compact approximation property. ■

Following [Joh3], we say that X has property $H(m, n, K)$ provided that there exists an m -codimensional subspace of X , all of whose n -dimensional subspaces are K -isomorphic to ℓ_2^n . So a Banach space X is *asymptotically Hilbertian* in Pisier’s terminology [Pis] or as. \mathcal{L}_2 in our terminology if and only if there exists a K so that for every n , X has $H(m, n, K)$ for some $m = m(n)$.

Lemma 7.5. *Given m, k, K , and a separable Banach space X , the following are equivalent:*

- (i) X has property $H(m, k, K)$.
- (ii) There is a paving $\{E_n\}_{n=1}^\infty$ for X such that for every n , E_n has property $H(m, k, K)$.
- (iii) Every finite dimensional subspace of X has property $H(m, k, K)$.
- (iv) Every subspace of X has property $H(m, k, K)$.
- (v) Every space which is finitely representable in X has property $H(m, k, K)$.

Proof. The implications (v) \Rightarrow (i) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii) are all obvious. To prove (ii) \Rightarrow (v), it is enough to observe that ultraproducts of $\{E_n\}_{n=1}^\infty$ have property $H(m, k, K)$ since any separable space finitely representable in X embeds isometrically into an ultraproduct of any paving of X . ■

Notice that the preceding lemma implies that asymptotically Hilbertian is a local property.

The next proposition is suggested by a result and proof due to P. Casazza [CS, Theorem Ae13].

Proposition 7.6. *Suppose that X is asymptotically Hilbertian, $\{E_n\}_{n=1}^\infty$ is a paving for X , and \tilde{E} is an ultraproduct of $\{E_n\}_{n=1}^\infty$. Then \tilde{E} is isomorphic to $X \oplus H$ for some (nonseparable) Hilbert space H .*

Proof. The subspace of \tilde{E} consisting of all Cauchy sequences $\{x_n\}_{n=1}^\infty$ in X with x_n in E_n is easily seen to be isometrically isomorphic to X . \tilde{E} is the direct sum of this subspace and the subspace of all $\{x_n\}_{n=1}^\infty$ in E_n for which $\{x_n\}_{n=1}^\infty$ weakly converges along \mathcal{U} to zero; this latter subspace is isomorphic to a Hilbert space.

Corollary 7.7. *If X is asymptotically Hilbertian and Y has a common paving with X , then $X \oplus \ell_2$ is isomorphic to $Y \oplus \ell_2$. Consequently, if also every infinite dimensional subspace of X contains a complemented subspace which is isomorphic to ℓ_2 (in particular, if X is an ℓ_2 sum of finite dimensional spaces), then X is determined by its pavings.*

Proof. If $\{E_n\}_{n=1}^\infty$ is a paving of X which is uniformly equivalent to a paving of Y , and \tilde{E} is an ultraproduct of $\{E_n\}_{n=1}^\infty$, then we get from Proposition 7.6 that $X \oplus H$ is isomorphic to $Y \oplus H$ for some nonseparable Hilbert space H , from which the first conclusion follows easily. For the “consequently” statement, we see that X is isomorphic to $X \oplus \ell_2$ since X contains a complemented copy of ℓ_2 . In fact, every subspace of $X \oplus \ell_2$ (in particular, Y) contains a complemented copy of ℓ_2 , so also Y is isomorphic to $Y \oplus \ell_2$.

This corollary implies that there are spaces not isomorphic to ℓ_2 which are determined by their pavings. In order to continue our investigation, we introduce a further notion:

We say that X is *finitely homogeneous* provided there is a K so that X is finitely crudely representable with constant K in every finite codimensional subspace of itself.

First we observe that finite homogeneity is determined by finite dimensional subspaces.

Proposition 7.8. *X is finitely homogeneous if and only if there is a K so that for every finite dimensional subspace E of X , there is a space $Z = Z(E)$ which has a monotonely*

subsymmetric and unconditional Schauder decomposition into spaces isometric to E and which is finitely crudely representable in X with constant K .

Proof. The forward direction involves just a combination of modern refinements of the Mazur technique for constructing basic sequences and the Brunel-Sucheston construction; see e.g. [Pis, Chpt. 14]. The reverse direction is even easier (if F is a finite subset of X^* and $E \oplus E \oplus \cdots \oplus E$ is a “good” Schauder decomposition of some subspace of X into N -copies of E with N large relative to the cardinality of F , then the action of F on two different copies of E is essentially the same. Now pass to differences.) ■

Proposition 7.9. *If the separable Banach space X is finitely homogeneous and X has the same finite dimensional subspaces as the separable Banach space Y , then X and Y have a common paving.*

Proof. The space Y is also finitely homogeneous by Proposition 7.8. Now it is a simple exercise to show that if $\{E_n\}_{n=1}^\infty$ is a paving for X and $\{F_n\}_{n=1}^\infty$ is a paving for Y , then there is a subsequence of $\{E_n \oplus F_n\}_{n=1}^\infty$ which is uniformly equivalent to pavings for both X and Y . ■

The next proposition crystalizes some of what has already been used.

Proposition 7.10. *If there is a K so that the separable Banach space Y is finitely crudely representable with constant K in every finite codimensional subspace of the separable Banach space X , then $X \oplus Y$ and X have a common paving.*

Corollary 7.11. *If X is finitely homogeneous and determined by its pavings, then every separable Banach space which is finitely representable in X is isomorphic to a complemented subspace of X . Moreover, X has an unconditional Schauder decomposition such that the summands form a sequence which is dense in the set of all finite dimensional subspaces of X , and hence every separable Banach space which is finitely representable in X has the bounded approximation property. Finally, if X has GL-l.u.st.; respectively, GL; then so does every subspace.*

See [T-J, §34] for the definitions of GL-l.u.st. and GL properties.

Using James’ characterization of reflexivity/superreflexivity [Jam] and the limiting technique of Brunel-Sucheston (again as in [Pis, Chpt. 14]), we get from Proposition 7.13:

Proposition 7.12. *If X is reflexive and is determined by its pavings, then X is super-reflexive.*

We do not know whether there exist nonreflexive spaces which are determined by their pavings.

Proposition 7.8 implies that if X is finitely homogeneous, then $X \oplus X$ is finitely crudely representable in X . The properties are not equivalent, however, since every Tsirelson type space is isomorphic to its square. The weaker condition gives similar conclusions to those in Corollary 7.12 for spaces which are determined by their finite dimensional subspaces:

Proposition 7.13. *If X is determined by its finite dimensional subspaces and $X \oplus X$ is finitely crudely representable in X , then X is isomorphic to $X \oplus Y$ for every space Y which is finitely representable in X . Hence if X has GL-lust, then so does every subspace.*

Corollary 7.7, Proposition 7.13 and [KT] yield:

Proposition 7.14. *Let $k_n \rightarrow \infty$ appropriately, set $E_n = \text{span}\{e_j\}_{j=k_n}^{k_{n+1}-1}$ in \mathcal{T}^2 , and let X be the ℓ_2 -sum of $\{E_n\}_{n=1}^\infty$. Then X is determined by its pavings but X is not determined by its finite dimensional subspaces.*

Proof. Corollary 7.7 says that X is determined by its pavings no matter how $\{k_n\}_{n=1}^\infty$ is defined. Next note that $E_n \oplus E_n$ is uniformly isomorphic to $E_n \oplus \ell_2^{k_{n+1}-k_n}$; again, no matter how $\{k_n\}_{n=1}^\infty$ is defined. This follows from the equivalence of $\{e_{2n}\}_{n=1}^\infty$ and $\{e_{2n-1}\}_{n=1}^\infty$ with $\{e_n\}_{n=1}^\infty$ and the fact that the last half of a finite string of e_j 's is equivalent to an orthonormal sequence. This yields that X is isomorphic to its square. Thus by Proposition 7.12, if X were determined by its finite dimensional subspaces, then every subspace of X would have GL-l.u.st., which easily implies that there is a K so that every finite dimensional subspace of X has GL-l.u.st. constant less than K . But by [KT], \mathcal{T}^2 has a subspace which fails GL-l.u.st., so for each N , $\text{span}\{e_j\}_{j=N}^\infty$ has finite dimensional subspaces with arbitrarily large GL-l.u.st. constant. Consequently, having defined k_n , X_n will have a subspace with GL-l.u.st. constant larger than n if k_{n+1} is sufficiently large. ■

Proposition 7.15. *Suppose that $p_n \downarrow 2$, $k_n \rightarrow \infty$, and let X be the ℓ_2 -sum of $\ell_{p_n}^{k_n}$. If X is determined by its finite dimensional subspaces, then X is isomorphic to ℓ_2 .*

Proof. Assume without loss of generality that all the p_n 's are smaller than 4. Now if $2 < p < 4$, then ℓ_p^k is uniformly isomorphic to $\ell_p^{k-\sqrt{k}} \oplus \ell_2^{\sqrt{k}}$ ([BDGJN], [JSc], which yields that X is isomorphic to $X \oplus W$, where W is the ℓ_2 -sum of $\ell_{p_n}^{\sqrt{k_n}}$. Now for $p > 2$, ℓ_p^k has a subspace whose GL-constant is of the same order as the distance of ℓ_p^k to ℓ_2^k . Thus if X is not isomorphic to a Hilbert space, then W has a subspace V which does not have the GL-property. But $X \oplus W$ is finitely crudely representable in X . Thus X is isomorphic to a Hilbert space if X is determined by its finite dimensional subspaces. ■

8. Problems

(1) Say that a Banach space *is determined by its uniform structure* if it is isomorphic to every Banach space to which it is uniformly equivalent. It seems that most “natural” or “classical” separable spaces are determined by their uniform structure but at present the tools to establish this fact are limited. Here are some specific problems, the first three being the most interesting.

- (a) Is $L_p(0, 1)$, $1 < p < \infty$, $p \neq 2$, determined by its uniform structure?
- (b) Are c_0 and ℓ_1 determined by their uniform structure? For c_0 partial results are given in section 3. For ℓ_1 we have practically no information beyond Ribe's theorem [Rib1].
- (c) Are separable $C(K)$ spaces and $L_1(0, 1)$ determined by their uniform structure? In particular, if K_1 and K_2 are compact metric spaces for which $C(K_1)$ is uniformly homeomorphic to $C(K_2)$, must $C(K_1)$ be isomorphic to $C(K_2)$?

More generally, one can ask

- (d) Is every separable \mathcal{L}_p space, $1 \leq p \leq \infty$, determined by its uniform structure? For the case $p = 1$ it is even open whether a uniform homeomorph of a \mathcal{L}_1 space must itself be a \mathcal{L}_1 space. For $1 < p < \infty$, $p \neq 2$, the simplest unknown case is the space $\ell_p \oplus \ell_2$.

In connection with the results of section 2 it is natural to ask

(e) Is $\ell_p \oplus \ell_r$ for $1 < p < 2 < r < \infty$ determined by its uniform structure?

(2) A general question related to uniform homeomorphisms is whether the uniform structure of a Banach space is determined by the structure of its discrete subsets. Suppose that A is an a -separated b -net for a Banach space X with $0 < a < b < \infty$; that is, every two points in A are of distance apart larger than a and every point in X is of distance less than b from some point of A . If T is a uniform homeomorphism from X onto Y , then $T[A]$ is a c -separated d -net for Y for some $0 < c < d < \infty$, and, since T and T^{-1} are “Lipschitz for large distances”, the restriction of T to A is bi-Lipschitz. We ask whether the converse is true:

If there exist b -nets A and B in X and Y , respectively, for some $b < \infty$ so that there is a bi-Lipschitz mapping from A onto B , then must X and Y be uniformly homeomorphic?

An affirmative answer to this problem would of course imply that ultrapowers of X and Y are Lipschitz equivalent; this much at least is true:

Proposition 8.1. *Suppose that A and B are b -nets in X and Y , respectively, and there is a bi-Lipschitz mapping T from X onto Y . Let \mathcal{U} be a free ultrafilter on the natural numbers. Then $X_{\mathcal{U}}$ is Lipschitz equivalent to $Y_{\mathcal{U}}$.*

Proof. Define a mapping $T_n : \frac{1}{n}A \rightarrow \frac{1}{n}B$ by $T_n(x) = \frac{1}{n}T(nx)$. Then for each n , T_n and T_n^{-1} have the same Lipschitz constants as T and T^{-1} , respectively. Given a bounded sequence $\tilde{x} = \{x_n\}_{n=1}^{\infty}$ in X , we can select a sequence $\tilde{y} = \{y_n\}_{n=1}^{\infty}$ with y_n in $\frac{1}{n}A$ so that $\|x_n - y_n\| < \frac{b}{n}$. So in the ultrapower $X_{\mathcal{U}}$, $\tilde{x} = \tilde{y}$. Since the T_n ’s are uniformly Lipschitz, if for \tilde{x} and \tilde{y} as above we set $\tilde{T}(\tilde{x}) = \{T_n(y_n)\}_{n=1}^{\infty}$, then \tilde{T} is a well defined Lipschitz mapping from $X_{\mathcal{U}}$ to $Y_{\mathcal{U}}$. Doing the the analogous thing with the T_n^{-1} ’s, we see that \tilde{T} is in fact a bi-Lipschitz mapping from $X_{\mathcal{U}}$ onto $Y_{\mathcal{U}}$. ■

(3) We recall here a well known problem. Are any two separable Lipschitz equivalent Banach spaces isomorphic?

(4) Besides uniformly continuous and Lipschitz maps, there are several other interesting kinds of nonlinear maps between Banach spaces which appear in the literature. A notion which arises naturally from the theory of quasiconformal maps and which was introduced and studied in the context of general metric spaces by Tukia and Väisälä [TV] is that of quasisymmetric maps. There are several variants of this notion, but for our purpose, in the context of Banach spaces, we can use the following definition. A map f from a subset G of a Banach space X into a Banach space Y is *quasisymmetric* if there is a constant M so that whenever u, v, w are in G with $\|u - v\| \leq \|u - w\|$, then $\|f(u) - f(v)\| \leq M \|f(u) - f(w)\|$. Väisälä asked the following: For $1 \leq p < q < \infty$, is there a homeomorphism between ℓ_p and ℓ_q which is bi-quasisymmetric? (More generally one can ask if ℓ_p is determined by its “quasisymmetric structure”.) In this connection we mention an unpublished observation of Väisälä and the second author: Suppose that G is an open subset of a separable Banach space X and f is a quasisymmetric Lipschitz map from G into a Banach space Y which has the Radon-Nikodym property (in particular, Y can be any separable conjugate space). Then, unless X is isomorphic to a subspace of Y , f must be a constant. Indeed, if T is the Gateaux derivative of f at some point, then T

is either 0 or an into isomorphism. Hence if X is not isomorphic to a subspace of Y , then the Gateaux derivative of f must be 0 whenever it exists. The assumptions on X , Y , and f guarantee that the Gateaux derivative exists almost everywhere (e.g. in the sense that the complement has measure 0 with respect to every nondegenerate Gaussian measure on X ; see [Ben]). By Fubini's theorem it follows that the restriction of f to a line $x + L$ has derivative 0 almost everywhere for almost all translates $x + L$ of L . This implies that f is constant.

The assumption that Y has the Radon-Nikodym property is essential, since Aharoni [Aha] proved that there is a Lipschitz embedding of $C(0, 1)$ into c_0 . Of course, every Lipschitz embedding is quasisymmetric.

(5) Theorem 5.8 gives examples of spaces whose uniform structure determine exactly 2^k isomorphism classes for $k = 0, 1, 2, \dots$. From the construction of [AL2] one easily gets examples of spaces whose uniform structure determine 2^{\aleph_0} isomorphism classes. If α is any cardinal less than the continuum which is not a power of two, in particular if $\alpha = 3$ or $\alpha = \aleph_0$, we do not know how to construct a space which determines exactly α isomorphism classes.

(6) If X is determined by its pavings (as defined in section 7), must X be determined by its uniform structure? More generally, if two spaces are uniformly homeomorphic, must they have a common paving? (In the terminology of section 7, they do have the same finite dimensional subspaces by Ribe's theorem [Rib1].) Notice that Corollary 7.7 and the second remark after Proposition 2.9 give examples of spaces which are determined both by their pavings and by their uniform structure.

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