

# Non-linear factorization of linear operators <sup>\*</sup>

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## Abstract

We show, in particular, that a linear operator between finite dimensional normed spaces, which factors through a third Banach space  $Z$  via Lipschitz maps, factors linearly through the identity from  $L_\infty([0, 1], Z)$  to  $L_1([0, 1], Z)$  (and thus, in particular, through each  $L_p(Z)$ ,  $1 \leq p \leq \infty$ ) with the same factorization constant. It follows that, for each  $1 \leq p \leq \infty$ , the class of  $\mathcal{L}_p$  spaces is closed under uniform (and even coarse) equivalences. The case  $p = 1$  is new and solves a problem raised by Heinrich and Mankiewicz in 1982. The proof is based on a simple local-global linearization idea.

## 1 Introduction

Let  $X$  be a pointed metric space with distinguished point, 0.  $X^\#$  denotes the Banach space of real valued Lipschitz functions  $f$  on  $X$  for which  $f(0) = 0$  under the norm  $\text{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|}$ . The evaluation map  $K_X : X \rightarrow X^{\#*}$  is then an isometric isomorphism. If  $X$  is a Banach space, then by [5],  $X^{\#*}$  is norm one complemented in  $X^{\#*}$  via a projection  $P$  that satisfies the identity  $PK_X = J_X$ , where  $J_X$  is the canonical embedding of  $X$  into  $X^{\#*}$ .

Given a Lipschitz function  $T : X \rightarrow Y$  with  $X$  and  $Y$  pointed metric spaces and  $1 \leq p \leq \infty$ , we define the Lipschitz  $L_p$  factorization norm  $\gamma_p^L(T)$  by

$$\gamma_p^L(T) := \inf \{ \text{Lip}(F) \text{Lip}(G) : K_Y \circ T = G \circ F \},$$

where the infimum is over all Lipschitz factorizations of  $K_Y \circ T$  through an arbitrary  $L_p$  space. By the result of Lindenstrauss mentioned in the previous paragraph, if  $Y$  is a normed space this is the same as taking the infimum over all Lipschitz factorizations of  $J_Y \circ T$  through an  $L_p$  space.

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The main result, Theorem 2, of this paper is that if  $T$  is a linear operator between Banach spaces and  $1 \leq p \leq \infty$ , then  $\gamma_p^L(T) = \gamma_p(T)$ . (Recall that  $\gamma_p(T)$  is defined to be the infimum of  $\|F\|\|G\|$  where the infimum is taken over all linear factorizations  $J_Y \circ T = G \circ F$  of  $J_Y \circ T$  through arbitrary  $L_p$  spaces.) The case  $p = 1$  of this theorem uses new ideas and gives as a corollary that a Banach space that is uniformly (or even coarsely) equivalent to a  $\mathcal{L}_1$  space is again a  $\mathcal{L}_1$  space. This answers a question Heinrich and Mankiewicz asked in their 1982 paper [3]. The main result is dealt with in section 3. The proof uses a local-global approach reminiscent of a method recently employed in [4]. Although the proof is fairly simple, easier special cases of Theorem 2 resisted for some time attacks by several experts; for example, it is not completely straightforward to show that if  $f_n : \ell_2^n \rightarrow L_1$ ,  $g_n : L_1 \rightarrow \ell_2^n$  are such that  $g_n f_n$  is the identity on  $\ell_2^n$ , then  $\text{Lip}(f_n)\text{Lip}(g_n) \rightarrow \infty$  as  $n \rightarrow \infty$ , and, indeed, understanding this example led us to Theorem 2.

In section 2 we present another factorization result the proof of which is based on much older and well known ideas but still produces some new information on the invariance of some classes of  $\mathcal{L}_1$  spaces under Lipschitz isomorphisms.

Although our research was motivated by problems concerning the classification of Banach spaces under non-linear equivalences, no specialized knowledge of Banach space theory is required for reading this paper. Let us just mention that one equivalence for a Banach space  $X$  to be a  $\mathcal{L}_p$  space is that the canonical injection  $J_X : X \rightarrow X^{**}$  factor (linearly) through  $L_p(\mu)$  for some measure  $\mu$  (and, if  $1 < p \neq 2 < \infty$ , that  $X$  not be isomorphic to a Hilbert space); see [6].

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## 2 The classical argument

Here we prove

**Theorem 1** *Assume  $T : X \rightarrow Y$  is a linear operator from a Banach space  $X$  to a dual Banach space  $Y$ . Assume  $T$  factors through a third Banach space  $Z$  via Lipschitz mappings:  $T = g \circ f$ ,  $f : X \rightarrow Z$ ,  $g : Z \rightarrow Y$  and assume in addition that  $f$  has a point of Gâteaux differentiability. Then there are linear operators  $A : X \rightarrow Z$ ,  $B : Z \rightarrow Y$  with  $T = BA$  and  $\|A\|\|B\| \leq \text{Lip}(f)\text{Lip}(g)$ .*

**Proof:** The proof uses a combination of an argument from [3] and a result from [5].

By making suitable translations in the domain and range we may assume that the point of differentiability of  $f$  is 0 and that  $f(0) = 0$ . Define  $f_n, g_n$  by  $f_n(x) = nf(\frac{x}{n})$ ,  $g_n(z) = ng(\frac{z}{n})$ . Since  $g \circ f = T$  is linear, we have that  $g_n \circ f_n = T$ . By hypothesis,  $\|f_n(x) - Ax\| \rightarrow 0$  for all  $x \in X$ , for some linear operator  $A : X \rightarrow Z$ . Let  $\tilde{g}(x)$  be the weak\* limit of  $g_n(x)$  through some fixed free ultrafilter of natural numbers. Using the

norm convergence of  $f_n(x)$  to  $Ax$ , we deduce that  $\tilde{g} \circ A = T$ . Of course,  $\|A\| \leq \text{Lip}(f)$  and  $\text{Lip}(\tilde{g}) \leq \text{Lip}(g)$ . So we have linear  $A$  and Lipschitz  $\tilde{g}$  such that

$$T = \tilde{g} \circ A$$

and  $\|A\|\text{Lip}(\tilde{g}) \leq \text{Lip}(f)\text{Lip}(g)$ . Also,  $\tilde{g}$  is linear on  $AE$ , so, by [5], there is a linear operator  $B : X \rightarrow Y$  with  $\|B\| \leq \text{Lip}(\tilde{g})$  so that  $BA = T$ . A softer proof of the result we used from [5] is contained in [7]; see also [1, Theorem 7.2]

■

As a corollary we get,

**Corollary 1** *Assume  $T : X \rightarrow Y$  is a linear operator between a separable Banach space  $X$  and a Banach space  $Y$ . Assume  $T$  factors through a third Banach space  $Z$  having the Radon–Nikodym property (RNP) via Lipschitz mappings:  $T = g \circ f$ ,  $f : X \rightarrow Z$ ,  $g : Z \rightarrow Y$ . Then there are linear operators  $A : X \rightarrow Z$ ,  $B : Z \rightarrow Y^{**}$  with  $JT = BA$  and  $\|A\|\|B\| \leq \text{Lip}(f)\text{Lip}(g)$ . Here  $J$  is the canonical embedding of  $Y$  into its second dual.*

Recall [1, section 6.4] that one equivalence to  $Z$  having the RNP is that every Lipschitz mapping from  $\mathbb{R}$  (or even from a general separable Banach space) into  $Z$  has a point of Gâteaux differentiability. Reflexive spaces and separable conjugate spaces, including  $\ell_1$ , have the RNP, while, for example,  $c_0$  and  $L_1[0, 1]$  fail the RNP.

Applying this to the case where  $Z = \ell_1$  and to the case where  $Z$  is a general separable  $\mathcal{L}_1$  space with the Radon–Nikodym property and using the fact that a Banach space  $X$  is  $\mathcal{L}_1$  whenever the canonical injection of  $X$  into its second dual factors through a  $\mathcal{L}_1$  space [6], we get

**Corollary 2** *The following two families are preserved under Lipschitz isomorphisms:*

1. *The  $\mathcal{L}_1$ -subspaces of  $\ell_1$ .*
2. *The separable  $\mathcal{L}_1$  spaces that have the Radon–Nikodym property.*

### 3 The local–global argument

Recall that any  $n$ -dimensional normed space  $X$  admits a normalized basis  $x_1, \dots, x_n$  such that  $n\|\sum_{i=1}^n a_i x_i\| \geq \sum_{i=1}^n |a_i|$ , for all  $a_1, \dots, a_n \in \mathbb{R}$ . Indeed, any Auerbach basis satisfies this. See e.g. [1, page 432] for the definition and proof of existence of an Auerbach basis. The next proposition is the main new tool in the proof of the main Theorem.

**Proposition 1** *Let  $X$  be an  $n$ -dimensional normed space,  $Y$  a Banach space and  $T : X \rightarrow Y$  a bounded linear map. Let  $S : X \rightarrow Y$  be Gâteaux differentiable everywhere and Lipschitz. Let  $x_1, \dots, x_n$  be an Auerbach basis for  $X$ ,  $\varepsilon > 0$ , and assume that*

$$\|Sx - Tx\| \leq \varepsilon/2n$$

for all  $x \in C = \{\sum_{i=1}^n t_i x_i ; 0 \leq t_i \leq 1, 1 \leq i \leq n\}$ . Then

$$\left\| \int_C D_x S dx - T \right\| \leq \varepsilon,$$

Where  $D_x S$  is the Gâteaux derivative of  $S$  at  $x$  and  $dx$  is the measure induced on  $C$  from the natural map from  $[0, 1]^n$  onto  $C$  and the Lebesgue measure on  $[0, 1]^n$ .

**Proof:** For every  $1 \leq i \leq n$

$$\begin{aligned} \int_C D_x S x_i dx &= \int_0^1 \int_0^1 \dots \int_0^1 D_{(\sum_{j \neq i} t_j x_j + t x_i)} S x_i dt dt_1 \dots dt_n \\ &= \int_0^1 \int_0^1 \dots \int_0^1 S(\sum_{j \neq i} t_j x_j + x_i) - S(\sum_{j \neq i} t_j x_j) dt_1 \dots dt_n. \end{aligned}$$

So

$$\begin{aligned} \left\| \int_C D_x S x_i dx - T x_i \right\| &\leq \int_0^1 \int_0^1 \dots \int_0^1 \|S(\sum_{j \neq i} t_j x_j + x_i) - T(\sum_{j \neq i} t_j x_j + x_i)\| dt_1 \dots dt_n \\ &\quad + \int_0^1 \int_0^1 \dots \int_0^1 \|S(\sum_{j \neq i} t_j x_j) - T(\sum_{j \neq i} t_j x_j)\| dt_1 \dots dt_n \\ &\leq \varepsilon/n \end{aligned}$$

For a general  $\sum_{i=1}^n a_i x_i \in C$ ,

$$\left\| \int_C D_x S \left( \sum_{i=1}^n a_i x_i \right) - T \left( \sum_{i=1}^n a_i x_i \right) \right\| \leq \frac{\varepsilon}{n} \sum_{i=1}^n a_i \leq \varepsilon \left\| \sum_{i=1}^n a_i x_i \right\|$$

since  $\{x_i\}$  is an Auerbach basis. ■

**Theorem 2** Let  $X$  be a finite dimensional normed space,  $Y$  a Banach space with the RNP and  $T : X \rightarrow Y$  a linear operator. Let  $Z$  be a separable Banach space and assume there are Lipschitz maps  $F_1 : X \rightarrow Z$  and  $F_2 : Z \rightarrow Y$  with  $F_2 \circ F_1 = T$ . Then for every  $\lambda > 1$  there are linear maps  $T_1 : X \rightarrow L_\infty(Z)$  and  $T_2 : L_1(Z) \rightarrow Y$  with  $T_2 \circ i_{\infty,1} \circ T_1 = T$  and  $\|T_1\| \cdot \|T_2\| \leq \lambda \text{Lip}(F_1) \text{Lip}(F_2)$ .

**Proof:** Here  $L_\infty(Z)$  and  $L_1(Z)$  are the  $Z$ -valued measurable functions on  $[0, 1]$  under the essential sup and  $L_1$  norms, respectively. However, it is equivalent to replace  $[0, 1]$  with any other separable, purely non atomic measure space, and in fact we use  $[0, 1]^m$  for a suitable  $m$  instead of  $[0, 1]$ . It would of course be better to factor  $T$  through  $Z$  itself, but it remains open whether this can be done even when  $Y$  is also finite dimensional and  $\lambda$  is replaced by any numerical constant.

Set  $n = \dim(X)$ , let  $\{x_i\}_{i=1}^n$  be an Auerbach basis for  $X$ , and let  $\delta > 0$ . First note that there are everywhere Gâteaux differentiable maps  $\tilde{F}_1 : X \rightarrow Z$  and  $\tilde{F}_2 : Z \rightarrow Y$  with  $\text{Lip}(\tilde{F}_i) \leq \text{Lip}(F_i)$  and  $\|F_i(x) - \tilde{F}_i(x)\| \leq \delta$  for all  $x$  in the domain of  $F_i$ ,  $i = 1, 2$ . The existence of  $\tilde{F}_1$  is classical: regard  $X$  as  $\mathbb{R}^n$  under some norm, let  $f$  be a non negative  $C^1$  function supported on a small neighborhood of 0 in  $X$  for which  $\int_X f(x) dx = 1$ , and

set  $\tilde{F}_1 = f * F_1$ . The function  $\tilde{F}_2$  exists because  $Y$  has the RNP; see, for example, [1, Corollary 6.43].

Define  $S = \tilde{F}_2 \circ \tilde{F}_1$ . Then  $\|Sx - Tx\| \leq \delta(1 + \text{Lip}(F_2))$  for all  $x$ . By Proposition 1,

$$\left\| \int_C D_{F_1(x)} F_2 \circ D_x F_1 - T \right\| = \left\| \int_C D_x S dx - T \right\| \leq 2n\delta(1 + \text{Lip}(F_2)), \quad (1)$$

where  $C = \{\sum_{i=1}^n t_i x_i ; 0 \leq t_i \leq 1, 1 \leq i \leq n\}$ . Define now  $H_1 : X \rightarrow L_\infty([0, 1]^n, Z)$  by

$$H_1 x(t) = D_{t_1 x_1 + \dots + t_n x_n} \tilde{F}_1(x)$$

and  $H_2 : L_1([0, 1]^n, Z) \rightarrow Y$  by

$$H_2 f = \int_{[0, 1]^n} D_{\tilde{F}_1(t_1 x_1 + \dots + t_n x_n)} \tilde{F}_2 f(t) dt.$$

Then,

$$\|H_1 u\| \leq \sup_{[0, 1]^n} \|D_{t_1 x_1 + \dots + t_n x_n} \tilde{F}_1(u)\| \leq \text{Lip}(\tilde{F}_1) \|u\|$$

and

$$\|H_2 f\| \leq \int_{[0, 1]^n} \|D_{\tilde{F}_1(t_1 x_1 + \dots + t_n x_n)} \tilde{F}_2 f(t)\| dt \leq \text{Lip}(\tilde{F}_2) \int_{[0, 1]^n} \|f(t)\| dt$$

so that

$$\|H_1\| \cdot \|H_2\| \leq \text{Lip}(F_1) \text{Lip}(F_2).$$

Also,

$$H_2(H_1 u) = \int_{[0, 1]^n} D_{\tilde{F}_1(t_1 x_1 + \dots + t_n x_n)} \tilde{F}_2(D_{t_1 x_1 + \dots + t_n x_n} \tilde{F}_1(u)) dt = \int_C D_x S u dx.$$

We thus have found linear  $H_1 : X \rightarrow L_\infty([0, 1]^n, Z)$  and  $H_2 : L_1([0, 1]^n, Z) \rightarrow Y$  with  $H := \int_C D_x S dx = H_2 \circ i_{\infty, 1} \circ H_1$ ,  $\|H_1\| \cdot \|H_2\| \leq \text{Lip}(F_1) \text{Lip}(F_2)$  and, by (1),  $\|H - T\| \leq 2n\delta(1 + \text{Lip}(F_2))$ . This implies that, if  $\delta$  is small enough, a similar factorization holds for  $T$ . To see this, we regard  $L_\infty([0, 1]^n, Z)$  as being those functions in  $L_\infty([0, 1]^{2n}, Z)$  which depend only on the first  $n$  coordinates and similarly for  $L_1$ . Define the operator  $\tilde{H}_2 : L_1([0, 1]^{2n}, Z) \rightarrow Y$  to be the composition of  $H_2$  with the (norm one) conditional expectation projection  $P$  from  $L_1([0, 1]^{2n}, Z)$  onto  $L_1([0, 1]^n, Z)$ . Of course, the kernel of  $P$  consists of mean zero  $Z$  valued integrable functions on  $[0, 1]^{2n}$  which depend only on the last  $n$  components. In order to get a factorization of  $T : X \rightarrow Y$  through the injection  $L_\infty([0, 1]^{2n}, Z) \rightarrow L_1([0, 1]^{2n}, Z)$  it is enough to show that for all  $\epsilon > 0$ , if  $U : X \rightarrow Y$  is a linear operator with sufficiently small norm, then there are operators  $A$  from  $X$  into the mean zero functions in  $L_\infty([0, 1]^{2n}, Z)$  which depend only on the last  $n$  coordinates and  $B$  from the the mean zero functions in  $L_1([0, 1]^{2n}, Z)$  which depend only on the last  $n$  coordinates into  $Y$  so that  $\|A\|, \|B\| < \epsilon$ . Indeed, setting  $U := T - \tilde{H}_2 H_1$ , we see that the

pair  $H_1 + A$  and  $\tilde{H}_2 + B(I - P)$  provides a factorization of  $T$  through  $L_\infty([0, 1]^{2n}, Z) \rightarrow L_1([0, 1]^{2n}, Z)$ , and obviously  $\|H_1 + A\| \leq \|H_1\| + \epsilon$  and  $\|\tilde{H}_2 + B(I - P)\| \leq \|H_2\| + 2\epsilon$ .

To define  $A$ , let  $z$  be any norm one vector in  $Z$ , let  $r := 1_{(0,1/2)} - 1_{(1/2,1)}$  and for  $1 \leq j \leq n$  let  $r_j$  be the function on  $[0, 1]^{2n}$  defined by  $r_j((t_1, \dots, t_{2n}) = r(t_{n+j})$ . Set  $Ax_j = (\epsilon/n)r_j \oplus z$  so that  $\|A\| \leq \epsilon$ . Let  $Q$  be a projection of norm at most  $\sqrt{n}$  from the mean zero functions in  $L_1([0, 1]^{2n}, Z)$  which depend only on the last  $n$  coordinates onto the span  $W$  of  $r_j \oplus z$ ;  $1 \leq j \leq n$ . Define  $\tilde{B}$  from  $W$  into  $Y$  by setting  $\tilde{B}(r_j \oplus z) = (n/\epsilon)Tx_j$  and define  $B := \tilde{B}Q$ . Clearly  $\|B\| \leq (n/\epsilon)\sqrt{n} \max\{\|Tx_j\| : 1 \leq j \leq n\}$ . ■

As Corollaries we get

**Corollary 3** *Let  $X$  and  $Y$  be Banach spaces,  $T$  a linear operator from  $X$  into  $Y$ . Then  $\gamma_1(T) = \gamma_1^L(T)$ .*

**Proof:** First we reduce to the case where  $X$  and  $Y$  are finite dimensional. By maximality of  $\gamma_1(T)$ , it is enough to show that for each finite dimensional subspace  $E$  of  $X$  and each finite codimensional subspace  $F$  of  $Y$  that  $\gamma_1(Q_F T i_E) \leq \gamma_1^L(T)$ , where  $i_E$  is the injection from  $E$  into  $X$  and  $Q_F$  is the quotient mapping from  $Y$  onto  $Y/F$  (see [2, Theorem 9.1]). Indeed, if  $J_Y T = G \circ F$  is a factorization of  $J_Y T$  through an  $L_1$  space, then  $Q_F^{**} G \circ F i_E$  is a factorization of  $Q_F T i_E$  through the same  $L_1$  space, which yields that  $\gamma_1^L(Q_F T i_E) \leq \gamma_1^L(T)$ .

So in the sequel we assume that  $X$  and  $Y$  are finite dimensional. In this case the desired conclusion follows from Theorem 2. Indeed, if  $T = G \circ F$  is a Lipschitz factorization of  $T$  through some (possibly non separable)  $L_1$  space  $Z$ , then we can replace  $Z$  with the (separable)  $L_1$  subspace of  $Z$  generated by  $FX$ , and then apply Theorem 2. ■

**Corollary 4** *Let  $X$  be a Banach space and assume that for some  $L_1$  space there are maps  $F : X \rightarrow L_1$  and  $G : L_1 \rightarrow X^{**}$  which are Lipschitz for large distances with  $G \circ F = I$  (the canonical embedding). Then  $\gamma_1(I) < \infty$ ; i.e., For some  $L_1$  space there are linear  $T_1 : X \rightarrow L_1$  and  $T_2 : L_1 \rightarrow X^{**}$  with  $T_2 \circ T_1 = I$ . Consequently, if  $X$  is uniformly or coarsely equivalent to a  $\mathcal{L}_1$  space then it is a  $\mathcal{L}_1$  space.*

**Proof:** Before starting the proof we recall that a map  $F$  is Lipschitz for large distances provided for all  $d > 0$  there is  $K_d < \infty$  so that  $d(F(x), F(y)) \leq K_d d(x, y)$  whenever  $d(x, y) \geq d$ . Uniformly continuous mappings whose domain is convex (or sufficiently close to being convex) are Lipschitz for large distances [1, Proposition 1.11]. A coarse equivalence between Banach spaces is just a biLipschitz equivalence between a pair of nets in the respective spaces; it is clear that such a map can be extended to a mapping that is Lipschitz for large distances.

We now turn to the proof. A standard ultraproduct argument shows that we may assume that  $F$  and  $G$  are Lipschitz. (Assume, without loss of generality, that  $F(0) = 0$ . Let  $\mathcal{U}$  be a free ultrafilter on the natural numbers and define  $F_n : X \rightarrow Y$  by  $F_n(x) = \frac{F(nx)}{n}$

and let  $\tilde{F} : X \rightarrow (L_1)_{\mathcal{U}}$  be the ultraproduct of the maps  $F_n$  into the ultrapower  $(L_1)_{\mathcal{U}}$  of  $L_1$ , which is another  $L_1$  space. Define  $G_n : L_1 \rightarrow Y$  by  $G_n(z) = G_n(x) = \frac{G(nz)}{n}$  and define  $\tilde{G} : (L_1)_{\mathcal{U}} \rightarrow Y^{**}$  by setting  $\tilde{G}(z_1, z_2, \dots) = w^* - \lim_{n \in \mathcal{U}} G_n(z_n)$ . The maps  $\tilde{F}$  and  $\tilde{G}$  are Lipschitz.)

Now use the previous corollary to get that  $\gamma_1(I) < \infty$ . By [6],  $X$  is an  $\mathcal{L}_1$  space. ■

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