Averages of norms and quasi-norms*

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Abstract

We compute the number of summands in q-averages of norms needed to approximate an Euclidean norm. It turns out that these numbers depend on the norm involved essentially only through the maximal ratio of the norm and the Euclidean norm. Particular attention is given to the case $q = \infty$ (in which the average is replaced with the maxima). This is closely connected with the behavior of certain families of projective caps on the sphere.

1. Introduction.

The starting point of this paper is a result from [MS1] which we would like to recall. Denote by $|\cdot|$ the canonical Euclidean norm on $\mathbb{R}^n$. Given another norm $\|\cdot\|$ on $\mathbb{R}^n$ denote by a and b the smallest constants such that

$$a^{-1}|x| \le \|x\| \le b|x|, \quad \text{for all } x \in \mathbb{R}^n. \tag{1.1}$$

For $X = (\mathbb{R}^n, \|\cdot\|)$, let $M = M(X) = \int_{S^{n-1}} \|x\| d\nu(x)$ (where ν is the normalized Haar measure on the Euclidean sphere). Let, as in [MS1], $k = k(X) \le n$ be the largest integer such that

$$\mu_{G_{n,k}}\left(\left\{E \ ; \quad \frac{M}{2}|x| \le \|x\| \le 2M|x|, \quad \text{for all } x \in E\right\}\right) > 1 - \frac{k}{n+k},$$

*Partially supported by BSF grants

where $\mu_{G_{n,k}}$ is the normalized Haar measure on the Grassmanian $G_{n,k}$, and let $t = t(X)$ be the smallest integer such that there are orthogonal transformations $u_1, \dots, u_t \in O(n)$ with

$$(1.2) \qquad \frac{M}{2}|x| \le \frac{1}{t}\sum_{i=1}^{t} \|u_i x\| \le 2M|x|, \qquad \text{for all } x \in \mathbb{R}^n .$$

For reasons that will become clear shortly, we shall also denote t by t_1. Combining Theorem 1.1 of [MS1] with the Observation following Theorem 2.2 there, we have

$$b \approx M\sqrt{\frac{n}{k(X)}} \approx M\sqrt{t_1},$$

or let us write this in the form

$$t_1 \approx (b/M)^2.$$

(Here and elsewhere in this paper, $\alpha \approx \beta$ means $c\beta \le \alpha \le C\beta$ for some absolute constants $0 < c < C < \infty$.) We shall see below that this last equivalence contains a lot of information about the set $K \cap M^{-1}S^{n-1}$, where K is the unit ball of X.

We would like to investigate in a similar manner the sets $K \cap rS^{n-1}$ for $r \in (b^{-1}, M^{-1})$. For that purpose we need to extend the result above to include ℓ_q-averages of the norms $\|u_i x\|$. First extend the definitions as follows: For $0 < q < \infty$, let $M_q = M_q(X) = (\int_{S^{n-1}} \|x\|^q d\nu(x))^{1/q}$ and let $t_q = t_q(X)$ be the smallest integer such that there are orthogonal transformations $u_1, \dots, u_t \in O(n)$ with

$$\frac{M_q}{2}|x| \le \left(\frac{1}{t}\sum_{i=1}^{t} \|u_i x\|^q\right)^{1/q} \le 2M_q|x|, \qquad \text{for all } x \in \mathbb{R}^n .$$

As is well known (and follows from the concentration of the function $\|x\|$ on S^{n-1}), for q not too large, M_q is almost a constant, as a function of q. It is also clear that as $q \to \infty$ $M_q \to b$. In Statement 3.1 below we show that the behavior of M_q can be described more precisely:

(i) $M_q \approx M_1$, for $1 \le q \le k(X)$,

(ii) $M_q \approx b\sqrt{\frac{q}{n}}$, for $k(X) \le q \le n$,

(iii) $M_q \approx b$, for $q > n$.

The main generalization of the result from [MS1] mentioned above is:

Theorem 3.4.

(i) $t_q \approx t_1$, *for* $1 \le q \le 2$,

(ii) $t_q^{2/q} \approx t_1 \left(\frac{M_1}{M_q}\right)^2$, *for* $2 \le q$.

Consequently,

(iii) $t_q^{2/q} \approx t_1 \approx \left(\frac{b}{M_1}\right)^2$, *for* $2 \le q \le k(X)$,

(iv) $t_q^{2/q} \approx \frac{n}{q}$, *for* $k(X) \le q \le n$.

Moreover, with the appropriate choice of constants (implicit in the notation $\approx$*), a random choice of the orthogonal transformations* $u_1, \dots, u_{t_q}$ *works with high probability. So, essentially, a random choice of orthogonal transformations gives the same result as the best choice.*

We now turn to the case $q = \infty$ in which case the norm $\max_{1\le i\le T} \|u_i^{-1}x\|$ corresponds to the body $K_{\infty,T} = K_{\infty,T}(u_1, \dots, u_T) = \cap_{i=1}^T u_i(K)$. Fix r with $b^{-1} < r \le M^{-1}$ and let $T(r) = T(r, X)$ be the smallest T for which there are T orthogonal transformations $u_1, \dots, u_T$ with $K_{\infty,T}(u_1, \dots, u_T) \subseteq rD$. We shall assume that $b/M \le C\sqrt{n/\log n}$ for some absolute constant C. This can be viewed as a condition of non-degeneracy, i.e., K is not an essentially lower dimensional body. Recall also that any symmetric convex body can be transformed, via a linear transformation, to a body satisfying this condition. In fact, it is enough to transform the body in such a manner that for the resulting body the Euclidean ball is the ellipsoid of maximal volume (see, e.g. the proof of Th. 5.8 of [MS2]). Theorem 4.1 below is a refinement of the following

Theorem. *Under the non-degeneracy condition above, for some universal constants* $0 < c < C < \infty$ *and for* r *in the interval* $[2b^{-1}, (2M)^{-1}]$,

$$\exp\left(\frac{cn}{r^2b^2}\right) \le T(r) \le \exp\left(\frac{Cn}{r^2b^2}\right).$$

The geometric interpretation of this theorem is that, for r as above and $t \geq \exp\left(\frac{Cn}{r^2b^2}\right)$, there are t rotations of the set $rS^{n-1} \setminus K$ whose union covers rS^{n-1} (and one can choose them randomly) while, for $t \leq \exp\left(\frac{cn}{r^2b^2}\right)$ no t rotations provide such a covering. It came as a surprise to us that the parameters involved in the Theorem and its geometric interpretation depends on the body K only through b and M. Moreover, the dependence on M is only to determine the range of r's for which the statement holds.

Next we would like to observe the relation between this theorem and Theorem 3.4. For r in the range above, pick q such that $M_q = r^{-1}$. Note that, by the formulas for M_q above, $q \approx \frac{n}{r^2b^2}$. It then follows from the formulation of Theorem 3.4 (iv), that $\log(t_q) \approx \frac{n\log(rb)}{r^2b^2}$. Notice the similarity between the two approximate formulas for t and t_q.

The results described up to now are contained in sections 3 and 4. In particular, in section 4 we treat the case $q = \infty$. In section 2, we gathered some of the more geometric preparatory results. It contains (Lemma 2.2.1) a separation lemma in a quasi convex setting. It also contains a theorem (2.3.1), which is an adaptation of Lemma 2.1 of [MS1], expressing b of (1.1) in terms of the corresponding quantity for the q-averaged norm (or quasi-norm) (1.2). We also give some geometric interpretation of this lemma, pertaining to the behavior of family of caps on the sphere. Let us mention here a curious application of the material in section 2.

Application. *Let* $\|\cdot\|_1, ..., \|\cdot\|_T$ *be norms on* $\mathbb{R}^n$. *Then*

$$\max_{S^{n-1}} (\|x\|_1 \cdot \|x\|_2 \cdot ... \cdot \|x\|_T) \geq \max_{S^{n-1}} \frac{\|x\|_1}{T} \cdot ... \cdot \max_{S^{n-1}} \frac{\|x\|_T}{T}.$$

In section 5 we adapt these results to the quasi-normed case and to the range $0 < q < 1$.

Recall that a body K is said to be quasi-convex if there is a constant C such that $K + K \subset CK$, and given a $p \in (0,1)$, a body K is called p-convex if for any $\lambda, \mu > 0$ satisfying $\lambda^p + \mu^p = 1$ and for any points $x, y \in K$ the point $\lambda x + \mu y$ belongs to K. Note that for the gauge $\|\cdot\| = \|\cdot\|_K$ associated with the quasi-convex (p-convex) body K the following inequality holds for all $x, y \in \mathbb{R}^n$

$$\|x + y\| \leq C \max\{\|x\|, \|y\|\} \quad (\|x + y\|^p \leq \|x\|^p + \|y\|^p\ )$$

and this gauge is called the quasi-norm (p-norm) if $K=-K$. In particular, every p-convex body K is also quasi-convex and $K+K\subset 2^{1/p}K$. A more delicate result is that for every quasi-convex body K, with the gauge $\|\cdot\|_K$ satisfying

$$\|x+y\|_K\le C\left(\|x\|_K+\|y\|_K\right),$$

there exists a q-convex body K_0 such that $K\subset K_0\subset 2CK$, where $2^{1/q}=2C$. This is the Aoki-Rolewicz theorem ([KPR], [R], see also [K], p.47). In this paper by a body we always mean a centrally-symmetric compact star-body, i.e. a body K satisfies $tK\subset K$ for any $t\in[-1,1]$.

Section 5 deals with averaging of general quasi-convex bodies while section 6, following [MS1], with averaging quasi-convex bodies in special positions.

Throughout this paper, c, C always denote absolute constants. These constants might be different in different instances.

We thank the staff of the MSRI where part of this research has been conducted while all three authors visited there. The second named author worked on this project during his stay in IHES also. The first named author thanks B.M. Makarov for useful conversations concerning the material of this paper.

An extended abstract of this work appeared in [LMS].

2. Behavior of family of projective caps on the sphere.

In this section we introduce a few auxiliary statements concerning choices of a special vectors on the sphere possessing certain special properties with respect to a given family of projective caps on the sphere. These statements will serve as a technical tool mainly in the next section, but we also use them in the proof of Theorem 2.3.1. Some geometric interpretation of these statements regarding the behavior of families of caps will be discussed towards the end of this section.

2.1. We begin with a standard inequality.

Lemma 2.1.1. *Let $\{x_i\}$ be a set of vectors in $\mathbb{R}^n$. Then*

$$\sup_{y\in S^{n-1}}\left(\sum_i|\langle y,x_i\rangle|^p\right)^{1/p}\ge\begin{cases}\sup_i|x_i| & \text{for } p\ge 2,\\ \left(\sum_i|x_i|^{\frac{2p}{2-p}}\right)^{\frac{2-p}{2p}} & \text{for } 1\le p<2.\end{cases}$$

Equality holds if and only if the $\{x_i\}$ are mutually orthogonal. Moreover, for every $p > 0$ we have

$$\sup_{y \in S^{n-1}} \left(\sum_i |\langle y, x_i \rangle|^p \right)^{1/p} \geq c_0^{-1} \cdot k^{-\frac{1}{2}} \cdot \left(\sum_i |x_i|^p \right)^{\frac{1}{p}},$$

where k is the dimension of $Y =$ span$\{x_i\}_i$ and $c_0 = \sqrt{2e^\gamma} < 2$ (γ is the Euler constant.)

Proof: Assume first $p \geq 1$ and let q be such that $1/q + 1/p = 1$. Then

$$\begin{aligned} A &= \sup_{y \in S^{n-1}} \left(\sum_i |\langle y, x_i \rangle|^p \right)^{1/p} = \sup_{y \in S^{n-1}} \sup_{\|a\|_q = 1} \langle y, \sum_i a_i x_i \rangle \\ &= \sup_{\|a\|_q=1} \left| \sum_i a_i x_i \right| = \sup_{\|a\|_q=1} \max_{\varepsilon_i = \pm 1} \left| \sum_i \varepsilon_i a_i x_i \right| \\ &\geq \sqrt{\sup_{\|a\|_q=1} \sum_i a_i^2 |x_i|^2} \end{aligned}$$

by the parallelogram equality. Here $a = \{a_i\}_i$ and $\|\cdot\|_q$ denotes norm in l_q. Equality holds if and only if the $\{x_i\}$ are mutually orthogonal. The desired result follows by duality.

Assume now $p > 0$. Let ν be the normalized rotation invariant measure on the Euclidean sphere $S^{k-1} = Y \cap S^{n-1}$. Then

$$A^p = \sup_{y \in S^{n-1}} \left(\sum_i |\langle y, x_i \rangle|^p \right) \geq \int_{S^{k-1}} \sum_i |\langle y, x_i \rangle|^p \, d\nu(y).$$

There is an absolute constant c_0 such that

$$c_0 \left(\int_{S^{k-1}} |\langle y, x_i \rangle|^p \, d\nu(y) \right)^{1/p} \geq \left(\int_{S^{k-1}} |\langle y, x_i \rangle|^2 \, d\nu(y) \right)^{1/2} = |x_i| \cdot k^{-1/2}. \tag{2.1}$$

Therefore,

$$A^p \geq c_0^{-p} k^{-p/2} \sum_i |x_i|^p,$$

which proves the lemma. $\square$

Remarks.

1. In fact c_0 can be exactly computed through the Γ-function and estimated by $c_0 = \sqrt{2e^\gamma} < 2$.

2. A straightforward computation gives, for $p > 1$,

$$\left(\int_{S^{k-1}} |\langle y, x_i \rangle|^p \, d\nu(y) \right)^{1/p} \geq c \left(\min(p,k) \int_{S^{k-1}} |\langle y, x_i \rangle|^2 \, d\nu(y) \right)^{1/2}.$$

Thus, for $p > 1$, we have also

$$\sup_{y \in S^{n-1}} \left(\sum_i |\langle y, x_i \rangle|^p \right)^{1/p} \geq c \sqrt{\frac{\min(p,k)}{k}} \cdot \left(\sum_i |x_i|^p \right)^{\frac{1}{p}}.$$

An immediate corollary is:

Corollary 2.1.2. *Let $\{x_i\}_{i=1}^T$ be a set of vectors on S^{n-1}. Then there exists a $y \in S^{n-1}$ such that*

$$\left(\frac{1}{T} \sum_i |\langle y, x_i \rangle|^p \right)^{1/p} \geq \begin{cases} T^{-1/p} & \text{for } p \geq 2, \\ T^{-1/2} & \text{for } 1 \leq p < 2, \\ c_0^{-1} T^{-1/2} & \text{for } 0 < p < 1. \end{cases}$$

2.2. The following lemma can be viewed as "non-linear" form of the Hahn-Banach theorem for p-convex sets.

Lemma 2.2.1. *Let $\|\cdot\|$ be a p-norm. Let $x_0 \in S^{n-1}$ be vector such that*

$$\|x_0\| = b = \max_{x \in S^{n-1}} \|x\|.$$

Then

$$\|x\| \ \geq \ \left(\frac{p}{2} \right)^{1/p} \cdot b \cdot \left(\frac{\langle x, x_0 \rangle}{|x|} \right)^{-1+2/p} \cdot |x|$$

for any $x \in I\!\!R^n$.

Proof: Denote the unit ball of $\|\cdot\|$ by K and the unit ball of $|\cdot|$ by D. Fix some $\alpha \in (0, \pi/2)$ and let x be a vector in $I\!\!R^n$ such that $\langle x, x_0 \rangle = |x| \sin\alpha$. Denote $r = 1/b$ then $rD \subseteq K$. To prove the claim we are going to find a lower bound on $|x|$ which will ensure that $x \notin K$. For that we represent the vector νx_0 for some ν as a p-convex combination of x and some $y \in rD$. The p-convexity of K and the maximality of b imply then that, if $\nu > r$, then $x \notin K$.

Without loss of generality we can assume that $n = 2$, $x_0 = (0,1)$, $x = (x_1, x_2) = |x|(\cos\alpha, \sin\alpha)$. Choose a vector $(v, w) = r(\cos\beta, \sin\beta)$, where $\beta \in (0, \pi/2)$ will be specified later.

By maximality of b the point $y = (-v, w) \in K$. Thus if $x \in K$ then $\lambda^{1/p}(x_1, x_2) + (1-\lambda)^{1/p}(-v, w) \in K$ for any $0 < \lambda < 1$. Take

$$\lambda = \frac{v^p}{x_1^p + v^p}$$

then

$$\lambda^{1/p}(x_1, x_2) + (1-\lambda)^{1/p}(-v, w) = \left(0, |x|\, r\, \frac{\sin(\alpha+\beta)}{(|x|^p \cos^p\alpha + r^p \cos^p\beta)^{1/p}}\right) \in K.$$

Hence, by the definition of x_0 we have, if $x \in K$,

$$|x|\, r\, \frac{\sin(\alpha+\beta)}{(|x|^p \cos^p\alpha + r^p \cos^p\beta)^{1/p}} \le r,$$

or, as long as, $\sin^p(\alpha+\beta) > \cos^p\alpha$,

$$|x|^p \le \frac{r^p \cos^p\beta}{\sin^p(\alpha+\beta) - \cos^p\alpha}\ .$$

Taking $\beta = \frac{\pi}{2} - \alpha$, we get

$$|x|^p \le \frac{r^p \sin^p\alpha}{1 - \cos^p\alpha}$$

and, since $1 - \cos^p\alpha = 1 - (1 - \sin^2\alpha)^{\frac{p}{2}} \ge \frac{p}{2}\sin^2\alpha$,

$$|x| \le r(2/p)^{1/p}(\sin\alpha)^{1-\frac{2}{p}} = r\left(\frac{2}{p}\right)^{1/p}\left(\frac{\langle x, x_0\rangle}{|x|}\right)^{1-\frac{2}{p}}$$

and the lemma is proved. □

2.3. Corollary 2.1.2 and Lemma 2.2.1 imply the following extension of Lemma 2.1 of [MS1].

Theorem 2.3.1. *Let $u_1, ..., u_T$ be orthogonal operators on $I\!R^n$. Let $\|\cdot\|$ be a p-norm on $I\!R^n$ and for some $q > 0$ put*

$$|||x||| = \left(\frac{1}{T} \sum_{i=1}^{T} \|u_i x\|^q \right)^{1/q}.$$

Assume $|||x||| \le C|x|$ for every x in $I\!R^n$ and some constant C. Then

$$\|x\| \le C(p,q)C|x| \cdot \begin{cases} T^{1/q} & \text{for } q \ge \frac{2p}{2-p}, \\ T^{1/p-1/2} & \text{for } q < \frac{2p}{2-p}, \end{cases}$$

where $C(p,q) = (C(q))^{\frac{2-p}{p}} C_1(p)$ with

$$C(q) = \begin{cases} 1 & \text{for } q \ge \frac{p}{2-p}, \\ c_0 & \text{for } q < \frac{p}{2-p}, \end{cases} \quad C_1(p) = \begin{cases} 1 & \text{for } p = 1, \\ (2/p)^{1/p} & \text{for } p < 1 \end{cases}$$

and $c_0 < 2$ is the same number as in Lemma 2.1.1.

The following two corollaries follow immediately from the statement of the theorem; the second one, by sending q to zero.

Corollary 2.3.2. *Under the condition of Theorem 2.3.1,*

(i) if $p = 1$, $0 < q < \infty$ then

$$\|x\| \le C(q)C|x| \cdot \begin{cases} T^{1/q} & \text{for } q \ge 2, \\ T^{1/2} & \text{for } q < 2, \end{cases} \qquad \text{for all } x \in I\!R^n,$$

(ii) if $0 < p \le 1$, $q = 2$, i.e. if

$$|||x||| = \left(\frac{1}{T} \sum_{i=1}^{T} \|u_i x\|^2 \right)^{1/2} \le C \cdot |x|,$$

then $\|x\| \le C_1(p) C T^{\frac{1}{2}} |x|$ for all $x \in I\!R^n$.

Corollary 2.3.3. *Let $u_1, ..., u_T$ be orthogonal operators on $I\!R^n$ and let $\|\cdot\|$ be a norm on $I\!R^n$ then*

$$\|x\| \le c_0 \sqrt{T} \left(\max_{x \in D} \prod_{i=1}^{T} \|u_i x\| \right)^{1/T} |x|, \qquad \text{for all } x \in I\!R^n.$$

Proof of Theorem 2.3.1: Let $x_0 \in S^{n-1}$ be a vector such that

$$\|x_0\| = b = \max_{x \in D} \|x\|$$

and set $x_i = u_i^{-1} x_0$. By Corollary 2.1.2 there is a $y \in S^{n-1}$, such that

$$\left(\frac{1}{T} \sum_i |\langle y, x_i \rangle|^{q \frac{2-p}{p}} \right)^{1/q} \geq \begin{cases} T^{-1/q} & \text{for } q\frac{2-p}{p} \geq 2, \\ T^{-1/p+1/2} & \text{for } 1 \leq q\frac{2-p}{p} < 2, \\ c_0^{\frac{p-2}{p}} \; T^{-1/p+1/2} & \text{for } q\frac{2-p}{p} < 1. \end{cases}$$

Hence, using Lemma 2.2.1,

$$\begin{aligned} C|y| &= C \geq \left(\frac{1}{T} \sum_{i=1}^{T} \|u_i y\|^q \right)^{1/q} \\ &\geq \left((C_1(p))^{-q} \frac{b^q}{T} \sum_i |\langle u_i y, x_0 \rangle|^{q \frac{2-p}{p}} \right)^{1/q} \\ &= (C_1(p))^{-1} b \left(\frac{1}{T} \sum_i |\langle y, x_i \rangle|^{q \frac{2-p}{p}} \right)^{1/q} \\ &\geq (C_1(p))^{-1} b (C(q))^{\frac{p-2}{p}} \cdot \begin{cases} T^{-1/q} & \text{for } q \geq \frac{2p}{2-p} \\ T^{-1/p+1/2} & \text{for } q < \frac{2p}{2-p}, \end{cases} \end{aligned}$$

which implies the theorem. □

2.4. Some of the results above have what seems to be an interesting geometric interpretation.

Fix a set of points $\{x_i\}$, $1 \leq i \leq T$, on the Euclidean sphere S^{n-1}, let $b \geq 1$ and define the family of seminorms

$$p_i(x) = b|\langle x_i, x \rangle| \,, \quad 1 \leq i \leq T \,.$$

Choose $\varepsilon_i = \pm 1$ such that the Euclidean norm of

$$z = \sum_{i=1}^{T} \varepsilon_i x_i$$

is maximal. Denote $\lambda = |z|$ and let $y = z/\lambda \in S^{n-1}$.

Lemma 2.4.1.

(i)

$$\sqrt{T} \;\le \lambda \;\le T,$$

$\lambda = \sqrt{T}$ *if and only if the points* $\{x_i\}$ *are mutually orthogonal and* $\lambda = T$ *if and only if* $x_i = \pm x_1$ *for every* i.

(ii)

$$\langle y, \varepsilon_i x_i \rangle \;\ge 1/\lambda \;\; \text{for every } i \;\; \text{and} \;\; \sum_{i=1}^{T} p_i(y) \;=\; b\lambda.$$

(iii)

$$\frac{b}{\lambda} \;\le\; p_i(y) \;\le b, \quad \text{for all } i.$$

Proof: (i) Since

$$|z|^2 \;\ge\; \mathrm{Ave}_{\varepsilon_i=\pm 1} \left| \sum_{i=1}^{T} \varepsilon_i x_i \right|^2 = T$$

we get the lower bound. The upper bound is obvious.

(ii) By the maximality of z, $\langle z - \varepsilon_i x_i, \varepsilon_i x_i \rangle \ge 0$. Hence $\langle z, \varepsilon_i x_i \rangle \ge 1$ for every i. Clearly,

$$\sum_{i=1}^{T} p_i(y) \;=\; b \sum_{i=1}^{T} \langle y, \varepsilon_i x_i \rangle \;=\; b\,\langle y, z \rangle \;\;=\; b\lambda.$$

(iii) follows from (ii) and the definitions. □

The following claim gives some information concerning the behavior of family of projective caps on the Euclidean sphere.

Claim 2.4.2. *Denote*

$$A_i(t) \;=\; \left(tS^{n-1}\right) \cap \{x \mid p_i(x) \ge 1\}$$

and

$$A_i \;=\; A_i(1) \;=\; S^{n-1} \cap \{x \mid p_i(x) \ge 1\}, \quad 1 \le i \le T.$$

Then

(i) the projective caps $A_i(\lambda/b)$, $1 \le i \le T$, have a common point

(ii) for $b > 1$ at least

$$k \ge \frac{b\lambda - T}{b-1}$$

of the projective caps A_i have a common point.

Proof: Lemma 2.4.1(iii) implies that $\frac{\lambda}{b}y \in A_i(\lambda/b)$ for all $1 \le i \le T$. Let $k = |\{i \mid p_i(y) \ge 1\}|$. Then, by Lemma 2.4.1(ii) and (iii),

$$b\,\lambda \;=\; \sum_{i=1}^{T} p_i(y) \;<\; T - k + bk,$$

from which (ii) follows easily. □

Remark. The case $T = 2$ is easier. One may directly check that $A_1(\sqrt{2}/b) \cap A_2(\sqrt{2}/b) \ne \emptyset$. We leave this easy exercise to the reader.

Corollary 2.4.3. *Let $\|\cdot\|_1, ..., \|\cdot\|_T$ be norms on $\mathbb{R}^n$. Then*

$$\max_{S^{n-1}} (\|x\|_1 \cdot \|x\|_2 \cdot ... \cdot \|x\|_T) \;\ge\; \max_{S^{n-1}} \frac{\|x\|_1}{T} \cdot ... \cdot \max_{S^{n-1}} \frac{\|x\|_T}{T}.$$

Proof: Without loss of generality we may assume that

$$b_i = \max_{S^{n-1}} \|x\|_i = 1$$

for all $i \le T$. Let $x_i \in S^{n-1}$, for $i \le T$, be such that $\|x_i\|_i = 1$. By Claim 2.4.2 there exist an $x \in \bigcap A_i(\lambda)$. Note that $x \in A_i(\lambda)$ implies that $\|x\|_i \ge 1$, so

$$\max_{S^{n-1}} (\|y\|_1 \cdot ... \cdot \|y\|_T) \ge \prod_{i=1}^{T} (\|x\|_i/\lambda) \ge \lambda^{-T}.$$

Lemma 2.4.1(i) now gives the result. □

For $T > 3$ we have a better result.

Proposition 2.4.4. *Let $\|\cdot\|_1, ..., \|\cdot\|_T$ be norms on $\mathbb{R}^n$. Then*

$$\max_{S^{n-1}} (\|x\|_1 \cdot \|x\|_2 \cdot ... \cdot \|x\|_T) \geq \left(c_0\sqrt{\min\{T,n\}}\right)^{-T} \max_{S^{n-1}} \|x\|_1 \cdot \ldots \cdot \max_{S^{n-1}} \|x\|_T.$$

Proof: Let $x_i \in S^{n-1}$ be such that

$$\|x_i\|_i = b_i = \max_{S^{n-1}} \|x\|_i.$$

Let k be the dimension of the span of the x_i's which, we assume without loss of generality, is $\mathbb{R}^k$. Then

$$\begin{aligned}
\sup_{y\in S^{n-1}} \prod_{i=1}^{T} |\langle y, x_i\rangle| &= \exp\left(\sup_{y\in S^{k-1}} \ln\left(\prod_{i=1}^{T} |\langle y, x_i\rangle|\right)\right) \\
&\geq \exp\left(\int_{y\in S^{k-1}} \sum_{i=1}^{T} \ln(|\langle y, x_i\rangle|)\, d\nu(y)\right) \\
&= \prod_{i=1}^{T} \exp\left(\int_{y\in S^{k-1}} \ln(|\langle y, x_i\rangle|)\, d\nu(y)\right) \\
&\geq \prod_{i=1}^{T} c_0^{-1} \left(\int_{y\in S^{k-1}} |\langle y, x_i\rangle|^2\, d\nu(y)\right)^{1/2} \\
&= \left(c_0\sqrt{k}\right)^{-T},
\end{aligned}$$

where the last inequality follows from (2.1) and c_0 is the same constant as in Lemma 2.1.1. The fact, already used above, that $b_i|\langle y, x_i\rangle| \leq \|y\|_i$ concludes the proof. □

Remarks. (i) It should be clear from the discussion above that the extreme case in Corollary 2.4.3 and Proposition 2.4.4 is attained for the seminorms $\|\cdot\|_i = |\langle x_i, \cdot\rangle|$. The optimal constant in the right hand side of the inequality of Proposition 2.4.4 is thus C_T^T where

$$C_T = \min_{x_1,\ldots,x_T\in S^{n-1}} \max_{x\in S^{n-1}} \left(\prod_{i=1}^{T} |\langle x_i, x\rangle|\right)^{1/T}.$$

Taking $x_1, \ldots, x_T$ to be orthogonal, if $T \leq n$, and an appropriate repitition of an orthogonal basis, if $T > n$, and using the inequality betweed the geometric

and arithmetic means, one gets easily that $C_T \leq 2\min\{T,n\}^{-1/2}$, i.e., the constant in Proposition 2.4.4 is optimal except for the choice of the absolute constant c_0. As one of the referees pointed out it may be of interest to determine the actual value of C_T.
(ii) A related question is the following: Given T and b find the configuration of T projective caps, $A_i = \{x \in S^{n-1}\,;\, b|\langle x_i, x\rangle| \geq 1\}$, with centers $x_i \in S^{n-1}$, for which the measure of their intersection is minimal. We remark that it is not hard to see that even for $T << n$ the extremal situation is not when the x_i are orthogonal. We may even have projective caps $\{A_i\}_{i=1}^T$ with orthogonal centers with non empty intersection but such that $\cap_{i=1}^T u_i(A_i) = \emptyset$ for some orthogonal transformations u_i, $i = 1,\dots,T$. The analogous question for (non-projective) caps was solved by Gromov [G] for $T \leq n$. For the best of our knowledge the case $T > n$ is still open.
(iii) Let $\|\cdot\|$ be a norm on $I\!\!R^n$ satisfying (1.1) then, specializing to the two norms $\|\cdot\|$ and $\|\cdot\|^*$, we have the following curious inequalities,

$$ab \geq \max_{S^{n-1}} \|x\|\|x\|^* \geq \frac{1}{2}ab.$$

The formal use of Corollary 2.4.3 gives 1/4 in the right side. However the case of two norms (i.e. $T = 2$) is simpler and stronger as we noted in the Remark after Claim 2.4.2. We may use $\sqrt{2}$ (instead of $T = 2$) twice in the right side of the displayed inequality of Corollary 2.4.3 which gives a factor of 1/2. Simple examples show that 1/2 can not be improved even in two-dimensional ($n = 2$) case.

3. q-averages of norms.

In this section we consider averages of norms under unitary rotations. The expression in Theorem 2.3.1 is a typical one.

We will study a normed spaces equipped, in addition, with an Euclidean norm, i.e. the spaces $X = (I\!\!R^n, \|\cdot\|, |\cdot|)$, where $\|\cdot\|$ is a norm and $|\cdot|$ is some fixed Euclidean norm on $I\!\!R^n$ which, without loss of generality, we assume is the canonical one. The parameter $M_q = M_q(X)$ below was introduced at the beginning of the Introduction. Throughout this section, as before, we denote

$$b = \|Id : (I\!\!R^n, |\cdot|) \longrightarrow (I\!\!R^n, \|\cdot\|)\| = \max_{S^{n-1}} \|x\|,$$

i.e. the best possible constant in the inequality $\|x\| \leq b|x|$, $x \in I\!\!R^n$.

Statement 3.1. *For $1 \le q \le n$ and any normed space $X = (\mathbb{R}^n, \|\cdot\|)$*

$$\max\left\{M_1, c_1 \frac{b\sqrt{q}}{\sqrt{n}}\right\} \;\le\; M_q \;\le\; \max\left\{2\,M_1, c_2\,\frac{b\sqrt{q}}{\sqrt{n}}\right\},$$

where c_1, c_2 are some absolute constants. Moreover,

$$\left|\frac{M_q}{M_1} - 1\right| \le C \frac{b}{M_1}\frac{\sqrt{q}}{\sqrt{n}}.$$

Proof: By the usual concentration inequalities ([MS2])

$$\nu\left(\left\{x \in S^{n-1}\ ;\ |\|x\| - M_1| > t\right\}\right) \le 2\exp\left(-ct^2 n/b^2\right).$$

So,

$$\int_{S^{n-1}} |\,\|x\| - M_1|^q \, d\nu(x) \le 2q\int_0^\infty t^{q-1}\exp\left(-ct^2n/b^2\right)dt =$$

$$= \left(\frac{\sqrt{cn}}{b}\right)^{-q} 2q\int_0^\infty s^{q-1}\exp\left(-s^2\right)ds \le C^q\left(\frac{b\sqrt{q}}{\sqrt{n}}\right)^q,$$

where C is an absolute constant. Thus

$$M_q - M_1 \;\le\; \|\;\|x\| - M_1\;\|_{L_q} \;\le\; C\frac{b\sqrt{q}}{\sqrt{n}},$$

which gives the right hand side inequality.

To prove the left hand side inequality, notice that the unit ball K of X is contained in a symmetric strip of width $1/b$. Indeed let $x_0 \in S^{n-1}$ be such that $\|x_0\| = b$ then $K \subset \{y |\, |\langle y, x_0\rangle| \le 1/b\}$. It follows that for every $t > 0$

$$tK \subset \{y\ ;\ |\langle y, x_0\rangle| \le t/b\}$$

and

$$\left\{x \in S^{n-1}\ ;\ \|x\| \ge t\right\} \supset S := \left\{y \in S^{n-1}\ ;\ |\langle y, x_0\rangle| \ge t/b\right\}.$$

So,

$$\nu\left(\left\{x \in S^{n-1}\ ;\ \|x\| \ge t\right\}\right) \ge \nu\left(S\right).$$

We shall show below that $\nu(S) \ge c\sqrt{n}\,\frac{t}{b}\exp\left(-cnt^2/b^2\right)$, for $t \le b/3$ and some absolute constant c. Thus, for every $t \in (b/\sqrt{n}, b/3)$,

$$M_q \ge t\left(\nu\left(\left\{x \in S^{n-1}\ ;\ \|x\| \ge t\right\}\right)\right)^{1/q} \;\ge\; c\,t\exp\left(\frac{-cnt^2}{qb^2}\right).$$

Choosing $t = \frac{1}{3}\frac{b\sqrt{q}}{\sqrt{n}}$ we get the result.

It remains to prove that $\nu(S) \geq c\sqrt{n}\,\frac{t}{b}\exp(-cnt^2/b^2)$ for $t \leq b/3$. Let

$$I_n = \int_{-\pi/2}^{\pi/2} \cos^n \theta d\theta,$$

then $1 \leq I_n\sqrt{n} \leq \sqrt{\pi/2}$ (see, e.g. ch. 2 of [MS2]) and

$$\nu(S) = \frac{2}{I_{n-2}} \int_{\varepsilon}^{\pi/2} \cos^{n-2} \theta d\theta,$$

for $\varepsilon = \arcsin(t/b)$. Hence,

$$\nu(S) \geq \sqrt{\frac{2(n-2)}{\pi}} \int_{\varepsilon}^{\varepsilon_1} \cos^{n-2} \theta d\theta \geq (\varepsilon_1 - \varepsilon)\sqrt{\frac{2(n-2)}{\pi}} \cos^{n-2} \varepsilon_1$$

for some $\varepsilon_1 \in (\varepsilon, \pi/2)$.

So, if $t \leq b/3$, we can chose $\varepsilon_1 = \arcsin(2t/b)$ and obtain

$$\nu(S) \geq c\frac{\sqrt{n}t}{b} \exp\left(-cnt^2/b^2\right)$$

for some absolute constant c and $n > 3$. □

Remarks. 1. Obviously $M_q \leq b$. It follows that if b is of order of magnitude larger than M_1 then M_q is of the same order as b if and only if q is larger than a constant times n.
2. It follows from a recent results of [La] that for every $0 < q < 1$ and every normed space X we have $cM_1 \leq M_q \leq M_1$.

Let $q > 0$ and let T be a positive integer. Denote

$$E_{q,T} = \mathbf{E} \left(\frac{1}{T}\sum_{i=1}^{T} \|x_i\|^q\right)^{1/q}$$

where $\mathbf{E}$ is the expectation with respect to the product measure on $(S^{n-1})^T$.

Lemma 3.2. *Let $1 \le q \le n$ and $\alpha = 1/\max\{q, 2\}$. There is an absolute constant C such that for any normed space $X = (\mathbb{R}^n, \|\cdot\|, |\cdot|)$*

$$0 \le M_q - E_{q,T} \le C\frac{b\sqrt{q}}{\sqrt{n}T^\alpha}.$$

In particular, there exists some absolute constant C_0 such that, if $T^\alpha \ge C_0$, then

$$E_{q,T} \le M_q \le 4/3\, E_{q,T}.$$

Proof: Since

$$\left(\frac{1}{T}\sum_{i=1}^{T}\|x_i\|^q\right)^{1/q} \le \frac{1}{T^\alpha}\left(\sum_{i=1}^{T}\|x_i\|^2\right)^{1/2}$$

we get from concentration inequalities for $(S^{n-1})^T$ (cf. [MS2], 6.5.2) that

$$\mathbf{Prob}\left(\left|\left(\frac{1}{T}\sum_{i=1}^{T}\|x_i\|^q\right)^{1/q} - E_{q,T}\right| > t\right) \le 2\exp\left(-c\,t^2\,n\,T^{2\alpha}/b^2\right)$$

for some absolute constant c. Following the first part of the previous proof we get

$$\left\| \left(\frac{1}{T}\sum_{i=1}^{T}\|x_i\|^q\right)^{1/q} - E_{q,T}\right\|_{L_q} \le C\frac{\sqrt{q}\,b}{\sqrt{n}\,T^\alpha}.$$

Thus, for $1 \le q \le n$,

$$E_{q,T} = \left\| \left(\frac{1}{T}\sum_{i=1}^{T}\|x_i\|^q\right)^{1/q}\right\|_{L^1} \le M_q = \left\| \left(\frac{1}{T}\sum_{i=1}^{T}\|x_i\|^q\right)^{1/q}\right\|_{L^q} \le E_{q,T} + C\frac{\sqrt{q}\,b}{\sqrt{n}\,T^\alpha}.$$

By (3.1)

$$M_q \ge c_1\frac{b\sqrt{q}}{\sqrt{n}}.$$

So, if

$$c_1\frac{b\sqrt{q}}{\sqrt{n}} > 4\,C\,\frac{\sqrt{q}\,b}{\sqrt{n}\,T^\alpha}\ ,$$

then

$$E_{q,T} \ge 3C\,\frac{\sqrt{q}\,b}{\sqrt{n}\,T^\alpha}\ ,$$

i.e. if $T^\alpha \ge C_0$ then $E_{q,T} \le M_q \le 4/3\, E_{q,T}$. □

Proposition 3.3. *Let $X = (I\!R^n, \|\cdot\|, |\cdot|)$ and let $q \geq 1$. For every $\varepsilon \in (0,1)$ there exists a constant C_ε, depending on ε only, such that if $T^\alpha > C_\varepsilon \frac{b}{M_q}$, with $\alpha = 1/\max\{q,2\}$, then there exist orthogonal operators $u_1, ..., u_T$ such that*

$$(1-\varepsilon)E_{q,T}|x| \leq \left(\frac{1}{T}\sum_{i=1}^{T}\|u_i x\|^q\right)^{1/q} \leq (1+\varepsilon)E_{q,T}|x| \tag{3.1}$$

for all $x \in I\!R^n$ and $E_{q,T} \leq M_q \leq 4/3\, E_{q,T}$. Moreover, one can take

$$C_\varepsilon = c\frac{\sqrt{\ln(3/\varepsilon)}}{\varepsilon}.$$

Proof: Let $\delta = \varepsilon/3$ and let $\mathcal{N}$ be a δ-net in S^{n-1}. By Lemma 2.6 of [MS2], $\mathcal{N}$ can be chosen with $|\mathcal{N}| \leq \left(\frac{3}{\delta}\right)^n$. Using again the concentration inequalities on $(S^{n-1})^T$ ([MS2], 6.5.2) we get, for the normalized Haar measure $\mathbf{Pr}$ on $(O(n))^T$, that

$$\mathbf{Pr}\left(\left|\left(\frac{1}{T}\sum_{i=1}^{T}\|U_i x\|^q\right)^{1/q} - E_{q,T}\right| > \delta\, E_{q,T}\right) \leq c\,\exp\left(-c\,\delta^2\, E_{q,T}^2\, n\, T^{2\alpha}/b^2\right)$$

for all $x \in S^{n-1}$. Hence, if

$$T^{2\alpha} > c\frac{\ln(1/\delta)}{\delta^2}\frac{b^2}{E_{q,T}^2},$$

then, with positive probability $\{u_1, ..., u_T\}$ satisfy

$$(1-\delta)E_{q,T} \leq \left(\frac{1}{T}\sum_{i=1}^{T}\|u_i x\|^q\right)^{1/q} \leq (1+\delta)E_{q,T}$$

for all x in $\mathcal{N}$. A standard successive approximation argument gives (3.1) for all $x \in S^{n-1}$ as long as

$$T^\alpha > c\frac{\sqrt{\ln(3/\varepsilon)}}{\varepsilon}\frac{b}{E_{q,T}}.$$

Take

$$C_\varepsilon = \max\left\{2c\frac{\sqrt{\ln(3/\varepsilon)}}{\varepsilon}, C_0\right\},$$

where C_0 is the constant from Lemma 3.2. By this lemma if $T^\alpha > C_\varepsilon\, b/M_q$ then $M_q \le 4/3\, E_{q,T}$ and hence

$$C_\varepsilon\,\frac{b}{M_q} \ge c\frac{\sqrt{\ln(3/\varepsilon)}}{\varepsilon}\frac{b}{E_{q,T}}.$$

Thus if $T^\alpha > C_\varepsilon\, b/M_q$ we get the result. □

Remark. The case $q = 1$ is implicitly contained in ([BLM]). The probabilistic estimates there are obtained in a different form (using the fact that one can estimate the ψ_2-norm of sums of independent random variables in term of the ψ_2-norms of the individual variables). A similar proof can also be used here. One needs to use a generalization, due to Schmuckenschläger ([S]), of the fact concerning ψ_2-norm above to the setting of general ψ_2-norms.

We turn now to the study of $k(X)$ and $t_q(X)$, which we introduced in the Introduction.

It was proved in [MS1] that for every n-dimensional normed space X the product $k(X)\cdot t_1(X)$ is of order n, or, in other words, $t_1(X) \approx n/k(X) \approx (b/M_1)^2$. We will study now the relation between $t_1(X)$ and $t_q(X)$ for any $q \ge 1$. This will provide a similar asymptotic formula (in n) for $t_q(X)$.

Theorem 3.4. *There are absolute constants c_1, c_2 such that for every normed space $X = (\mathbb{R}^n, \|\cdot\|)$*

(i) if $q > 2$ then

$$c_1 \cdot t_1(X)\cdot\left(\frac{M_1}{M_q}\right)^2 \le t_q^{2/q}(X) \le c_2\cdot t_1(X)\cdot\left(\frac{M_1}{M_q}\right)^2,$$

(ii) if $1 \le q \le 2$ then

$$c_1 \cdot t_1(X) \le t_q(X) \le c_2\cdot t_1(X).$$

Proof: By Corollary 2.3.2 we have for every $q > 0$ that

$$t_q(X) \geq \left(\frac{b}{2A\,M_q}\right)^s,$$

where

$$A = \begin{cases} 1 & \text{for } q \geq 1, \\ c_0 & \text{for } q < 1 \end{cases} \qquad \text{and} \qquad s = \max\{2,\, q\}.$$

Since $t_1(X) \approx (b/M_1)^2$ we get the left side inequality.

Proposition 3.3 and Lemma 3.2 imply that, if $T^\alpha > C\,\frac{b}{M_q}$, with $\alpha = 1/\max\{q, 2\}$ and $C = C_{1/3}$ an absolute constant, then there exist orthogonal operators $u_1, ..., u_T$ such that

$$\frac{1}{2} M_q |x| \leq \frac{2}{3} E_{q,T}|x| \leq \left(\frac{1}{T}\sum_{i=1}^{T} \|u_i x\|^q\right)^{1/q} \leq 4/3\, E_{q,T}|x| \leq 2M_q|x|$$

for all $x \in \mathbb{R}^n$. Hence we get that, for $q \leq 1$,

$$t_q^\alpha(X) \leq C\,\frac{b}{M_q}.$$

Thus

$$t_q^{2\alpha}(X) \;\approx\; \left(\frac{b}{M_q}\right)^2 \;\approx\; t_1(X)\left(\frac{M_1}{M_q}\right)^2.$$

□

4. ∞-averages (intersection of rotated bodies).

Let $\|\cdot\|$ be a norm and K be the unit ball of this norm. Let $q > 0$ and, given orthogonal operators $u_1, ..., u_T$, denote

$$\|x\|_{qT} = \left(\frac{1}{T}\sum_{i=1}^{T} \|u_i x\|^q\right)^{1/q} \quad \text{and} \quad \|x\|_{\infty T} = \max_{i \leq T} \|u_i x\|.$$

To avoid cumbersome notation, we ignore, in this notation, the concrete choice of the operators $\{u_i\}$, which of course influence the resulted norms.

We shall be mostly interested in the dependence of the quantities above on T. Let K_{qT} denote the unit ball of $\|\cdot\|_{qT}$. Of course,

$$K_{\infty T} = \bigcap_{i=1}^{T} u_i^{-1} K$$

and, for $q \geq \ln T$, $K_{\infty T} \subset K_{qT} \subset eK_{\infty T}$.

The question we would like to study is: Given r with $M_1^{-1} \geq r > b^{-1}$, how many orthogonal operators we need in order to have

$$K_{\infty T} \subset rD \ ?$$

More precisely, what is a correct order of the minimal number $T(r)$ such that there exist $T = T(r)$ orthogonal operators such that

$$K_{\infty T} \subset rD \ ?$$

In the following theorem M denotes the median of the function $\|\cdot\|$ on the sphere S^{n-1}. A similar statement with almost identical proof holds when M denotes the average of $\|\cdot\|$ (in which case $M = M_1$).

Theorem 4.1. *There are absolute constants c_1, C_1, c_2, C_2 such that if $b > 1/r > M$ then*

$$C_1 \exp\left(\frac{n}{2}\frac{(1-rM)^2}{(rb)^2}\right) \leq T(r) \leq C_2 n^{3/2} \log(1+n)\left(1 - \frac{1}{(rb)^2}\right)^{-n/2}.$$

In particular, if $\frac{b}{2} > 1/r > 2M$ (say) and $\frac{M}{b} > \sqrt{\frac{\log n}{n}}$, then

$$\exp\left(\frac{c_1 n}{(rb)^2}\right) \leq T(r) \leq \exp\left(\frac{c_2 n}{(rb)^2}\right).$$

Moreover, for

$$T = \left[C_2 n^{3/2} \log(1+n)\left(1 - \frac{1}{(rb)^2}\right)^{-n/2}\right]$$

(or $T = \exp\left(\frac{c_2 n}{(rb)^2}\right)$ in the case $\frac{b}{2} > 1/r > 2M$ and $\frac{M}{b} > \sqrt{\frac{\log n}{n}}$), a random choice $u_1, \ldots, u_T$ satisfies, with high probability, $K_{\infty T} \subset rD$.

Proof: Let $x_0 \in S^{n-1}$ be such that $\|x_0\| = b$ and let $\theta \in [0, \frac{\pi}{2}]$ be such that $\cos\theta = \frac{1}{rb}$. For $x \in S^{n-1}$ and $\varepsilon \in [0, \frac{\pi}{2}]$ denote by $S(x, \varepsilon)$ the cap

$$S\left(x, \varepsilon\right) = \left\{y \in S^{n-1}\ ;\ \rho(y, x) \leq \varepsilon\right\} = \left\{y \in S^{n-1}\ ;\ \langle y, x\rangle\ \geq\ \cos\varepsilon\right\},$$

where ρ is the geodesic distance on S^{n-1}.

For β to be chosen later, choose $\delta = \beta\theta$-net, $\mathcal{N}$, on the sphere S^{n-1} (with respect to the geodesic distance ρ) satisfying $|\mathcal{N}|\ \leq\ \left(\frac{3\,\pi}{2\,\delta}\right)^n$(cf. Lemma 2.6 of [MS2]). If, for some set $\{x_i\}_{i=1}^T$ of points on the sphere, the union of the caps $S\left(x_i, \theta - \delta\right)$ covers $\mathcal{N}$, then the union of the caps $\{S\left(x_i, \theta\right)\}_{i=1}^T$ covers S^{n-1}. In this case for any orthogonal operators u_i with $x_i = u_i x_0$ we have

$$\max_{1\leq i\leq T} \|u_i^{-1}x\|\ \geq\ b\cos\theta = \frac{1}{r}$$

for any $x \in S^{n-1}$, i.e. $K_{\infty T} \subseteq rD$.

Let $\mathbf{Pr}$ be the normalized Haar measure on $(O(n))^T$. Then

$$\begin{aligned}
\mathbf{Pr}\left(\exists x \in \mathcal{N}\ ;\ x \notin \bigcup_{i=1}^{T} S\left(u_i x_0, \theta - \delta\right)\right) \ &\leq\ \sum_{x\in\mathcal{N}} \mathbf{Pr}\left(x \notin \bigcup_{i=1}^{T} S\left(u_i x_0, \theta - \delta\right)\right) \\
&=\ |\mathcal{N}|\left(\ \mathbf{Pr}\ \left(x \notin S\left(u_1 x_0, \theta - \delta\right)\right)\right)^T \\
&=\ |\mathcal{N}|\left(1 - \nu\left(S\left(x_0, \theta - \delta\right)\right)\right)^T \\
&\leq\ \exp\left(n\ln\left(\frac{3\,\pi}{2\,\delta}\right) - T\,\nu\left(S\left(x_0, \theta - \delta\right)\right)\right).
\end{aligned}$$

So if

$$T > B := \frac{n\ln\left(\frac{3\,\pi}{2\,\delta}\right)}{\nu\left(S\left(x_0, \theta - \delta\right)\right)},$$

then there are u_i such that

$$\max_{1\leq i\leq T} \|u_i^{-1}x\| \geq \frac{1}{r}$$

for all $x \in S^{n-1}$. To estimate B note that as we saw in the proof of Statement 3.1

$$\nu\left(S\left(x_0, \varepsilon\right)\right)\ \geq\ \sqrt{\frac{2(n-2)}{\pi}}\left(\varepsilon - \varepsilon_1\right)\sin^{n-2}\varepsilon_1,$$

for any $0 < \varepsilon_1 < \varepsilon$. Therefore

$$\begin{aligned} A &:= \nu\left(S\left(x_0, \theta - \delta\right)\right) \\ &\geq \sqrt{\frac{2\,(n-2)}{\pi}}\left(\left(\theta - \beta\theta\right) - \left(\theta - \beta\theta - \alpha\theta\right)\right)\sin^{n-2}\left(\theta - \beta\theta - \alpha\theta\right) \end{aligned}$$

as long as $\alpha + \beta < 1$. Set $\alpha = \beta = \frac{1}{2\,n}$ then, since $\sin\gamma\,\theta \geq \gamma\,\sin\theta$ for $\gamma \in (0,1)$ and $\theta \in [0, \frac{\pi}{2}]$,

$$A \geq \sqrt{\frac{2\,(n-2)}{\pi}}\left(1 - \frac{1}{n}\right)^{n-2}\frac{\theta}{2\,n}\sin^{n-2}\theta.$$

Hence,

$$B = \frac{n\,\ln\left(\frac{3\pi}{2\delta}\right)}{A} \leq \frac{n\left(\ln\left(\frac{3\pi}{2\theta}\right) + \ln\left(2n\right)\right)}{A} \leq \frac{c\,n^{3/2}\,\ln\left(1+n\right)}{\theta^2\,\sin^{n-2}\theta}.$$

Since $\theta^2 \geq 1 - \cos^2\theta$ for $\theta \in [0, \frac{\pi}{2}]$, we have

$$B \leq \frac{c\,n^{3/2}\,\ln\left(1+n\right)}{\left(1 - \cos^2\theta\right)^{n/2}}$$

and we get that, for

$$T = \left[C_2 n^{3/2}\log\left(1+n\right)\left(1 - \frac{1}{\left(rb\right)^2}\right)^{-n/2}\right]$$

with an absolute constant C_2, there are orthogonal operators $u_1, ..., u_T$ such that $K_{\infty T} \subset rD$.

To prove the lower bound let us point out that if

$$\bigcap_{i=1}^{T} u_i K \subset rD$$

then, for $S_i = \{x \in S^{n-1}\ ;\ rx \notin \text{ int } u_i K\}$, where int K is the interior of K, $\bigcup_{i=1}^{T} S_i$ covers S^{n-1}. So, if $A := \nu\left(S_i\right) = \nu\left(S\right)$ for

$$S = \left\{x \in S^{n-1}\ ;\ \|x\| \geq \frac{1}{r}\right\}$$

then $T \cdot A \geq 1$, i.e. $T \geq \frac{1}{A}$.

Denote $\alpha = \frac{1}{rM} - 1$, i.e. $r = \frac{1}{(\alpha+1)M}$. Recall the concentration inequality (see, for example, ch. 2 of [MS2]. See also Proposition V.4 in the same book for a similar inequality when M denotes the average):

$$\nu\left(\{x\ ;\ \|x\| > M + b\varepsilon\}\right) \leq \sqrt{\frac{\pi}{8}} \exp\left(-\varepsilon^2 \frac{n-2}{2}\right)$$

for any $\varepsilon > 0$. Take $\varepsilon = \frac{M\alpha}{b}$, then, since $b\varepsilon = \frac{1}{r} - M$,

$$A \leq \sqrt{\frac{\pi}{8}} \exp\left(-\frac{M^2\alpha^2}{b^2}\frac{n-2}{2}\right) = \sqrt{\frac{\pi}{8}} \exp\left(-\frac{n-2}{2}\frac{(1-rM)^2}{(rb)^2}\right).$$

Hence

$$T \geq \sqrt{\frac{8}{\pi}}\frac{1}{e} \exp\left(\frac{n}{2}\frac{1}{(rb)^2}\,(1-rM)^2\right),$$

which proves the theorem. □

Remarks. 1. Let K be a strip $\{x\ ;\ |x_1| \leq 1\}$ (or a bounded approximation of the strip). Then $M/b \approx 1/\sqrt{n}$ and we need at least n rotations to get a bounded K_∞. This shows that a condition ensuring that M/b is of order of magnitude larger than $1/\sqrt{n}$ is necessary. In fact a more careful examination shows that $M/b > c\sqrt{\frac{\ln n}{n}}$ is the right condition.

2. It may be instructive to notice that, for any (fixed) $C > 2$, the inequality in the "In particular" part of the theorem for r in the interval $[(CM_1)^{-1}, (2M_1)^{-1}]$ can be written as

$$(rM)^{-c_1 k(X)} \leq T(r) \leq (rM)^{-c_2 k(X)}.$$

The left hand side inequality continues to hold also for r close (but smaller than) M^{-1}. More precisely, for $1/r = (1+\theta)M$ one has

$$C_1 \exp\left(c\theta^2 k(X)\right) \leq T(r)$$

for C_1 being the constant from Theorem 4.1 and some absolute constant c. Note that in contrast to the exponential behavior of $T(r)$ above, if $1/r = (1-\theta)M$, $0 < \theta < 1$, then

$$T(r) \leq C\theta^{-2}\left(\frac{b}{M}\right)^2.$$

This easily follows from the main result of [BLM] (see also [S]) together with the fact that

$$\max_{1\le i\le T} \|u_i^{-1}x\| \ge \frac{1}{T}\sum_{i=1}^{T} \|u_i^{-1}x\|.$$

5. Averages of quasi-norms.

We explained the notion of p-norms in section 2. The results of section 2 can be applied to the case of p-norms in a similar manner as they were applied for norms. However, the use of the p-triangle inequality, leads sometimes to gaps between the upper and the lower estimates. We still get some interesting versions of the convex case. As the proofs are quite similar to the respective ones in the convex case, we do not repeat them here. The real difference between the proofs here and those in the convex case is the use of the non-linear separation result, Lemma 2.2.1, in the proof of Theorem 2.3.1 and through it in Corollary 5.4 below.

Claim 5.1. *Let $1 \le q \le n$ and let $X = (\mathbb{R}^n, \|\cdot\|)$ be a p-normed space. Then*

$$\max\left\{M_1, c_1\, b\left(\frac{q}{n}\right)^{1/p-1/2}\right\} \ \le\ M_q\ \le\ \max\left\{2\,M_1, c_2\,\frac{b\sqrt{q}}{\sqrt{n}}\right\},$$

and

$$E_{q,T}\ \le\ M_q\ \le\ \max\left\{2E_{q,T}, \frac{c\,b\sqrt{q}}{\sqrt{n}T^{\alpha}}\right\},$$

where $\alpha = 1/\max\{q,2\}$ and c, c_1, c_2 depend on p only.

Let us point out, that using ideas of [La] one can get that for every $1 > q > 0$ and every p-normed space X

$$cM_1 \le M_q \le M_1,$$

where c depends on p but not on q ([L1], chapter 5). In [La] this is proved in the normed case.

Note also that if $\|\cdot\|$ is p-norm then for every integer T

$$\|x\|_{qT} := \left(\frac{1}{T}\sum_{i=1}^{T}\|u_i x\|^q\right)^{1/q}$$

is s-norm for $s = \min\{p, q\}$. For s-norm it is more convenient to use concentration inequalities not for the function $\|x\|_{qT}$ but for the function $\|x\|_{qT}^s$. Thus if we define

$$L_{q,T} = \left(\mathbf{E}\ \|x\|_{qT}^s\right)^{1/s},$$

where $\mathbf{E}$ is the expectation with respect to the product measure on $(O(n))^T$, which is equivalent to $E_{q,T}$ by Latała's theorem, then repeating the proof of Proposition 3.3 we obtain

Proposition 5.2. *Let $q > 0$, $1 \geq p > 0$ and $X = (\mathbb{R}^n, \|\cdot\|)$ be a p-normed space. For $1 > \varepsilon > 0$ denote*

$$C_\varepsilon = \left(\frac{c}{s\varepsilon}\right)^{1/s} \sqrt{\log\frac{2}{\varepsilon}},$$

where $s = \min\{p, q\}$. If $T^\alpha > C_\varepsilon \frac{b}{M_q}$ with $\alpha = 1/\max\{q, 2\}$ then there are orthogonal operators $u_1, ..., u_T$ such that

$$(1-\varepsilon)L_{q,T}|x| \leq \left(\frac{1}{T}\sum_{i=1}^{T}\|u_i x\|^q\right)^{1/q} \leq (1+\varepsilon)L_{q,T}|x|$$

for all $x \in \mathbb{R}^n$.

These statements allow us to extend the results of [MS1] and of the previous section concerning the relation between $k(X)$, $t_1(X)$, and $t_q(X)$ in the following way.

Corollary 5.3. *Let $q > 0$. Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a p-normed space for some $p \in (0, 1]$. Denote $s = \min\{p, q\}$ then there exists an absolute constant c, such that for $q > 2$*

$$t_q(X) \leq \left(\frac{c}{p}\right)^{q/p}\left(\frac{b}{M_q}\right)^q$$

and for $0 < q \leq 2$

$$t_q(X) \leq \left(\frac{c}{s}\right)^{2/s}\left(\frac{b}{M_q}\right)^2.$$

As we noted after Claim 5.1 in the second case, $q \leq 2$, the term $(b/M_q)^2$ can be estimated by $c_p\,(b/M_1)^2$, where c_p is a constant depending on p only.

The proof of this fact is analogous to the proof of Theorem 3.4.

The following fact is a corollary of Theorem 2.3.1. Recall that, for the p-convex part of Theorem 2.3.1 a crucial role was played by the "non-linear" form of Hahn-Banach theorem for p-convex sets (Lemma 2.2.1).

Corollary 5.4. *Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a p-normed space for some $p \in (0,1]$. Let $q > 0$. Then*

$$t_q(X) \geq \left(\frac{b}{2A\, M_q}\right)^{\beta},$$

where

$$\beta = \begin{cases} q & \text{for } q \geq \frac{2p}{2-p}, \\ \frac{2p}{2-p} & \text{for } q < \frac{2p}{2-p} \end{cases} \qquad \text{and} \qquad A = C_1(p)\,(C(q))^{\frac{2-p}{p}}$$

for $C_1(p)$ and $C(q)$ defined in Theorem 2.3.1.

Claim 5.5. *Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a p-normed space for some $p \in (0,1]$. Then there exists a constant $C_2(p)$, depending on p only, such that*

$$C_2(p)\, n \left(\frac{M_1}{b}\right)^2 \leq k(X) \leq n \left(\frac{2M_1}{b}\right)^{\frac{2p}{2-p}}.$$

Proof: The standard concentration-phenomena methods ([MS2]) on the sphere implies the lower bound. This fact was already used by S.J. Dilworth in [D].

Using the same scheme as in the proof of Theorem 2.2.b of [MS1] and p-convexity of $\|\cdot\|$ we get that

$$\|x\| \leq c \left(\frac{n}{k(X)}\right)^{1/p-1/2} |x|$$

for all $x \in \mathbb{R}^n$. That proves the upper bound. □

We conclude this section with a variant of Theorem 4.1 for p-norms.

Proposition 5.6. *Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a p-normed space for some $p \in (0,1]$. There are absolute constants C_1, C_2 such that if $b > 1/r > M$ then*

$$C_1 \exp\left(\frac{n}{2}\frac{(1-(rM)^p)^2}{(rb)^{2p}}\right) \leq T(r) \leq C_2 n^{3/2} \log(1+n) \left(1 - (c_p rb)^{-\frac{2p}{2-p}}\right)^{-n/2},$$

where $c_p = (p/2)^{1/p}$. *In particular, if* $c_p b/2 > 1/r > 2M$ *(say) and* $\frac{M}{b} > \left(\frac{\log n}{n}\right)^{\frac{2-p}{2p}}$, *then there are constants* c'_p, c''_p *depending on* p *only, such that*

$$\exp\left(c'_p n\,(rb)^{-2p}\right) \le T(r) \le \exp\left(c''_p n\,(rb)^{-\frac{2p}{2-p}}\right).$$

Proof: The proof of this proposition essentially repeats the proof of Theorem 4.1. To prove the upper bound we only need to substitute the inclusion

$$\left\{x \in S^{n-1}\ ;\ \|x\| \ge \frac{1}{r}\right\} \supset \left\{x \in S^{n-1}\ ;\ |\langle x, x_0\rangle| \ge \frac{1}{rb}\right\}$$

with the inclusion

$$\left\{x \in S^{n-1}\ ;\ \|x\| \ge \frac{1}{r}\right\} \supset \left\{x \in S^{n-1}\ ;\ |\langle x, x_0\rangle| \ge \left(\frac{1}{c_p rb}\right)^{\frac{p}{2-p}}\right\},$$

which follows from Lemma 2.2.1. In other words, in the proof of Theorem 4.1 one should chose $\theta \in [0, \frac{\pi}{2}]$ such that $\cos\theta = \left(\frac{1}{c_p rb}\right)^{\frac{p}{2-p}}$.

To prove the lower bound we have to apply concentration inequalities to the function $\|\cdot\|^p$. □

6. Averages of quasi-convex bodies in M-position.

Recall that for any subsets K_1, K_2 of $\mathbb{R}^n$ the covering number $N(K_1, K_2)$ is the smallest number N such that there are N points $y_1, ..., y_N$ in $\mathbb{R}^n$ such that

$$K_1 \subset \bigcup_{i=1}^{N} (y_i + K_2).$$

Define the volume radius r of a star-body K by the formula $|K| = |rD|$, where $|K|$ denotes the n-dimensional volume of K.

Let C_p (for $p \in (0, 1]$) be the constant from J. Bastero, J. Bernués, and A. Peña's extension [BBP] of the second named author's reverse Brunn-Minkowski inequality for p-convex bodies, i.e.

$$C_p = \left(\frac{2}{p}\right)^{c/p}$$

(see [L2] for the dependence of the constant on p). Let us recall this result.

Theorem 6.1. *Let $0 < p \le 1$. For all $n \ge 1$ and all symmetric p-convex bodies $K_1, K_2 \subset \mathbb{R}^n$ there exists a linear operator $U : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $|det(U)| = 1$ and*

$$|UK_1 + K_2|^{1/n} \le C_p(|K_1|^{1/n} + |K_2|^{1/n}) \ .$$

In terms of covering numbers this theorem can be formulated in the following way.

Theorem 6.1′. *For every symmetric p-convex body K in $\mathbb{R}^n$ with volume radius r there exists a linear operator $U : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $|det(U)| = 1$ such that*

$$\max\{N(UK, trD),\ N(rD, tUK)\} \le \begin{cases} \exp\left(n\,(C_p/t)^{p/2}\right) & \text{for } t \ge C_p, \\ \exp\left(n \ln\left(3^{1/p}C_p/t\right)\right) & \text{for } 1 < t < C_p, \\ C_p^n & \text{for } t = 1. \end{cases}$$

If the operator U in Theorem 6.1′ can be taken to be the identity operator then we say that the body K is in M-position.

Remark. If the bodies $K_1, K_2, ..., K_l$ are in M-position then, as in the convex case ([P], pp. 120-121),

$$|K_1 + K_2 + ... + K_l|^{1/n} \le C_p \cdot l^{2/p}\left(|K_1|^{1/n} + |K_2|^{1/n} + ... + |K_l|^{1/n}\right).$$

The following theorem has been recently proved in [MS1].

Theorem 6.2. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and assume that its unit ball K is in M-position. Assume further that for some T orthogonal operators $u_1, ..., u_T$ and for some constant C,*

$$|x| \ \le\ \frac{1}{T}\sum_{i=1}^{T}\|u_i x\| \ \le\ C|x|$$

for all $x \in \mathbb{R}^n$. Then there are an orthogonal operator u and a constant C_1, depending on T and C only, such that for some R

$$R|x| \ \le\ \|x\| + \|ux\| \ \le\ C_1 R|x|$$

for all $x \in \mathbb{R}^n$.

By duality this theorem is equivalent to the following statement.

Theorem 6.2′. *Let a symmetric convex body K be in M-position. Assume that for some T orthogonal operators $u_1, ..., u_T$ and for some constant C,*

$$D \subset \frac{1}{T}\sum_{i=1}^{T} u_i K \subset CD.$$

Then there exist an orthogonal operator u and a constant C_1, depending on T and C only, such that for some R

$$RD \subset K + uK \subset C_1 RD.$$

In this section we shall extend both theorems to the quasi-convex case. Because duality arguments can not be applied in the non-convex case these two theorems become different statements.

Lemma 6.3. *Let $q > 0$ and $B > 0$. Let K be a star body such that for any $t \geq B$*

$$N(K, tD) \leq \exp\left(n\left(\frac{B}{t}\right)^q\right).$$

Then there exists an orthogonal operator u such that

$$K \cap uK \subseteq C^{1+\frac{1}{q}} BD.$$

Remark. An immediate corollary is that if a p-convex body K is in M-position then there exists an orthogonal operator u such that for all $x \in \mathbb{R}^n$

$$\|x\|_K + \|ux\|_K \geq \frac{1}{r\,C(p)}|x|,$$

where r is the volume radius of K. Here and everywhere in this section we denote a function of the type $(2/p)^{c/p}$ by $C(p)$. The absolute constant c may be different in different places. Therefore the product of two functions of that type is again a function denoted by $C(p)$.

Proof of Lemma 6.3: Let the constant C_0 satisfy

$$\alpha = \left(\frac{B}{C_0}\right)^q < 1.$$

By the definition of covering numbers for $N = [e^{\alpha n}]$, there exist $\{x_i\}_1^N$ such that

$$K \subset \bigcup_{i=1}^{N} (x_i + C_0 D).$$

Consider the normalized rotation invariant measure on the sphere $R\, C_0\, S^{n-1}$, where $R > 0$ will be specified later. Since the measure of the intersection

$$(x_i + 4C_0 D) \bigcap R\, C_0\, S^{n-1}$$

does not exceed

$$A = \sqrt{\frac{\pi}{8}} \exp\left(-\frac{\pi^2\, (1 - 4/R)^2\, (n-2)}{8} \right)$$

for any x_i ([MS2], ch. 2), we obtain that if $N^2 A < 1$ then there exists an orthogonal operator, u, such that

$$R\, C_0\, u(x_i/|x_i|) \notin x_j + 2C_0 D$$

for any i and j. But the union of $(ux_i + C_0 D)$ covers

$$u\left(K \bigcap R\, C_0\, S^{n-1} \right) = u(K) \bigcap R\, C_0\, S^{n-1}.$$

Therefore

$$K \bigcap u(K) \bigcap R\, C_0\, S^{n-1} = \emptyset.$$

Take

$$C_0 = \left(\sqrt{5}\, (1 + q/2) \right)^{2/q} B \quad \text{and} \quad R = \frac{4}{1 - \sqrt{5\alpha}}$$

then $N^2 A < 1$ and $RC_0 \le 4e5^{1/q} \frac{2+q}{q} B$. This completes the proof. □

Lemma 6.4. *Let $B > 0$. Let K_i, $i = 1, ..., T$, be symmetric p-convex bodies such that $|K_i| = |D|$ and for any $t \ge B$*

$$N(D, tK_i) \le \exp\left(n \left(\frac{B}{t} \right)^q \right)$$

for all $1 \le i \le T$. Then

$$\left| \bigcap_i K_i \right|^{1/n} \ge f(T) \cdot |D|^{1/n},$$

where

$$f(T) \,=\, \left(c \cdot B \cdot 2^{T/p} \cdot A^{1/q}\right)^{-1}$$

for

$$A \,=\, \min\left\{T,\, \max\left\{2,\, \frac{2}{q(-1+1/p)}\right\}\right\}.$$

In particular, if all the K_i's are in M-position then

$$f(T) \,\geq\, \left(C(p) \cdot 2^{T/p} \cdot \min\left\{T,\, \frac{4}{1-p}\right\}^{2/p}\right)^{-1}.$$

This lemma easily follows from the following claim.

Claim 6.5. *Under the assumptions of Lemma 6.4*

$$N(D, t\, 2^{T/p} \bigcap_i K_i) \leq \exp\left(n\left(\frac{B}{t}\right)^q A\right).$$

Proof: Let two star-bodies B_1 and B_2 satisfy

$$B_1 \,\subset\, \bigcup_{i=1}^{N} (x_i + B_2).$$

Then it is not hard to see that there are points y_i in B_1 such that

$$B_1 \,\subset\, \bigcup_{i=1}^{N} \left(y_i + (B_2 - B_2)\right). \tag{6.1}$$

Indeed,

$$B_1 \,\subset\, \bigcup_{i=1}^{N} \left((x_i + B_2) \bigcap B_1\right)$$

and if

$$z_i \,\in\, B_1 \bigcap (x_i + B_2)$$

then

$$B_1 \bigcap (x_i + B_2) \,\subset\, z_i + (B_2 - B_2)\,.$$

Analogously,

$$B_1 \bigcap (x + B_2) \,\subset\, x + (B_1 - B_1) \bigcap B_2 \tag{6.2}$$

for any $x \in B_1$. Hence, using p-convexity and the assumptions of the claim, we get from (6.1) that, for $N_1 = N(D, t_1 K_1)$ and for $x_i \in D$,

$$D \subset \bigcup_{i=1}^{N_1} \left(\left(x_i + t_1\, 2^{1/p}\, K_1 \right) \bigcap D \right) \subset \bigcup_{i=1}^{N_1} \left(x_i + \left(t_1\, 2^{1/p}\, K_1 \bigcap 2D \right) \right).$$

For $N_2 = N(D, t_2 K_2)$ and for $y_i \in 2D \bigcap t_1\, 2^{1/p}\, K_1$, we get by (6.1) and (6.2), that

$$D \subset \bigcup_{i=1}^{N_1} \left(x_i + \left(2D \bigcap t_1\, 2^{1/p}\, K_1 \bigcap \bigcup_{j=1}^{N_2} \left(y_j + 2\, 2^{1/p}\, t_2\, K_2 \right) \right) \right) \subset$$

$$\subset \bigcup_{i=1}^{N_1 N_2} \left(z_i + \left(4D \bigcap t_1\, 2^{2/p}\, K_1 \bigcap t_2\, 2\, 2^{1/p}\, K_2 \right) \right)$$

for $z_i = x_j + y_k$. Continuing in this way we get

$$D \subset \bigcup_{i=1}^{N_1 \ldots N_T} \left(v_i + \left(2^T\, D \bigcap t_1\, 2^{T/p}\, K_1 \bigcap t_2\, 2\, 2^{(T-1)/p}\, K_2 \ldots \bigcap t_T\, 2^{T-1}\, 2^{1/p}\, K_T \right) \right),$$

for some v_i's, where

$$N_i \;=\; N(D, t_i\, K_i) \;\leq \exp \left(n \left(\frac{B}{t_i} \right)^q \right).$$

Setting

$$t_i \;=\; t \cdot 2^{(i-1)(-1+1/p)},$$

we obtain that

$$N(D, t\, 2^{T/p} \bigcap K_i) \;\leq\; \prod_{i=1}^{T} N(D, t_i K_i) \;\leq$$

$$\leq \exp \left(n \left(\frac{B}{t} \right)^q \sum_{i=1}^{T} 2^{-q(i-1)(-1+1/p)} \right) \;\leq\; \exp \left(n \left(\frac{B}{t} \right)^q A \right).$$

□

Lemma 6.6. *Let K_i, $i = 1,...,T$, be symmetric p-convex bodies in M-position. Let $\|\cdot\|_i$ denote the gauge of K_i. Assume that for any $x \in I\!R^n$*

$$|x| \le \max_{i \le T} \|x\|_i \, .$$

Then there exist a number $k \in \{1,...,T\}$ and an orthogonal operator u such that

$$\frac{f(T)}{C(p)}|x| \le \|x\|_k + \|ux\|_k,$$

where $f(T)$ was defined in Lemma 6.4.

Proof: Since $|x| \le \max_{i \le T} \|x\|_i$,

$$\bigcap K_i \subset D.$$

Let $r_1,...,r_T$ denote the volume radii of $K_1,...,K_T$ and

$$r_k = \min_{i \le T} r_i.$$

Then by Lemma 6.4

$$f(T) \le \frac{\left|\bigcap \frac{1}{r_i} K_i\right|^{1/n}}{|D|^{1/n}} \le \frac{1}{r_k} \frac{|\bigcap K_i|^{1/n}}{|D|^{1/n}} \le \frac{1}{r_k}.$$

Hence by Lemma 6.3, there exists an orthogonal operator such that

$$\|x\|_K + \|ux\|_K \ge \frac{|x|}{r_k C(p)} \ge \frac{|x|}{f(T)C(p)}.$$

□

Now we are ready to extend Theorem 6.2 to the quasi-convex case.

Theorem 6.7. *Let $\|\cdot\|$ be a p-norm on $I\!R^n$ and assume that its unit ball K is in M-position. Assume further that for some T orthogonal operators $u_1,...,u_T$ and for some constant C,*

$$|x| \;\le\; |||x||| \;=\; \left(\frac{1}{T}\sum_i \|u_i x\|^p\right)^{1/p} \le C|x|$$

for all $x \in I\!R^n$. Then there exists an orthogonal operator u such that

$$\frac{f(T)}{C(p)}|x| \le \; \|x\| + \|ux\| \le 2\, T^{1/p}\, C\, |x| \, .$$

This theorem is a direct consequence of the previous lemma. To extend Theorem 6.2′ we need the following two lemmas.

Lemma 6.8. *Let $B > 0$. Let K be a symmetric p-convex body such that for any $t \geq B$*

$$N(rD, tK) \leq \exp\left(n\left(\frac{B}{t}\right)^q\right),$$

where r is the volume radius of K. Then there exists an orthogonal operator u such that

$$D \subset \frac{C(p) \cdot B \cdot c^{1/q}}{r}\,(K + uK)\,.$$

The proof of this lemma is almost identical to the proof of Theorem 2′ of [LMP]. Note also that if the body K is in M-position then $B \cdot c^{1/q} = C(p)$ and $C(p) \cdot B \cdot c^{1/q}$ can be replaced by $C(p)$.

Lemma 6.9. *Let K_i, $i = 1, ..., T$, be symmetric p-convex bodies in M-position. Assume that for some constant $C > 0$*

$$\frac{1}{C} D \subset \frac{1}{T}\sum_{i=1}^{T} K_i \subset D.$$

Then there exist a number $k \in \{1, ..., T\}$ and an orthogonal operator u such that

$$D \;\subset\; C\,C(p)\,T^{2/p}\,(K_k + uK_k)\,.$$

Proof: Let $r_1, ..., r_T$ denote volume radii of $K_1, ..., K_T$ and

$$r_k = \max_{i \leq T} r_i\,.$$

By the assumption of the lemma and the remark after Theorem 6.1′ we have

$$\frac{1}{C} \;\leq\; \frac{\left|\sum_{i=1}^{T} K_i/T\right|^{1/n}}{|D|^{1/n}} \;\leq\; C(p)\,T^{2/p}\,\frac{\sum_{i=1}^{T} r_i}{T} \;\leq\; C(p)\,T^{2/p}\,r_k\,.$$

Therefore, by Lemma 6.8, we get

$$D \subset \frac{C(p)}{r_k}\,(K_k + uK_k) \subset\; C\,C(p)\,T^{2/p}\;(K_k + uK_k)$$

for some orthogonal operator u. □

This lemma gives us the following extension of Theorem 6.2′.

Theorem 6.10. *Let a symmetric convex body K be in M-position. Assume that for some T orthogonal operators $u_1, ..., u_T$ and for some constant C,*

$$D \ \subset \ \frac{1}{T}\sum_{i=1}^{T} u_i K \ \subset \ C D.$$

Then there exists an orthogonal operator u such that

$$\frac{1}{C\, C(p)\, T^{-1+2/p}} D \subset K + uK \subset 2\, T\, D.$$

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