Eidelheit’s Theorem Revisited

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Godefroy’s Fest June 2022

Based on an observation by Johnson, Phillips and Schechtman
Eidelheit’s Theorem

We say that two Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as Banach algebras if there is an injective and surjective homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ such that $\|\Phi\|, \|\Phi^{-1}\| < \infty$. $\Phi$ is called a Banach algebras isomorphism.

For a Banach space $X$, $L(X)$ denotes the Banach algebra of all bounded linear operators on $X$.

A theorem of Meier Eidelheit from 1940 states that if $L(X)$ and $L(Y)$ are isomorphic as Banach algebras then $X$ and $Y$ are isomorphic as Banach spaces. Moreover, if $\Phi : L(X) \to L(Y)$ is the Banach algebra isomorphism then there is a Banach space isomorphism $U : X \to Y$ such that for all $A \in L(X)$

$$\Phi(A) = UAU^{-1}.$$
Meier Eidelheit, 1910–1943, was a student of Banach in Lwów (=Lvov=Lviv).
He was murder in the Holocaust.

We (Bill Johnson, Chris Phillips and I) noticed that digging a bit into the proof of Eidelheit theorem one can prove a stronger theorem.
Theorem. (JPS)

Let $\mathcal{A}$ be a Banach subalgebra of $L(X)$ and $\mathcal{B}$ a Banach subalgebra of $L(Y)$. Assume that $\mathcal{A} \supseteq F(X)$ (the finite rank operators) and that $\mathcal{B} \supseteq F(Y)$. Assume that $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is a Banach algebra isomorphism. Then, there is an isomorphism $U : X \rightarrow Y$ such that for all $A \in \mathcal{A}$

$$\Phi(A) = UAU^{-1}.$$ 

Moreover $\|U\| \leq \|\Phi\|$ and $\|U^{-1}\| \leq \|\Phi^{-1}\|$. 

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Actually, as we’ll see, the continuity of \( \Phi \) and \( \Phi^{-1} \) is automatic and need not be assumed so a stronger theorem holds.

**Theorem.** (JPS)

Let \( A \) be a Banach subalgebra of \( L(X) \) and \( B \) a Banach subalgebra of \( L(Y) \). Assume that \( A \supseteq F(X) \) (the finite rank operators) and that \( B \supseteq F(Y) \). Assume that

\[
\Phi : A \rightarrow B
\]

is an homomorphism of algebras. Then, \( \Phi \) and \( \Phi^{-1} \) are bounded and there is an isomorphism \( U : X \rightarrow Y \) such that for all \( A \in A \)

\[
\Phi(A) = UAU^{-1}.
\]

Moreover \( \|U\| \leq \|\Phi\| \) and \( \|U^{-1}\| \leq \|\Phi^{-1}\| \).
Recall that a (two sided) ideal in $L(X)$ is a subspace $\mathcal{I}$ such that for all $A \in \mathcal{I}$ and $B \in L(X)$, $AB$ and $BA$ are in $\mathcal{I}$.

**Corollary.**

*If $\mathcal{I}$ and $\mathcal{J}$ are closed ideals in $L(X)$ which are isomorphic as Banach algebras then $\mathcal{I} = \mathcal{J}$.***

**Proof:** Assume $\mathcal{I}$ and $\mathcal{J}$ are isomorphic as Banach algebras. Since any ideal in $L(X)$ contains $F(X)$, it follows that there is a Banach space isomorphism $U$ of $X$ onto itself such that $A \rightarrow UAU^{-1}$ maps $\mathcal{I}$ onto $\mathcal{J}$. If $B \in \mathcal{J}$ there is an $A \in \mathcal{I}$ such that

$$B = UAU^{-1}.$$

Since $\mathcal{I}$ is an ideal $B \in \mathcal{I}$. So $\mathcal{J} \subseteq \mathcal{I}$. Similarly, $\mathcal{I} \subseteq \mathcal{J}$. ■

**Remark:** Enough to assume that $\mathcal{I}$ and $\mathcal{J}$ are homomorphic as algebras.
This corollary can be used to boost several recent theorems concerning the number of closed ideals in $L(X)$ for some classical spaces $X$.

**Theorem. (JPS +)**

There is a continuum of closed ideals in $L(L_1(0, 1))$ each two of which are non-isomorphic as Banach algebras. The same holds for $L(C(0, 1))$ and $L(L_{\infty}(0, 1))$.

**Theorem. (JS +)**

For each $1 < p \neq 2 < \infty$ there are exactly $2^{\text{continuum}}$ closed ideals in $L(L_p(0, 1))$ up to isomorphism of Banach algebras.
Theorem. (Freeman, Schlumprecht, Zsak +)

For $1 < p < q < \infty$ there are exactly $2^{\text{continuum}}$ closed ideals in $L(\ell_p \oplus \ell_q)$ up to isomorphism of Banach algebras. Same holds for $L(\ell_1 \oplus \ell_q)$ and $L(\ell_p \oplus c_0)$.

Remark: In the last three theorems one can replace “isomorphism of Banach algebras” by “homomorphism of algebras”.

In the rest of this talk I’ll give the, hopefully complete, proof of the Theorem.
Lemma

The following are equivalent for $V_0 \in L(X)$:

1. $V_0$ is of rank at most one.
2. For all $V \in L(X)$ there is a $\lambda \in \mathbb{C}$ such that
   
   $$(VV_0)^2 = \lambda VV_0$$

3. For all $V \in L(X)$ of rank 2 there is a $\lambda \in \mathbb{C}$ such that
   
   $$(VV_0)^2 = \lambda VV_0.$$ 

Proof. $1 \Rightarrow 2$: 
Let $V_0$ be of rank 1. For each $V \in L(X)$, $W = VV_0$ is of rank at most 1. Assume it is of rank 1. $W = x^* \otimes x$, $W(y) = x^*(y)x$.

$$W^2(y) = x^*(y)W(x) = x^*(y)x^*(x)x = x^*(x)W(y).$$
Lemma

2 $\Rightarrow$ 3 is clear.
3 $\Rightarrow$ 1:
Assume that the range of $V_0$ contains two independent vectors $y_1$ and $y_2$. Let $x_1$ and $x_2$ be such that

$$V_0(x_1) = y_1, \quad V_0(x_2) = y_2.$$ 

Let $f_1$ and $f_2$ be linear functionals such that

$$f_1(y_1) = 1, \quad f_1(y_2) = 0, \quad f_2(y_1) = 0, \quad f_2(y_2) = 1$$

and put

$$V(x) = f_1(x)x_2 + f_2(x)x_1.$$ 

Then,

$$VV_0(x_1) = x_2, \quad VV_0(x_2) = x_1.$$
Lemma

\[ VV_0(x_1) = x_2, \quad VV_0(x_2) = x_1. \]

Since, \( (VV_0)^2 = \lambda VV_0, \)

\[ x_1 = (VV_0)^2(x_1) = \lambda VV_0(x_1) = \lambda x_2 \]

and thus

\[ y_1 = \lambda y_2. \]

A contradiction. ■
Proof of the Theorem

Theorem. (JPS)

Let \( \mathcal{A} \) be a Banach subalgebra of \( L(X) \) and \( \mathcal{B} \) a Banach subalgebra of \( L(Y) \). Assume that \( \mathcal{A} \supseteq F(X) \) (the finite rank operators) and that \( \mathcal{B} \supseteq F(Y) \). Assume that

\[
\Phi : \mathcal{A} \rightarrow \mathcal{B}
\]

is a Banach algebra isomorphism. Then, there is an isomorphism \( U : X \rightarrow Y \) such that for all \( A \in \mathcal{A} \)

\[
\Phi(A) = UAU^{-1}.
\]

Moreover \( \|U\| \leq \|\Phi\| \) and \( \|U^{-1}\| \leq \|\Phi^{-1}\| \).
Proof of the Theorem

Let $V_0$ be a rank one projection:

$$V_0(x) = f_0(x)x_0$$

where $x_0 \in X$, $f_0 \in X^*$, $\|x_0\| = \|f_0\| = 1$, $f_0(x_0) = 1$.

By the Lemma for all $V \in A$ there is a $\lambda \in \mathbb{C}$ such that

$$(VV_0)^2 = \lambda VV_0.$$ 

Applying $\Phi$ we get that for all $W \in B$ there is a $\lambda \in \mathbb{C}$ such that

$$(W\Phi(V_0))^2 = \lambda W\Phi(V_0).$$

In particular this holds for all $W \in L(Y)$ of rank 2. So, By the Lemma $W_0 = \Phi(V_0)$ is of rank one. Say,

$$W_0(y) = g_0(y)y_0,$$

for some $y_0 \in Y$, $g_0 \in Y^*$, $\|y_0\| = 1$, $\|g_0\| \leq \|\Phi\|$. 
Proof of the Theorem

\[ V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0. \]

We now define the isomorphism \( U : X \rightarrow Y. \)
Given a \( z \in X \) choose an arbitrary \( V \in \mathcal{A} \) such that \( V(x_0) = z \) and define

\[ U(z) = \Phi(V)(y_0). \]

We need to show:

a. \( U \) is well defined (doesn’t depend on the choice of \( V \))
b. \( U \) is injective
c. \( U \) is surjective
d. \( U \) is linear
e. \( U \) is bounded, \( \| U \| \leq \| \Phi \| \)
f. \( U^{-1} \) is bounded, \( \| U^{-1} \| \leq \| \Phi^{-1} \| \)
g. For all \( A \in \mathcal{A} \),

\[ \Phi(A) = UAU^{-1}. \]
Proof of the Theorem, \( a. \) Well defined

\[
V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0.
\]

\[
U(z) = \Phi(V)(y_0) \text{ for } V \text{ such that } V(x_0) = z.
\]

Assume \( V_1(x_0) = V_2(x_0) = z, \quad V_1, V_2 \in \mathcal{A}. \) Then, for all \( x \in X, \)

\[
V_1(V_0(x)) = f_0(x)z = V_2(V_0(x)).
\]

So, \( V_1 V_0 = V_2 V_0 \)

and \( \Phi(V_1) W_0 = \Phi(V_2) W_0, \) i.e., for all \( y \in Y, \)

\[
g_0(y)\Phi(V_1)(y_0) = g_0(y)\Phi(V_2)(y_0).
\]

So,

\[
\Phi(V_1)(y_0) = \Phi(V_2)(y_0).
\]
Proof of the Theorem, b. Injectivity

\[ V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0. \]
\[ U(z) = \Phi(V)(y_0) \] for \( V \) such that \( V(x_0) = z \).

Let \( z_1 \neq z_2 \). Let \( V_1, V_2 \in \mathcal{A} \) such that
\[ V_1(x_0) = z_1, \quad V_2(x_0) = z_2. \]

Then \( V_1 V_0 \neq V_2 V_0 \), so,
\[ \Phi(V_1)W_0 \neq \Phi(V_2)W_0 \]

Implying
\[ \Phi(V_1)(y_0) \neq \Phi(V_2)(y_0) \]
i.e.,
\[ U(z_1) \neq U(z_2). \]
Proof of the Theorem c. Surjectivity

\[ V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0. \]

\[ U(z) = \Phi(V)(y_0) \] for \( V \) such that \( V(x_0) = z \).

Let \( y \in Y \) and let \( W \) be a rank one operator in \( B \) such that \( W(y_0) = y \). Let \( V \in \mathcal{A} \) be such that \( \Phi(V) = W \) and let \( z = V(x_0) \). Then,

\[ U(z) = \Phi(V)(y_0) = W(y_0) = y. \]
Proof of the Theorem d. Linearity

\[ V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0. \]

\[ U(z) = \Phi(V)(y_0) \quad \text{for} \quad V \quad \text{such that} \quad V(x_0) = z. \]

Let \( z_1, z_2 \in X, \quad \lambda \in \mathbb{C}. \) Let \( V_1(x_0) = z_1, \quad V_2(x_0) = z_2. \) Then,

\[ (V_1 + V_2)(x_0) = z_1 + z_2, \quad \text{so}, \]

\[ U(z_1 + z_2) = \Phi(V_1 + V_2)(y_0) = \Phi(V_1)(y_0) + \Phi(V_2)(y_0) = U(z_1) + U(z_2). \]

Also, Since \( \lambda V_1(x_0) = \lambda z_1, \)

\[ U(\lambda z_1) = \Phi(\lambda V_1)(y_0) = \lambda \Phi(V_1)(y_0) = \lambda U(z_1). \]
Proof of the Theorem e. boundedness

\[ V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0. \]

\[ U(z) = \Phi(V)(y_0) \text{ for } V \text{ such that } V(x_0) = z. \]

Give \( z \in X \) let \( V \in \mathcal{A} \) with \( V(x_0) = z, \| V \| = \| z \|. \) Then,

\[ \| U(z) \| = \| \Phi(V)(y_0) \| \leq \| \Phi \| \| V \| = \| \Phi \| \| z \|. \]
Proof of the Theorem f. boundedness of $U^{-1}$

$V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0.$

$U(z) = \Phi(V)(y_0)$ for $V$ such that $V(x_0) = z.$

Following c. Let $y \in Y$ and let $W$ be a rank one operator in $B$ such that $W(y_0) = y,$ $\|W\| = \|y\|.$ Let $V \in A$ be such that $\Phi(V) = W$ and let $z = V(x_0).$ Then,

$U(z) = \Phi(V)(y_0) = W(y_0) = y.$

$\|U^{-1}(y)\| = \|z\| = \|\Phi^{-1}(W)(x_0)\| \leq \|\Phi^{-1}\|\|W\| = \|\Phi^{-1}\||y||.$
Proof of the Theorem g. $\Phi(A) = UAU^{-1}$

\[ V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0. \]

\[ U(z) = \Phi(V)(y_0) \text{ for } V \text{ such that } V(x_0) = z. \]

Let $z \in X$ and $A \in \mathcal{A}$. Let $V \in \mathcal{A}$ such that $V(x_0) = z$. Then, $AV(x_0) = A(z)$ so,

\[ UA(z) = U(A(z)) = \Phi(AV)(y_0) = \Phi(A)\Phi(V)(y_0) = \Phi(A)U(z). \]

It follows that

\[ UA = \Phi(A)U, \]

or,

\[ \Phi(A) = UAU^{-1}. \]
Another theorem

**Theorem. (JPS)**

Let $X$ and $Y$ be Banach spaces and let $\mathcal{A}$ be a Banach subalgebra of $L(X)$ containing $F(X)$. Let

$$\Phi : \mathcal{A} \to L(Y)$$

be injective bounded homomorphism.

Then, there is an isomorphism $U : X \to Y_0 \subseteq Y$ such that for all $A \in \mathcal{A}$

$$UA = \Phi(A)U.$$

Moreover, for all $x \in X$,

$$\|\Phi\|^{-1}\|x\| \leq \|Ux\| \leq \|\Phi\|\|x\|.$$ 

Also, if $X$ is complemented in $X^{**}$, $Y_0$ is complemented in $Y$.
**Proposition.**

Let $X$ and $Y$ be Banach spaces and let $\Phi : \overline{F(X)} \to L(Y)$ be injective bounded homomorphism. Then there are $U \in L(X, Y)$ and $W \in L(Y, X^{**})$ such that $WV$ is the natural injection of $X$ into $X^{**}$ and $\|U\|, \|W\| \leq \|\Phi\|$.

**Proof:** Let $V_0 = f_0 \otimes x_0$ be a norm one projection in $L(X)$ ($\|f_0\| = \|x_0\| = f_0(x_0) = 1$). For $z \in X$ and $h \in X^*$ define $A_z$ and $B_h$ by

$$A_z(x) = f_0(x)z, \quad B_h(x) = h(x)x_0.$$  

Note that $A_z$ and $B_h$ are rank one operators, that $\|A_z\| = \|z\|$, $\|B_h\| = \|h\|$, and that $z \to A_z$ and $h \to B_h$ are continuous linear operations. Also, $A_{x_0} = B_{f_0} = V_0$. 

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**Eidelheit's Theorem Revisited**
For all $z \in X$, $h \in X^*$ and all $x \in X$,

$$B_h A_z(x) = f_0(x)B_h(z) = f_0(x)h(z)x_0 = h(z)A_{x_0}(x),$$

i.e., $B_h A_z = h(z)A_{x_0}$.

Fix $y_0 \in \text{Ran}(\Phi(V_0))$ and $g_0 \in Y^*$ with $\|g_0\| = \|y_0\| = g_0(y_0) = 1$

and Define $U : X \to Y$ by

$$U(z) = \Phi(A_z)(y_0), \quad \text{for} \quad z \in X.$$

Define also $W : Y \to X^{**}$ by

$$W(y)(h) = g_0(\Phi(B_h)(y)), \quad \text{for} \quad h \in X^*.$$

Note, $\|U\|, \|W\| \leq \|\Phi\|$. 
Then, for all $z \in X$, $h \in X^*$,

$$(WU(z))(h) = g_0(\Phi(B_h)(U(z))) = g_0(\Phi(B_h)\Phi(A_z)(y_0))$$

$$= g_0(\Phi(B_hA_z)(y_0)) = g_0(h(z)\Phi(A_{x_0})(y_0))$$

$$= h(z)g_0(\Phi(V_0)(y_0)) = h(z),$$

i.e.,

$$WU(z) = i(z)$$

where $i : X \to X^{**}$ is the natural injection.  ■
Theorem. (JPS)

Let $X$ and $Y$ be Banach spaces and let $\mathcal{A}$ be a Banach subalgebra of $L(X)$ containing $F(X)$. Let

$$\Phi : \mathcal{A} \to L(Y)$$

be injective bounded homomorphism. Then, there is an isomorphism $U : X \to Y_0 \subseteq Y$ such that for all $A \in \mathcal{A}$

$$UA = \Phi(A)U.$$

Moreover, for all $x \in X$,

$$\|\Phi\|^{-1}\|x\| \leq \|Ux\| \leq \|\Phi\|\|x\|.$$

Also, if $X$ is complemented in $X^{**}$, $Y_0$ is complemented in $Y$. 

Proof of "Another theorem"
Proof of "Another theorem"

Proof: Define $U$ and $W$ as in the proof of the Proposition. If $C \in \mathcal{A}$ and $z \in X$ then

$$UC(z) = \Phi(A_{Cz})(y_0) = \Phi(CA_z)(y_0)$$
$$= \Phi(C)\Phi(A_z)(y_0) = \Phi(C)U(z).$$

So,

$$UC = \Phi(C)U.$$  

We already showed that $\|Ux\| \leq \|\Phi\| \|x\|$. To prove the lower bound, for all $z \in X$,

$$\|\Phi\| \|Uz\| \geq \|W\| \|Uz\| \geq \|WUz\| = \|z\|.$$  

Finally, if $P$ is a projection from $X^{**}$ onto $X$ (identified with $iX$), then $UPW$ is a projection from $Y$ onto $UX$.  ■
Remarks

- **Open:** How many closed ideals are there in $L(L_p)$, $p \neq 2$, up to Banach space isomorphism?

- Automatic continuity: The main theorem and its corollaries can be strengthened. The continuity of $\Phi$ and $\Phi^{-1}$ is automatic and need not be assumed.

**Theorem. (JPS)**

Let $\mathcal{A}$ be a Banach subalgebra of $L(X)$ and $\mathcal{B}$ a Banach subalgebra of $L(Y)$. Assume that $\mathcal{A} \supseteq F(X)$ (the finite rank operators) and that $\mathcal{B} \supseteq F(Y)$. Assume that

$$\Phi : \mathcal{A} \to \mathcal{B}$$

is injective and surjective homomorphism. Then, there is an isomorphism $U : X \to Y$ such that for all $A \in \mathcal{A}$

$$\Phi(A) = UAU^{-1}.$$
Proof: Assume first that $I_X \in \mathcal{A}$. Then, since $\Phi(I_X)$ commutes with all finite rank operators, $\Phi(I_X) = I_Y$. Assume that there are $A_n \in \mathcal{A}$, with $\|A_n\| = 1$ and $\|\Phi(A_n)\| \to \infty$. By Banach–Steinhaus, there is a $y_0 \in Y$ such that $\|\Phi(A_n)(y_0)\| \to \infty$ and an $g_0 \in Y^*$ such that $g_0(\Phi(A_n)(y_0)) \to \infty$.

Let $A \in \mathcal{A}$ be such that $\Phi(A) = g_0 \otimes y_0$. Since $\{AA_n\}$ is a bounded sequence there is a $\lambda_0 > 0$ such that, if $|\lambda| \leq \lambda_0$, $I_X - \lambda AA_n$ are all invertible. It follows that $I_Y - \lambda \Phi(A)\Phi(A_n)$ are all invertible for $|\lambda| \leq \lambda_0$. But, $\lambda_n = (g_0(\Phi(A_n)(y_0)))^{-1} \to 0$ and

$$(I_Y - \lambda_n \Phi(A)\Phi(A_n))(y_0) = y_0 - \lambda_n g_0(\Phi(A_n)(y_0))y_0 = 0.$$ 

A contradiction. So $\Phi$ is bounded. Similarly, $\Phi^{-1}$ is bounded. If $I_X \notin \mathcal{A}$ (and then necessarily $I_Y \notin \mathcal{B}$) extend $\Phi$ naturally to the algebra generated by $\mathcal{A}$ and $I_X$. ■
The End