Matrix Subspaces of L_1 *

Gideon Schechtman†

Abstract

If $E = \{e_i\}$ and $F = \{f_i\}$ are two 1-unconditional basic sequences in L_1 with E r-concave and F p-convex, for some $1 \le r < p \le 2$, then the space of matrices $\{a_{i,j}\}$ with norm $\|\{a_{i,j}\}\|_{E(F)} = \big\|\sum_k \|\sum_l a_{k,l} f_l\| e_k\big\|$ embeds into L_1. This generalizes a recent result of Prochno and Schütt.

1 Introduction

Recall that a basis $E = \{e_i\}_{i=1}^N$ of a finite ($N < \infty$) or infinite ($N = \infty$) dimensional real or complex Banach space is said to be K-unconditional if $\|\sum_i a_i e_i\| \le K \|\sum_i b_i e_i\|$ whenever $|a_i| = |b_i|$ for all i. Given a finite or infinite 1-unconditional basis, $E = \{e_i\}_{i=1}^N$, and a sequence of Banach spaces $\{X_i\}_{i=1}^N$ denote by $(\sum \bigoplus X_i)_E$ the space of sequences $x = (x_1, x_2, \dots)$, $x_i \in X_i$, for which the norm $\|x\| = \big\|\sum_i \|x_i\| e_i\big\|$ is finite.

If X has a 1-unconditional basis $F = \{f_j\}$ then $(\sum \bigoplus X)_E$ can be represented as a space of matrices $A = \{a_{i,j}\}$, denoted $E(F)$, with norm

$$\|A\|_{E(F)} = \Big\|\sum_i \|\sum_j a_{i,j} f_j\| e_i\Big\|.$$

In [PS], Prochno and Schütt gave a sufficient condition for bases E and F of two Orlicz sequence spaces which assure that $E(F)$ embeds into L_1. Here we generalize this result by giving a sufficient condition on two unconditional

*AMS subject classification: 46E30, 46B45, 46B15. Key words: subspaces of L_1, unconditional basis, r-concavity, p-convexity

†Supported in part by the Israel Science Foundation.

bases E, F, which assure that $E(F)$ embeds into L_1. As we shall see this condition is also "almost" necessary.

Recall that an unconditional basis $\{e_i\}$ is said to be p-convex (resp. r-concave) with constant K provided that for all n and all $x_1, x_2, \ldots, x_n$ in the span of $\{e_i\}$,

$$\| \sum_{i=1}^{n} (|x_i|^p)^{1/p} \| \leq K (\sum_{i=1}^{n} \|x_i\|^p)^{1/p}$$

(resp.

$$(\sum_{i=1}^{n} \|x_i\|^r)^{1/r} \leq K \| \sum_{i=1}^{n} (|x_i|^r)^{1/r} \| \ ).$$

Here, for $x = \sum x(j)e_j$ and a positive α, $|x|^\alpha = \sum |x(j)|^\alpha e_j$.

L_p will denote here $L_p([0,1], \lambda)$, λ being the Lebesgue measure. As is known and quite easy to prove, any 1-unconditional basic sequence in L_p, $1 \leq p \leq 2$ (resp. $2 \leq p < \infty$), is p-convex (resp. p-concave) with constant depending only on p. It is also worthwhile to remind the reader that any K-unconditional basic sequence in L_p is equivalent, with a constant depending only on p and K to a 1-unconditional basic sequence in L_p. It is due to Maurey [Ma] (see also [Wo, III.H.10]), that for every $1 \leq r < p \leq 2$, the span of every p-convex 1-unconditional basic sequence in L_1 embeds into L_p and also embeds into L_r after change of density; i.e., there exists a probability measure μ on $[0,1]$ so that this span is isomorphic (with constants depending on r, p and the p-convexity constant only) to a subspace of $L_r([0,1], \mu)$ on which the $L_r(\mu)$ and the $L_1(\mu)$ norms are equivalent.

If M is an Orlicz function then the Orlicz space ℓ_M embeds into L_p if and only if $M(t)/t^p$ is equivalent to an increasing function and $M(t)/t^2$ is equivalent to a decreasing function. This happens if and only if the natural basis of ℓ_M is p-convex and 2-concave.

Theorem 1 below states in particular that if E and F are two 1-unconditional basic sequences in L_1 with E r-concave and F p-convex for some $1 \leq r < p \leq 2$ then $E(F)$ embeds into L_1. When specializing to Orlicz spaces, this implies the main result of [PS].

2 The main result

Theorem 1 *Let $E = \{e_i\}$ be a 1-unconditional basic sequence in L_1 with $\{e_i\}$ r-concave with constant K_1 and let X be a subspace of $L_1([0,1], \mu)$ for*

some probability measure μ *satisfying* $\|x\|_{L_r([0,1],\mu)} \le K_2\|x\|_{L_1([0,1],\mu)}$ *for some constant* K_2 *and all* $x \in X$. *Then* $(\sum \bigoplus X)_E$ *embeds into* L_1 *with a constant depending on* K_1, K_2 *and* r *only.*
Consequently, if $E = \{e_i\}$ *and* $F = \{f_i\}$ *are two 1-unconditional basic sequences in* L_1 *with* E r*-concave with constant* K_1 *and* F p*-convex with constant* K_2*, for some* $1 \le r < p \le 2$*, then the space of matrices* $A = \{a_{k,l}\}$ *with norm*

$$\|A\|_{E(F)} = \|\sum_k \|\sum_l a_{k,l} f_l\| e_k\|$$

embeds into L_1 *with a constant depending only on* r, p, K_1 *and* K_2.

Proof: The p-convexity of $\{f_i\}$ implies that after a change of density the L_1 and L_r norms are equivalent on the span of $\{f_i\}$. See [Ma]. That is, there is a probability measure μ on $[0,1]$ and a constant K_3, depending only on r, p and K_2 such that $\|\sum a_j \tilde{f}_j\|_{L_r([0,1],\mu)} \le K_3 \|\sum a_j \tilde{f}_j\|_{L_1([0,1],\mu)}$ for some sequence $\{\tilde{f}_j\}$ 1-equivalent, in the relevant L_1 norm, to $\{f_j\}$, and for all coefficients $\{a_i\}$. It thus follows that the second part of the theorem follows from the first part.

To prove the first part, in $L_1([0,1] \times [0,1], \lambda \times \mu)$ consider the tensor product of the span of $\{e_i\}$ and X, that is the space of all functions of the form $\sum_i e_i \otimes x_i$, $x_i \in X$ for all i, where $e_i \otimes x_i(s,t) = e_i(s)x_i(t)$. Then, by the 1-unconditionality of $\{e_i\}$ and the triangle inequality,

$$\begin{aligned}
\|\sum_i e_i \otimes x_i\|_1 &= \int \|\sum_i |x_i(t)| e_i\|_{L_1([0,1],\lambda)} d\mu(t) \\
&\ge \|\sum_i (\int |x_i(t)| d\mu(t)) e_i\|_{L_1([0,1],\lambda)} \\
&= \|\sum_i \|x_i\| e_i\|.
\end{aligned}$$

On the other hand, by the 1-unconditionality and the r-concavity with constant K_1 of $\{e_i\}$ (used in integral instead of summation form),

$$\begin{aligned}
\|\sum_i e_i \otimes x_i\|_1 &= \int\int \big|\sum_i |x_i(t)| e_i(s)\big| d\lambda(s) d\mu(t) \\
&\leq (\int(\int \big|\sum_i |x_i(t)| e_i(s)\big| d\lambda(s))^r d\mu(t))^{1/r} \\
&= (\int \|\sum_i |x_i(t)| e_i\|^r_{L_1([0,1],\lambda)} d\mu(t))^{1/r} \\
&\leq K_1 \|\sum_i (\int |x_i(t)|^r d\mu(t))^{1/r} e_i\|_{L_1([0,1],\lambda)} \\
&\leq K_1 K_2 \|\sum_i \int |x_i(t)| d\mu(t) e_i\|_{L_1([0,1],\lambda)} \\
&= K_1 K_2 \big\|\sum_i \|x_i\| e_i\big\|
\end{aligned}$$

■

As is explained in the introduction the main result of [PS] follows as corollary.

Corollary 1 *If M and N are Orlicz functions such that $M(t)/t^r$ is equivalent to a decreasing function, $N(t)/t^p$ is equivalent to an increasing function and $N(t)/t^2$ is equivalent to a decreasing function then $\ell_M(\ell_N)$ embeds into L_1.*

Remark 1 *The role of L_1 in Theorem 1 can easily be replaced with L_s for any $1 \leq s \leq r$.*

Remark 2 *If the bases E and F are infinite, say, and the smallest r such that E is r-concave is larger than the largest p such that F is p-convex, then $E(F)$ doesn't embed into L_1. This follows from the fact that in this case it is known that ℓ_r^n uniformly embed as blocks of E and ℓ_p^n uniformly embed as blocks of F, for some $r > p$, while it is known that in this case $\ell_r^n(\ell_p^n)$ do not uniformly embed into L_1.*

This still leaves the case $r = p$, which is not covered in Theorem 1, open:

- *If E and F are two 1-unconditional basic sequences in L_1 with E r-concave and F r-convex, does $E(F)$ embed into L_1?*

In the case that E is an Orlicz space the problem above has a positive solution. We only sketch it. By the factorization theorem of Maurey mentioned above ([Wo, III.H.10] is a good place to read it), and a simple compactness argument (to pass from the finite to the infinite case), it is enough to consider the case that F is the ℓ_r unit vector basis. If the basis of ℓ_M is r-concave, then the $2/r$-convexification of ℓ_M (which is the space with norm $\|\{|a_i|^{2/r}\}\|_{\ell_M}^{r/2}$) embeds into $L_{2/r}$. This is again an Orlicz space, say, $\ell_{\tilde{M}}$. Now, tensoring with the Rademacher sequence (or a standard Gaussian sequence) we get that $\ell_{\tilde{M}}(\ell_2)$ embeds into $L_{2/r}$. We now want to $2/r$ concavify back, staying in L_1, so as to get that $\ell_M(\ell_r)$ embeds into L_1. This is known to be possible (and is buried somewhere in [MS]): If $\{x_i\}$ is a 1-unconditional basic sequence in L_s, $1 < s \leq 2$ then its s-concavification (which is the space with norm $\|\{|a_i|^{1/s}\}\|_{\ell_M}^{s}$) embeds into L_1. Indeed, Let $\{f_i\}$ be a sequence of independent $2/s$ symmetric stable random variables normalized in L_1 and consider the span of the sequence $\{f_i \otimes |x_i|^s\}$ in L_1.

References

[Ma] B. Maurey, Théorémes de factorisation pour les opérateurs linéaires à valeurs dans les espaces Lp. (French) With an English summary. Astérisque, No. 11. Société Mathématique de France, Paris, ii+163 pp (1974).

[MS] B. Maurey, G. Schechtman, Some remarks on symmetric basic sequences in L_1. Compositio Math. 38, no. 1, 67–76 (1979).

[PS] J. Prochno, C. Schütt, Combinatorial Inequalities and Subspaces of L_1, Studia Math. 211, 21-39 (2012).

[Wo] P. Wojtaszczyk, Banach spaces for analysts. Cambridge Studies in Advanced Mathematics, 25. Cambridge University Press, Cambridge, xiv+382 pp. (1991).

G. Schechtman
Department of Mathematics
Weizmann Institute of Science
Rehovot, Israel
gideon@weizmann.ac.il