GLOBAL VS. LOCAL ASYMPTOTIC THEORIES OF FINITE DIMENSIONAL NORMED SPACES

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1. Introduction

The paper is devoted to the comparison between global properties and local properties of symmetric convex bodies of high dimension. By global properties we refer to properties of the original body in question and its images under linear transformations while the local properties pertain to the structure of lower dimensional sections and projections of the body, i.e., to the linear structure of a normed space in the spirit of functional analysis. In both theories we are interested in the asymptotic behavior, as the dimension grows to infinity, of the relevant quantities. Also, as is common in such questions, the approach we consider does not yield exact isometric results but rather falls into the isomorphic (and for some results the almost isometric) category.

Unexpectedly, it appears that there is an exact parallelism between the two theories; the global (geometric) asymptotic theory and the Local Theory. We will demonstrate that several well known facts of Local type have corresponding equivalent geometric results and vice versa. Let us describe, as an example, a well known classical Local Theory fact – Dvoretzky's theorem.

A well known version of Dvoretzky's theorem known today states: *Let $X = (\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional normed space. Let $|\cdot|$ denote the canonical Euclidean norm on $\mathbb{R}^n$. Put $M = \int_{S^{n-1}} \|x\| d\mu(x)$ (where μ is the normalized Haar measure on the Euclidean sphere); and assume $\|x\| \leq b|x|$ for all $x \in \mathbb{R}^n$. Let $\varepsilon > 0$, then for some absolute*

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constant c and for $k = \left[c\varepsilon^2 n\left(\frac{M}{b}\right)^2\right]$, there exists a subspace $E \in G_{n,k}$ (the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$) for which

$$\frac{M}{1+\varepsilon}\|x\| \leq |x| \leq (1+\varepsilon)M\|x\| \quad \text{for all} \quad x \in E \ . \tag{1}$$

Moreover, the subset of $G_{n,k}$ satisfying this has measure tending fast to 1 as $n \to \infty$. (Here, the measure is the normalized Haar measure on $G_{n,k}$.) (cf. [Mi1], [FLM], [MS], [Pi], [G], [Sc].)

To this statement, and others similar to it pertaining to the behavior of a section (subspace) of convex bodies (normed spaces), we shall refer to as "local statements". In particular, the above is a local form of Dvoretzky's theorem.

In [BLM] (see also [Schm]) it was proved (although not stated – we shall return to this in the proof of Theorem 2.2 below) that, under the same conditions, for some integer $t \leq \frac{C}{\varepsilon^2}\left(\frac{b}{M}\right)^2$, with C an absolute constant, there are t orthogonal transformations u_i, $i = 1, \ldots, t$, such that

$$\frac{M}{1+\varepsilon}|x| \leq \frac{1}{t}\sum_{i=1}^{t}\|u_i x\| \leq (1+\varepsilon)M|x| \quad \text{for all } x \in \mathbb{R} \ . \tag{2}$$

Moreover, with high probability a random choice of the u_is will do. (This time the probability measure is the normalized Haar measure on the orthogonal group $O(n)$.

An equivalent statement to (2) is that there are t orthogonal transformations such that

$$\frac{M}{1+\varepsilon}D \subseteq \frac{1}{t}\sum_{i=1}^{t} u_i(B_{\|\cdot\|^*}) \subseteq (1+\varepsilon)MD \ . \tag{3}$$

Here $B_{\|\cdot\|}$ or B_X denote the unit ball of X $B_{\|\cdot\|^*} = B_{X^*}$ denotes the unit ball of X^* and $D = B_{\ell_2^n}$. To statements of this nature, pertaining to the structure of the whole convex body or all space (here $B_{\|\cdot\|^*}$ or X^*), we shall refer to as "global statements". In particular, the above is a global statement of Dvoretzky's theorem.

The known proofs of the statements above use similar tools but, except for that, do not relate the local statements directly to the global ones or vice versa. The purpose of this paper is to remedy this in the particular situation above as well as in several other instances.

To describe the remedy for the particular case of Dvoretzky's theorem described in detail above, we introduce the following two definitions:

Definition. Let X be $\mathbb{R}^n$ with norm $\|\cdot\|$. Let $k = k(X) \le n$ be the largest integer such that

$$\mu_{G_{n,k}}\left(\left\{E\ ;\quad \frac{M}{2}|x| \le \|x\| \le 2M|x| \text{ for all } x \in E\right\}\right) > 1 - \frac{k}{n+k}\ .$$

Let t be the smallest integer such that there are orthogonal transformations $u_1, \dots, u_t \in O(n)$ with

$$\frac{M}{2}|x| \le \frac{1}{t}\sum_{i=1}^{t}\|u_i x\| \le 2M|x|\ , \qquad \text{for all } x \in \mathbb{R}^n\ .$$

The main result of section 2 below is Theorem 2.2 which is a more precise version of:

Theorem 1.1. *For some universal constant C*

$$C^{-1}n \le kt \le Cn\ .$$

In section 3 we deal with symmetric convex bodies which are in M-position (i.e., satisfy the inverse Brunn-Minkowski inequality, see exact definition in section 3). Recall that every symmetric convex body can be put in such a position by applying an invertible linear transformation. The main result in this section (Theorem 3.3) states that if such a body K has the property that for some t and some orthogonal transformations $u_1, \dots, u_t$

$$D \subseteq \frac{1}{t}\sum_{i=1}^{t} u_i(K) \subseteq 2D$$

then there is ONE orthogonal transformation u such that

$$D \subseteq K + u(K) \subseteq CD$$

where C depends on t only.

Section 4 deals with properties related to finite volume ratio. Corollary 4.5 there gives a global property of a space implied by the (local) random version of the subspace of quotient theorem of the first named author ([Mi2]).

2. The Equivalence of the Local and Global Forms of Dvoretzky's Theorem

The new ingredient in the proof is the following simple lemma.

Lemma 2.1. *Let $u_1, u_2, \dots, u_t$ be orthogonal transformations of $\mathbb{R}^n$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and let $|\cdot|$ denote the canonical Euclidean norm on $\mathbb{R}^n$. Put $|||x||| = \frac{1}{t}\sum_{i=1}^t \|u_i x\|$ and assume $|||x||| \le C|x|$ for all $x \in \mathbb{R}^n$ and some $0 < C < \infty$. Then*

$$\|x\| \le C\sqrt{t}|x| \ , \qquad \text{for all } x \in \mathbb{R}^n \ .$$

Proof: Let K denote the unit ball of $\|\cdot\|$. Let $b = \max\{\|x\|; x \in S^{n-1}\}$ and let $x_1 \in S^{n-1}$ be such that $\|x_1\| = b$. Since $\frac{x_1}{b} \in \frac{1}{b}S^{n-1} \cap \partial K$ it follows that the functional $\frac{x_1}{b}$ supports K, i.e. $\langle x, x_1\rangle \le 1/b$, for all $x \in K$.

Put $x_i = u_i^{-1}x_1$, $i = 1, 2, \dots, t$, and, for $\lambda_i > 0$, $i = 1, 2, \dots, t$, consider the $2t$ caps

$$A_i^\varepsilon = \left\{ y \in S^{n-1} \ ; \ \varepsilon\,\langle y, x_i\rangle \ge \frac{\lambda_i}{b} \right\} \ , \qquad i = 1, \dots, t \ , \quad \varepsilon = \pm \ .$$

Note that if the t sets $\{A_i^+ \cup A_i^-\}_{i=1}^t$ have a point of intersection then $t^{-1}\sum \lambda_i \le C$. Indeed, if $|\langle u_i y, x_1\rangle| = |\langle y, x_i\rangle| \ge \lambda_i/b$ for some $y \in S^{n-1}$ and all $i = 1, \dots, t$, then $\|u_i y\| \ge \lambda_i$, $i = 1, \dots, t$ and thus $|||y||| \ge t^{-1}\sum \lambda_i$.

Choose $\varepsilon_i = \pm$ such that $\left|\sum \varepsilon_i x_i\right|$ is maximal with respect to all possible choices of signs. It is easily proved then that, for all $i = 1, 2, \dots, t$, $\sum_{j\ne i}^t \langle \varepsilon_i x_i, \varepsilon_j x_j\rangle \ge 0$. Thus, putting $y = \sum \varepsilon_i x_i / \left|\sum \varepsilon_i x_i\right|$ and $\lambda_i = b|\langle y, x_i\rangle|$, we get that the caps $\{A_i^{\varepsilon_i}\}_{i=1}^t$ have y in their intersection. Consequently, $C \ge \frac{b}{t}\sum |\langle y, x_i\rangle| \ge \frac{b}{t}\langle y, \sum \varepsilon_i x_i\rangle = \frac{b}{t}\left|\sum \varepsilon_i x_i\right| \ge \frac{b}{\sqrt{t}}$. □

Remarks. a. The last part of the proof of the lemma above is similar to the proof of a special case of a lemma of Bang [Ba] (see also [B]).

b. Note that $M = \int_{S^{n-1}} \|x\| = \int_{S^{n-1}} |||x|||$ and also that in Lemma 2.1 necessarily $C \ge M$. Up to a constant factor the converse of Lemma 2.1 also holds:

There exists a universal constant K such that if $\|\cdot\|$ is a norm on $\mathbb{R}^n$ satisfying $\|x\| \le C|x|$ for all $x \in \mathbb{R}^n$, then, given a positive integer t, there exist t orthogonal transformations $u_1, u_2, \dots, u_t$ such that putting $|||x||| = \frac{1}{t}\sum_{i=1}^t \|u_i x\|$,

$$|||x||| \le K \max\left\{\frac{C}{\sqrt{t}}, M\right\} |x| \ , \qquad \text{for all} \quad x \in \mathbb{R}^n \ .$$

The proof of this assertion follows the proof of Theorem 2 in [BLM] (or see Proposition 1 in [Schm]).

c. Note that Lemma 2.1 is informative only if $C\sqrt{t}$ is not too large. It is always true that, for some absolute constant A, $\|x\| \le AM\sqrt{n}|x|$.

d. One can generalize Lemma 2.1 so as to replace the norms $\|x\|_i = \|u_i x\|$ with general norms. The proof is similar:

Let $\|\cdot\|_i$, $i = 1, \ldots, t$, *be* t *norms on* $\mathbb{R}^n$. *Put* $b_i = \max_{x \in S^{n-1}} \|x\|_i$, *and* $|||x||| = \frac{1}{t}\sum_{i=1}^t \|x\|_i$. *Assume* $|||x||| \le C|x|$ *for all* $x \in \mathbb{R}^n$. *Then* $C \ge \frac{1}{t}\big(\sum_{i=1}^t b_i^2\big)^{1/2}$.

We are now ready to state the main theorem of this section which is a more precise version of Theorem 1.1.

Theorem 2.2. a. *If, for some orthogonal transformations* $u_1, u_2, \ldots, u_t$ *and all* $x \in \mathbb{R}^n$, $|x| \le t^{-1}\sum_{u=1}^t \|u_i x\| \le C|x|$. *Then* $(\mathbb{R}^n, \|\cdot\|)$ *contains, for each* $\varepsilon > 0$, *a subspace of dimension* $k = \big[\frac{\eta\varepsilon^2}{C^2}\frac{n}{t}\big]$ *on which the norm is* $1+\varepsilon$ *equivalent to a multiple of the Euclidean norm* $|\cdot|$. $\eta > 0$ *is a universal constant.*

Moreover, the collection of all subspaces of dimension k *having this property has probability* $\ge 1 - \exp(-c(\varepsilon)n/C^2 t)$. *Here* $c(\varepsilon) > 0$ *depends only on* ε *and the probability measure is the normalized Haar measure on the relevant Grassmanian.*

b. *Conversely, there exists an absolute constant* $c > 0$ *such that if, for some* $1 \le t \le n$ *and some* $\varepsilon > 0$, *the collection of all* $\big[\frac{n}{c^2\varepsilon^2 t} + 1\big]$*-dimensional subspaces* V *of* $(\mathbb{R}^n, \|\cdot\|)$, *satisfying* $|x| \le \|x\| \le 2|x|$ *for all* $x \in V$, *has probability larger than* $1 - 1/(2c^2\varepsilon^2 t)$, *then there exist orthogonal transformations* $u_1, \ldots, u_t$, *such that the norm* $|||x||| = \frac{1}{t}\sum_{i=1}^t \|u_i x\|$ *is* $1+\varepsilon$ *equivalent to a multiple of the Euclidean norm.*

Proof: Except for the exact dependence on ε, a. is a straightforward application of Lemma 2.1 and [Mi1] (or see [MS] Theorem 4.2 and Remark b. following it). To get the dependence on ε to be of order ε^2, one needs to use [Go] (or [Sc]) instead of [Mi1].

To prove b., note first that if, for some absolute constant K and some $1 \le t \le n$, the collection of all $\big[\frac{n}{t} + 1\big]$-dimensional subspaces V of $(\mathbb{R}^n, \|\cdot\|)$, satisfying $\|x\| \le 2|x|$ for all $x \in V$, has probability larger than $1 - 1/2t$ then $\|x\| \le 2\sqrt{t}|x|$ for all $x \in \mathbb{R}^n$.

Indeed the assumption easily implies the existence of t orthogonal subspaces, $V_1, \dots, V_t$, each of dimension approximately n/t, for which the inequality $\|x\| \le 2|x|$ is satisfied for each $x \in \bigcup_{i=1}^t V_i$. For a general x apply the triangle inequality.

It follows that $\|x\| \le 2c\varepsilon\sqrt{t}|x|$ for all $x \in \mathbb{R}^n$. Clearly also $M = \int \|x\| d\mu \ge 1$. It follows now from the method of [BLM] (or see [Schm] Proposition 1 for a more precise statement) that there exist orthogonal $u_1, \dots, u_t$ satisfying

$$(1+\varepsilon)^{-1} M|x| \le \frac{1}{t}\sum_{i=1}^t \|u_i x\| \le (1+\varepsilon)M|x| \quad \text{for all } x \in \mathbb{R}^n \ . \qquad \square$$

We conclude this section with an observation which gives a formula for (the order of magnitude of) the norm of an operator from an Euclidean space to an n-dimensional normed space in terms of parameters of probabilistic nature related to the operator. The proof is basically a repetition of the proof of Theorem 2.2.

Observation. *There exist universal constants c and C such that if $T : \ell_2^n \to X = (\mathbb{R}^n, \|\cdot\|)$, $M(T) = \int_{S^{n-1}} \|Tx\|$ and $k = k(T)$ is the largest integer such that*

$$\mu_{G_{n,k}} \left(\{ E \ ; \quad \|Tx\| \le 2M(T)|x| \ \ \textit{for all } x \in E \} \right) > 1 - \frac{ck}{n} \ .$$

Then,

$$C^{-1} M(T)\sqrt{n/k(T)} \le \|T\| \le C M(T) \sqrt{n/k(T)}.$$

Note that the definition of $k(T)$ here is different from the definition of $k = k(X)$ given in the introduction even for the formal identity operator because in $k(T)$ we consider only upper bounds. However, using the by now standard techniques from the Local Theory it is easy to see that, if we change the definition as to require that $\|Tx\|$ is bounded from below by $(1/2M(T))/|x|$, say, the new $k(T)$ will be universally equivalent to the old one.

This observation may have some significance in computational geometry. In its dual form, it reduces the problem of estimating the diameter of a convex (symmetric) body to that of deciding whether the diameter is within a factor of 2 of the mean width of the body.

3. Behavior of a Convex Body in M-position

The results we discussed in section 2 connect an arbitrary norm (convex body) with a fixed Euclidean structure (ellipsoid). We did not choose a special Euclidean structure connected with the norm. Note that the Euclidean structure was involved in these results in a "global" form: through the orthogonal groups induced by the Euclidean norm and through the Haar measures on the Grassmannians.

However, there is a Euclidean structure which induces the Haar probability measure on Grassmannians which behaves as a "natural" measure. As we described in the Introduction, we always prefer to choose a presentation of a norm $\|\cdot\|$ in $\mathbb{R}^n$ such that the standard Euclidean structure of $\mathbb{R}^n$ is this natural structure for the norm $\|\cdot\|$. To describe this Euclidean structure, we start with some geometric inequality.

Recall that the inverse Brunn-Minkowski inequality of [Mi2] states that, given k symmetric convex bodies $B_1, B_2, \dots, B_k$ in $\mathbb{R}^n$ one can find positions $\widetilde{B}_1, \widetilde{B}_2, \dots, \widetilde{B}_k$ of the bodies (i.e. $\widetilde{B}_i = T_i B_i$ for some volume preserving operators T_i) which satisfy the inequality

$$\big(\mathrm{vol}(t_1\widetilde{B}_1 + \cdots + t_k\widetilde{B}_k)\big)^{1/n} \le C_k\big[(\mathrm{vol}(t_1\widetilde{B}_1)^{1/n} + \cdots + (\mathrm{vol}(t_k\widetilde{B}_k))^{1/n}\big]$$

for all $t_1 > 0, \dots, t_k > 0$ and for some constant C_k depending merely on k. The position $\widetilde{B}_i$ depends only on the body B_i and not on B_j for $j \ne i$, if B_i is already in this special position, i.e. if $T_i = I$, we shall say that B_i is in M-position. The dependence of C_k on k is discussed in [Pi], p. 120: For every $\alpha > 1/2$ every symmetric convex body B has a position $\widetilde{B}$ (which we shall refer to as α-regular M-position) satisfying

$$\big(\mathrm{vol}(t_1\widetilde{B}_1 + \cdots + t_k\widetilde{B}_k)\big)^{1/n} \le \chi(\alpha)k^{\alpha}\big[(\mathrm{vol}(t_1\widetilde{B}_1)^{1/n} + \cdots + \mathrm{vol}(t_k\widetilde{B}_k))^{1/n}\big]$$

for all symmetric convex bodies $B_1, B_2, \dots, B_k$ in $\mathbb{R}^n$ where $\chi(\alpha)$ depends only on α. Following the proofs in [Pi] one gets $\chi(\alpha) \le \exp\big(A/\big(\alpha - \frac{1}{2}\big)^{1/2\alpha}\big)$ for some absolute constant A. If B is in this position so is its polar, any multiple of B and any image of B under an orthogonal transformation.

The main result of this section is

Theorem 3.1. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and assume its unit ball K is in M position. Assume further that for some t orthogonal transformations $u_1, \dots, u_t$ and for some*

$0 < r, C < \infty$,

$$r|x| \leq \frac{1}{t}\sum_{i=1}^{t} \|u_i x\| \leq Cr|x| \tag{3.1}$$

for all $x \in \mathbb{R}^n$. *Then there is a* C', *depending on* t *and* C *only and an orthogonal transformation* u *such that, for some* r',

$$r'|x| \leq \|x\| + \|ux\| \leq C'r'|x| \tag{3.2}$$

for all $x \in \mathbb{R}^n$.

Remarks.. a. In geometric language, the theorem states that if K is in the above position and, for some t, there are t orthogonal rotations $\{u_i\}_1^t$ such that $d(T, D) \leq 2$ for $T = \frac{1}{t}\sum_1^t u_i K$, then there is another rotation $v \in O(n)$ such that $d(K + vK, D) \leq C(t)$. Here $d(A, B)$ denotes the smallest ratio R/r such that $rA \subseteq B \subseteq RA$.

b. The requirement that K be in M-position is crucial. Otherwise, the result cannot be true. Consider the n-dimensional normed space $X = \ell_1^{n/10} \oplus \ell_\infty^{\frac{9}{10}n}$. Then it is known that there are, say, 20 rotations u_i such that (3.1) is satisfied for X. At the same time it is also not hard to show that no 5 rotations in (3.1) can give us the norm uniformly (i.e., independent of the dimension n) equivalent to ℓ_2^n. In this particular example an M-position can be taken to be the unit ball of the space $Y = L_1^{n/10} \oplus L_\infty^{\frac{9}{10}n}$ which is isometric to X. In this realization, Y, of the same space X, it is already impossible to find a number of rotations, independent of dimension n, satisfying (3.1). We thank Bill Johnson who pointed up this example to us.

We prove below, in fact, a stronger statement than Theorem 3.1. By section 2 we know that both (3.1) and (3.2) are equivalent to information about almost Euclidean sections of the space X. The conclusion of Theorem 3.1' is stated in this local form. Clearly, Theorem 3.1 follows from it. Given a subspace Y of the space $X = (\mathbb{R}^n, \|\cdot\|)$ with unit ball K, we denote by $\bar{d}_Y = \bar{d}_K$ the natural distance of Y to the natural Euclidean space (Y equiped with $|\cdot|$), i.e., $\bar{d}_Y = d(K, B_{|\cdot|})$. (See remark a. above.)

Theorem 3.1'. *Under the assumptions of Theorem 3.1, for every* $0 < \beta < 1$, *there is a subspace* Y *of dimension* $[\beta n]$ *of* $(\mathbb{R}^n, \|\cdot\|)$ *for which* $\bar{d}_Y$ *is bounded by a constant* *depending on β, t and C only. Moreover, the collection of all $[\beta n]$-subspaces satisfying the conclusion has probability tending to one as n tends to ∞.*

Proof: We may assume $r = 1$. By Lemma 2.1, we know that

$$\|x\| \le C\sqrt{t}|x| \ .$$

Let ρ be the volume radius (v.rad(K)) of K, i.e. $|\rho D| = |K|$. Then, by the Blaschke-Santaló inequality v.rad$(K^\circ) \le 1/\rho$.

By duality (3.1) implies

$$D \subseteq \frac{1}{t} \sum_{1}^{t} u_i K^\circ \equiv T \subseteq C \cdot D$$

and, by the reverse Brunn-Minkowski inequality

$$1 \le \text{v.rad}\, T \le C(t)\, \text{v.rad}\, K^\circ \le C(t)/\rho \ .$$

So, $\rho \le C(t)$.

Finally, $\frac{1}{\sqrt{t}C} D \subseteq K$ and the canonical volume ratio of K $(= (|K|/\rho|D|)^{1/n}$ for the largest ρ such that $\rho D \subseteq K)$ satisfies

$$\text{v.r.}\, K \le \sqrt{t} C \left(\frac{|K|}{|D|} \right)^{1/n} = \sqrt{t} C \rho \le \sqrt{t} C(t) \cdot C \equiv C_1(t) \ .$$

If K is in an α-regular M-position for some $\alpha > \frac{1}{2}$, then $C(t) \le C_\alpha t^\alpha$. So, for every $\varepsilon > 0$ there is $C(\varepsilon)$ such that

$$\text{v.r.}\, K \le C(\varepsilon) t^{1+\varepsilon} \ .$$

Now Theorem 3.1' follows from the standard knowledge about spaces with finite volume ratio. □

This simple proof has an unfortunate disadvantage. The known estimates of the distance of "random" λn-dimensional subspaces to Euclidean spaces as a function of $\lambda < 1$ (and of the v.r. K) are very poor.

We present now another proof, more involved but which implies much better estimates.

Theorem 3.1". *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and assume its unit ball K is in α-regular M-position for some $\alpha > 1/2$. Assume further that for some t orthogonal transformations $u_1, \ldots, u_t$ and for some $0 < r, C < \infty$,*

$$r|x| \le \frac{1}{t}\sum_{i=1}^{t} \|u_i x\| \le Cr|x|$$

for all $x \in \mathbb{R}^n$. Then for every $0 < \beta < 1$, there is a subspace Y of dimension $[\beta n]$ of $(\mathbb{R}^n, \|\cdot\|)$ for which $\bar{d}_Y \le KC^2\chi^2(\alpha)(1-\beta-o(1-\beta))^{1/2-\alpha}t^{1/2+\alpha}$, for some absolute constant K.

Proof. We may and shall assume that $\operatorname{vol} K = \operatorname{vol} D$.

We would like to give first the motivation for the computations below. From Lemma 2.1 we know that $b = \sup_{S^{n-1}} \|x\| \le Cr\sqrt{t}$. Also, $M = \int_{S^{n-1}} \|x\| \ge r$ (and $M \le Cr$). It follows that for each R there is a subspace of dimension $k = [\min\{\frac{\eta R^2}{C^2}\frac{n}{t}, n\}]$ (η an absolute constant) on which the $\|x\| \le RM|x|$. Actually, most subspaces of this dimension will do. Also, the "M^* lower bound theorem" of the first named author (cf. [MS] or [Pi]), implies that for every $0 < \delta < 1$ most subspaces of dimension δn satisfy the lower bound $\|x\| \ge \frac{c_\delta}{M^*}|x|$, where $c_\delta > 0$ depends only on δ and $M^* = \int_{S^{n-1}} \|x\|^*$. We can of course find a large subspace satisfying both conditions. The problem of course is that we do not know, and it is not always true that MM^* is bounded. Our plan is to replace M^* with the respective quantity, denoted M_s^*, for a related norm, show that it is good enough to consider that related norm, when dealing with the lower bound and show that MM_s^* is bounded.

We first estimate r and, since $r \le M \le Cr$, also M. The dual ball to that of the norm $\frac{1}{t}\sum_{i=1}^t \|u_i x\|$ is $\frac{1}{t}\sum_{i=1}^t u_i K^o$. We thus get

$$rD \subset \frac{1}{t}\sum_{i=1}^{t} u_i K^o \subset rCD. \tag{3.3}$$

By Brunn-Minkowski inequality and its inverse stated above,

$$(\operatorname{vol} K^o)^{1/n} \le (\operatorname{vol}(\frac{1}{t}\sum_{i=1}^{t} u_i K^o))^{1/n} \le \chi(\alpha)t^\alpha(\operatorname{vol} K^o)^{1/n}. \tag{3.4}$$

Also, by Blaschke-Santaló's inequality,

$$(\mathrm{vol}\, K^o)^{1/n} \le (\mathrm{vol}\, D)^{1/n} \tag{3.5}$$

for some universal $c > 0$. (3.3),(3.4) and (3.5) now imply

$$r \le \chi(\alpha) t^\alpha \quad \text{and} \quad M \le C\chi(\alpha) t^\alpha. \tag{3.6}$$

We turn now to the lower bound. Fix a $\lambda > 1$. For $0 < s < \infty$, let $K_s = K \cap sD$, let $\|x\|_s = \max\{\|x\|, s^{-1}|x|\}$ be the norm corresponding to K_s and let $M_s^* = \int_{S^{n-1}} \|x\|_s^*$. Let s be such that $\frac{s}{M_s^*} = \lambda$. (Note that $\frac{s}{M_s^*} = 1$ if s is small enough and, since $M_s^* = M^*$ for s large enough, $\lim_{s\to\infty} \frac{s}{M_s^*} = \infty$).

We first want to estimate M_s^*. By [BLM] we get that there exist k orthogonal transformations $u_1 \dots u_k$, with $k = k_\lambda$ depending only on λ (actually k is bounded by a universal constant times λ^2,because the expression "b/M" for the norm $\|\cdot\|_s^*$ is λ), such that

$$\frac{M_s^*}{2}|x| \le \frac{1}{k}\sum_{i=1}^k \|u_i x\|_s^* \le 2M_s^*|x| \tag{3.7}$$

i.e.,

$$\frac{M_s^*}{2} D \subset \frac{1}{k}\sum_{i=1}^k u_i K_s \subset 2M_s^* D. \tag{3.8}$$

By the inverse Brunn-Minkowski inequality as stated above,

$$\begin{aligned}\frac{M_s^*}{2}(\mathrm{vol}\, D)^{1/n} &\le (\mathrm{vol}(\frac{1}{k}\sum_{i=1}^k u_i K_s))^{1/n} \le (\mathrm{vol}(\frac{1}{k}\sum_{i=1}^k u_i K))^{1/n} \\ &\le \chi(\alpha) k^{\alpha-1} (\mathrm{vol}\, K)^{1/n} = \chi(\alpha) k^{\alpha-1}(\mathrm{vol}\, D)^{1/n}. \end{aligned} \tag{3.9}$$

Consequently,

$$M_s^* \le 2\chi(\alpha) k^{\alpha-1}. \tag{3.10}$$

By the M^* lower bound theorem ([MS] or [Pi]), for any $0 < \beta < 1$ we get for an appropriate λ with $\lambda \to \infty$ if $\beta \to 1^-$ (actually, one can take $\lambda = (1-\beta-o(\beta))^{-1/2}$ see [Mi4], Proposition 2.ii.2, p. 24), that most $[\beta n]$-dimensional subspaces X of $\mathbb{R}^n$ satisfy $\frac{1}{s}|x| = \frac{1}{\lambda M_s^*}|x| < \|x\|_s$ for all $x \in \mathbb{R}^n$. But $\frac{1}{s}|x| < \|x\|_s$ implies $\|x\|_s = \|x\|$ so

$$(2\lambda\chi(\alpha)k_\lambda^{\alpha-1})^{-1}|x| \le \|x\| \quad \text{for all} \quad x \in X. \tag{3.11}$$

Combining this with (3.6) and the discussion at the beginning of this proof we get that for every $1 < R < \infty$ there exists a subspace X of $\mathbb{R}^n$ of dimension $[min\{\beta, \frac{\eta R^2}{tC^2}\}n]$ on which

$$(2\lambda\chi(\alpha)k_\lambda^{\alpha-1})^{-1}|x| \le \|x\| \le C\chi(\alpha)t^\alpha R|x|. \tag{3.12}$$

Where $\lambda = (1-\beta-o(\beta))^{-1/2}$. Choosing $R = (\beta t C^2/\eta)^{1/2}$ we get the conclusion.

□

Another fact in the same spirit we would like to demonstrate here is

Theorem 3.2. *Assume the unit ball K of $X = (\mathbb{R}^n, \|\cdot\|)$ is in M-position. Fix positive integers t and τ. If $[n/t]$-dimensional subspaces of X are 2-Euclidean with probability at least $1 - \frac{1}{t}$, and $[n/\tau]$-dimensional quotient spaces are 2-Euclidean with probability at least $1 - \frac{1}{\tau}$, then the space X is $C(t,\tau)$-Euclidean , i.e. $\bar{d}_X \le C(t,\tau)$ or equivalently $\bar{d}(K,D) \le C(t,\tau)$.*

Remark. The global form of this theorem, as follows from the results of section 2, is:

Let K be in M-position and $|K| = |D|$. If for some r and ρ

$$\rho D \subset K_\tau = \frac{1}{\tau}\sum_i^\tau u_i K \subset 2\rho D \quad (\text{for some } u_i \in O(n)) \tag{3.13}$$

and

$$rD \subset (K^\circ)_t = \frac{1}{t}\sum_1^t v_i K^\circ \subset 2rD \quad (\text{for some } v_i \in O(n)) \; . \tag{3.14}$$

Then $\bar{d}(D,K) \le C(t,\tau)$.

Moreover, if for some $\alpha > 1/2$, K is in an α-regular M-position, then there is a constant $C(\alpha)$ such that $d(K,D) \le C(\alpha)t^{\alpha+\frac{1}{2}}\tau^{\alpha+\frac{1}{2}}$.

Proof of the Theorem. Let a and b be the smallest numbers such that $\frac{1}{a}|x| \le \|x\| \le b|x|$. (3.13) and (3.14) mean that for some orthogonal operators $\{v_i\}_1^t, \{u_j\}_1^\tau \subset O(n)$

$$r|x| \le \frac{1}{t}\sum_1^t \|v_i x\| \le 2r|x| \; ,$$

$$\rho|z| \le \frac{1}{\tau}\sum_1^\tau \|u_j x\|^* \le 2\rho|x| \; .$$

By Lemma 2.1, $\|x\| \leq 2r\sqrt{t}|x|$ and $\|x\|^* \leq 2\rho\sqrt{\tau}|x|$, for all $x \in \mathbb{R}^n$. This means that $a \leq 2\rho\sqrt{\tau}$ and $b \leq 2r\sqrt{t}$. Also, by the reverse Brunn-Minkowski inequality (we use here that K is in M-position) we have from (3.13) and (3.14)

$$r \leq \text{v.rad}(K^\circ)_t \leq c(t)\,\text{v.rad}\,K^\circ \leq c(t)$$

(Blaschke-Santaló inequality is used here) and

$$\rho \leq \text{v.rad}\,K_\tau \leq c(\tau)\ .$$

Therefore, $\bar{d}(K,D) = a \cdot b \leq 4c(t)c(\tau)\sqrt{t\tau}$.

Note that if K is in an α-regular M-position for some $\alpha > \frac{1}{2}$ then $c(t) \leq C_\alpha t^\alpha$ and $c(\tau) \leq C_\alpha \tau^\alpha$. So, in this case there is $C(\alpha)$ such that

$$\bar{d}(K,D) \leq C(\alpha) t^{\frac{1}{2}+\alpha} \tau^{\frac{1}{2}+\alpha}\ . \qquad \square$$

Remarks. **a.** In the appendix we introduce another ellipsoid that can serve as the M-ellipsoid in the proof above. Using it one gets another version of the global form of Theorem 3.2. We state it as Corollary A.3 below.

b. We would like to make some comments on the proof above. By Theorem 2.2.b and its proof

$$(M/b)^2 \gtrsim 1/t \quad \text{and} \quad (M^*/a)^2 \gtrsim 1/\tau\ . \tag{3.15}$$

By Theorem 3.2 this implies that K is isomorphic to the Euclidean ball (of course, under the condition that K is in M-position). Note, that the space $\ell_1^{n/2} \oplus \ell_\infty^{n/2}$ satisfies (3.15) and is not uniformly isomorphic to the Euclidean. So, M-position plays a decisive role here too. From another point of view,

$$(M/b)^2 + (M^*/a)^2 = \theta > 1$$

is known ([Mi3]) to imply $d(K,D) \leq \frac{1}{\theta-1}$ without any restriction on special position. So, if both quantities M/b and M^*/a are close to one this always implies isomorphism to Euclidean space; however, in M-position already $\frac{MM^*}{a \cdot b} \geq \eta > 0$ implies $\bar{d}_X \leq C(\eta)$ for some constant $C(\eta)$ depending on $\eta > 0$ only.

4. More Global and Local Equivalences

We will present two more examples of parallel statements in the Global and Local Theories. This time the global (geometric) form was known and we develop the Local direction.

Theorem 4.1. *Let the space $X = (\mathbb{R}^n, \|\cdot\|, |\cdot|)$ have the following property: $\|x\| \le |x|$ for all $x \in \mathbb{R}^n$ and for some $C > 0$ and $t > 0$, for every integer k, $n/2 \le k < n$, there are subspaces $S_k \in G_{n,k}$, such that*

$$|x| \le C\left(1 - \frac{k}{n}\right)^{-t} \|x\|, \tag{4.1}$$

for all $x \in S_k$, and moreover, for every such k, the probability that our element of $G_{n,k}$ satisfies this inequality is at least $1 - \frac{1}{n}$. Then there is a $V = V(C,t)$ such that $(|B_{\|\cdot\|}|/|B_{|\cdot|}|)^{1/n} \le V$.

Remark. Recall the fact (discovered in [TS] based on [Ka]. cf. [To]) that if a space X satisfies $\|\cdot\| \le |\cdot|$ and $(|B_{\|\cdot\|}|/|B_{|\cdot|}|)^{1/n} \le V$ then for every k there is a large measure of k-dimensional subspaces on which the two norms $\|\cdot\|$ and $|\cdot|$ are equivalent up to some constant $f(k/n)$.

The Proof of the Theorem is standard and we just sketch it. Note first that the assumption on measure of subspaces satisfying (4.1) implies that there is a flag of subspaces $S_{\frac{n}{2}} \subseteq S_{\frac{n}{2}+1} \cdots \subseteq S_{n-1} \subseteq X$ ($\dim S_i = i$) satisfying condition (4.1). Let a be the smallest such that $\frac{1}{a}|x| \le \|x\| \le |x|$ then $a \le Cn^t$ (by (4.1)). We use Lemma 3.7 from [BM1] (or rather its reformulation as Lemma 4.7 in [BM2] which is a better form for our purposes). It states that for $k_1 = n - \frac{n}{(\log n)^2}$ any k_1-dimensional subspace, in particular our subspace S_{k_1}, has the volume radius of its unit ball almost exactly the same as the volume radius of X:

$$\text{v.rad}\, S_{k_1} \simeq \text{v.rad}\, X \ .$$

Note that $\bar{d}_{S_{k_1}} \le C(\log n)^{2t}$. Then, by the same lemma, any k_2-dimensional subspace of S_{k_1} for

$$k_2 \ge k_1 - \frac{k_2}{(\log d_{S_{k_1}})^2} \sim n - \frac{n}{(\log\log n)^2}$$

satisfies v.rad $S_{k_2} \simeq$ v.rad S_{k_1} ($\simeq$ v.rad X). applying it to our subspace S_{k_2} from the chosen flag we have also $\bar{d}_{S_{k_2}} \leq C(\log\log n)^{2t}$. So, continuing along our flag of subspaces we reach, after p steps, where p is the first to satisfy $\log^{(p)} n \leq 2$, a subspace S_{k_p} which is Euclidean up to a constant $K(C,t)$ without significant change of v.rad.: v.rad $S_{k_p} \simeq$ v.rad X. (For that conclusion one needs to consult [BM2] to get the right constant of equivalence in each step of the procedure above and check that the product of these constants converge.) Also, $(|S_{k_p} \cap S_{k_p} \cap B_{\|\cdot\|}|/|B_{|\cdot|}|)^{1/k_p} \leq K(C,t)$ which implies that $(|B_{\|\cdot\|}|/|B_{|\cdot|}|)^{1/n}$ is bounded by a constant depending on C and t only. □

In our last result, Proposition 4.3, we describe the global property of a space X being in M-position by local information on some quotient spaces of X.

Let $X = (\mathbb{R}^n, \|\cdot\|, |\cdot|)$. For a subspace q we define the normed space qX as the one whose unit ball is the orthogonal projection on q of the unit ball of X. If $K = B_{\|\cdot\|}$ we denote by qK the unit ball of qX. For two bodies K and H in $\mathbb{R}^n$ we let $N(K,H)$ denote the minimal number of translates of H needed to cover K. The next lemma is known (part a. is implicitly contained in [Mi4]) although we could not locate a good reference. We include a sketch of a proof.

Lemma 4.2. *Let K be a convex symmetric body in $\mathbb{R}^n$ such that $|K| = |D|$ and $N(D,K) \leq e^{cn}$. Then*

a. *With high probability a $[(n+1)/2]$-dimensional subspace of $\mathbb{R}^n$ satisfy that $\delta qD \subseteq qK$ for $\delta > 0$ depending only on c.*

b. *$N(K, 2D) \leq e^{c'n}$ with c' depending only on c. Consequently, $(|qK|/qD|)^{1/n}$ is bounded by constant depending only on c for every $[(n+1)/2]$- dimensional subspace.*

Sketch of a proof. Assume n is even. It follows from the assumption that $|D \cap K| \geq e^{-cn}$ this implies, via the Blaschke-Santaló inequality, that the volume ratio of $(D \cap K)^\circ$ is bounded by e^c. It follows that a random $n/2$-dimensional section of $(D \cap K)^\circ$ is contained in a multiple, depending only on c, of D. Dualizing again we get a.

$N(D,K) \leq e^{cn}$ implies $|K + D| \leq e^{cn} 2^n |K|$ consequently the maximal cardinality of a 2-separated family in K is bounded by $|K + D|/|K| \leq e^{c'n}$. This proves b. □

The next proposition shows that the inverse statement also holds.

Proposition 4.3. *Let the space* $X = (\mathbb{R}^n, \|\cdot\|, |\cdot|)$ *be such that, for* $K = B_{\|\cdot\|}$, $|K| = |D|$ *and with probability at least* $1/2$, *an* $[(n+1)/2]$*-dimensional quotient* qX *satisfies* $(|qK|/qD|)^{1/n} \leq C$ *and* $\delta qD \subseteq qK$. *Then* $N(D,K) \leq e^{cn}$ *with* c *depending on* δ *and* C *only.*

Remark. It follows from Lemma 4.2 that also $N(K,D)$ is bounded by $e^{c'n}$. For most purposes this (i.e., both $N(K,D)$ and $N(D,K)$ are at most exponential in n) can serve as a definition of "K is in M position".

The proof of the proposition follows easily from the following lemma

Lemma 4.4. *Let* $H \subset K \subseteq \mathbb{R}^n$ *with* H *convex symmetric and such that* $(|K|/|H|)^{1/n} \leq C$. *Then,*

$$N(K, tH) \leq e^{n \log(\frac{t}{t-2}C)}$$

for all $t > 2$.

Proof. Let $x \in K$ then

$$K \cap (\frac{2x}{t} + H) \supseteq \frac{t-2}{t}H + \frac{2}{t}(K \cap (x+H)).$$

Indeed, if $h_1 \in H$ and $h_2 \in H$ with $k = x + h_2 \in K$, then $\frac{t-2}{t}h_1 + \frac{2}{t}k \in K$ and $\frac{t-2}{t}h_1 + \frac{2}{t}k \in K = \frac{2}{t}x + \frac{t-2}{t}h_1 + \frac{2}{t}h_2 \in \frac{2}{t}x + H$.

Since $K \cap (x+H)$ is not empty,

$$|K \cap (\frac{2x}{t} + H)| \geq |\frac{t-2}{t}H| = (\frac{t-2}{t})^n |H|.$$

Let now $\{x\}_{x \in I}$ be a maximal family of points in K such that $\|x-y\|_H > t$ for all $x \neq y$, $x, y \in I$. Then $\cup_{x \in I}(x + tH) \supseteq K$ and $\{\frac{2x}{t} + H\}_{x \in I}$ are disjoint. It follows that

$$|I| \leq |K| / \min_{x \in I} |K \cap (\frac{2x}{t} + H)| \leq (\frac{t}{t-2})^n |K|/|H| \leq e^{n \log(\frac{t}{t-2}C)}.$$

□

Proof of Proposition 4.3. Assume n is even. Let $q_1 X$ and $q_2 X$ be two orthogonal $n/2$-dimensional quotient spaces of X satisfying the assumption. By the Lemma, for some C depending only on δ and c, $N(q_i K, \delta q_i D) \leq e^{Cn}$, $i = 1, 2$. It follows that $N(K,D) \leq N(K, \delta D) \leq e^{2Cn}$. □

Remark.. It follows from Lemma 4.4 that, if D is the maximal volume ellipsoid inscribed in K and K has volume ratio v (i.e. $(|K|/|D|)^{1/n} = V$), then $N(K,D) \times N(D,K) \le e^{cn}$ with c depending only on V. i.e., for bodies with "finite volume ratio" the maximal volume ellipsoid can serve also as the M-ellipsoid.

A result of the first named author ([Mi2]) states that when B_X is in M position a random subspace of quotient of proportional dimension is Euclidean (cf [MS] or [Pi]). The following is a converse to this statement and is another instance in which we get a global result from local one.

Corollary 4.5. *Let the space $X = (\mathbb{R}^n, \|\cdot\|, |\cdot|)$ satisfy $|B_{\|\cdot\|}| = |B_{|\cdot|}|$ and, for some $C, t > 0$ and all $k < n/2$, a k-dimensional subspace of an $[(n+1)/2]$-dimensional quotient of X, $Y = sqX$, satisfies, with probability at least $1 - \frac{1}{n}$,*

$$C^{-1}\left(1 - \frac{2k}{n}\right)^{-t} \|x\| \le |x| \le C\left(1 - \frac{2k}{n}\right)^{-t} \|x\|, \tag{4.2}$$

for all $x \in Y$. Then,

a. With probability larger than 1/2, an $[(n+1)/2]$-dimensional quotient satisfies $(|qB_{\|\cdot\|}|/|qD|)^{1/n} \le V$ for some V depending on C and t only. ($D = B_{|\cdot|}$)

b. X is in M position. i.e., $N(B_{\|\cdot\|}, D) \times N(D, B_{\|\cdot\|}) \le e^{cn}$ with c depending on C and t only.

Proof: Assume n is even. The assumption implies that, with probability larger than 1/2, an $n/2$-dimensional quotient, Y, satisfies that a k-dimensional subspace of it satisfies (4.2) with probability larger than $1 - \frac{2}{n}$. In particular, applying the upper bound in (4.2) for two orthogonal subspaces of Y of dimension approximately $n/4$ (as in the proof of Theorem 2.2) we get that $\|x\| \le \delta^{-1}|x|$ for all $x \in Y$ ($\delta > 0$ depends on C and t only). i.e., $\delta qD \subseteq B_Y$. Applying Theorem 4.1, we get that, with probability larger than 1/2, $(|B_Y|/|qD|)^{1/n} \le V(C,t)$. Which proves a. Applying Proposition 4.3 and the remark following its statement we also get b. ◻

5. Appendix

Here we prove

Proposition A.1. *Let K be a convex symmetric body and let $e < \lambda_0$ then there exits an ellipsoid D such that, for all $\log\lambda_0 < \lambda \le \lambda_0$*

$$N(K, \lambda D) \le e^{Cn\frac{\log\lambda_0}{\lambda^2}} \quad \text{and} \quad N(K^\circ, \lambda D) \le e^{Cn\frac{\log\lambda_0}{\lambda^2}}.$$

Where K° is the polar to K with respect to D.

Before we turn to the proof of the proposition Let us state two corollaries. The first deals with estimating the volume of sums of few bodies with common ellipsoid satisfying the conclusion above. For simplicity we do it only for bodies which are rotations of one body. The second corollary is the variant of Theorem 3.2 promised in the remark following its proof.

Corollary A.2. *Let K be in M position for some λ_0 as in Proposition A.1 (meaning that the D in the proposition is the standard Euclidean ball of $\mathbb{R}^n$). Let $\log\lambda_0 \le t \le \lambda_0^2/\log\lambda_0$. Then for any $\{u_i\}_{i=1}^t \subset O(n)$,*

$$|t^{-1}\sum_{i=1}^t u_i K|^{1/n} \le c\sqrt{t\log\lambda_0}|D|^{1/n}.$$

Proof. Let $\lambda_0 > \lambda > \log\lambda_0$. Let $\{x_{ij}\}_{j=1}^N$ be the set of centers of a minimal covering of $u_i K$ by λD, i.e., $u_i K \subseteq \cup_{j=1}^N (x_{ij} + \lambda D)$ with $N \le e^{Cn\log\lambda_0/\lambda^2}$. Then $T = t^{-1}\sum_{i=1}^t u_i K$ is covered by $e^{Ctn\log\lambda_0/\lambda^2}$ λD balls. Therefore $|T|^{1/n} \le \lambda e^{Ct\log\lambda_0/\lambda^2}|D|^{1/n}$. Now choose λ such that $t = \lambda^2/\log\lambda_0$ and we get $|T|^{1/n} \le C\sqrt{t\log\lambda_0}|D|^{1/n}$. □

Corollary A.3. *Let K be in M position defined for a fixed λ_0 as in Proposition A.1. Let t and τ be integers such that $\log\lambda_0 \le t, \tau \le \lambda_0^2/\log\lambda_0$. If for some orthogonal transformations $\{u_i\}_{i=1}^\tau$ and $\{v_i\}_{i=1}^t$*

$$\rho D \subset K_\tau = \frac{1}{\tau}\sum_i^\tau u_i K \subset R_1 \rho D$$

and

$$rD \subset (K^\circ)_t = \frac{1}{t}\sum_1^t v_i K^\circ \subset R_2 r D$$

Then $\bar{d}(D,K) \le R_1 R_2 t\tau \log\lambda_0$.

The proof is the same as the proof of (the Remark after) Theorem 3.2. The only difference is that we substitute the new bounds for $c(t)$ and $c(\tau)$ given in Corollary A.2.

Proof of Proposition A.1. For a convex body T (not necessarily symmetric) and for a (symmetric) ellipsoid D we put

$$M_D^*(T) = \int_{\partial D} max\{< x, y >_D;\ y \in T\} d\mu_D(x)$$

where $< \cdot, \cdot >_D$ is the inner product defined by D and μ_D is the unique probability measure on the boundary of D which is invariant under the orthogonal (w.r.t. D) group.

Clearly, $M_D^*(\alpha T_1 + \beta T_2) = \alpha M_D^*(T_1) + \beta M_D^*(T_2)$ for all $\alpha, \beta \ge 0$, $M_D^*(-T) = M_D^*(T)$ and $M_D^*(T_1) \le M_D^*(T_2)$ if $T_1 \subseteq T_2$.

Sudakov's inequality states that $N(T, \lambda D) \le e^{Cn(M_D^*(T)/\lambda)^2}$ for all $\lambda > 0$. Below we shall use this in a situation in which $\frac{T+(-T)}{2} \subseteq K$ for some symmetric convex set K. From the above it follows that $M_D^*(T) \le M_D^*(K)$ and that $N(T, \lambda D) \le e^{Cn(M_D^*(K)/\lambda)^2}$ for all $\lambda > 0$.

For all K symmetric there is a D (which we will call an ℓ-ellipsoid) for which $1 \le M_D^*(K) M_D(K) \le C \log d_K \le C \log n$, C an absolute constant. Here $M_D(K) = M_D^*(K^\circ)$ where K° denotes the polar body to K with respect to D. We can, of course, assume that $M_D(K) = 1$.

Step 1: Given K we let D_1 be an ℓ-ellipsoid of K and denote

$$M_{D_1}^*(K) = M_1^*, \qquad M_{D_1}(K) = M_{D_1}^*(K^\circ) = M_1$$

so that $M_1 M_1^* \le C \log n$. Here K° denotes the polar body to K with respect to D_1. We will also assume without loss of generality that $M_1 = 1$. Then $N(K, \lambda D_1) \le e^{Cn(M_1^*/\lambda)^2}$ for all $\lambda > 0$ and $N(K^\circ, \lambda D_1) \le e^{Cn(1/\lambda)^2}$ for all $\lambda > 0$. Fix $a_1 > 1$ and put

$$K_1 = conv\left((K \cap a_1 M_1^* D_1) \cup \frac{1}{a_1 M_1} D_1\right).$$

Denote by $L^\bullet$ the polar body to L with respect to an arbitrary ellipsoid E to be specified later. Then

$$K_1^\bullet = conv(K^\bullet \cup \frac{1}{a_1 M_1^*} D_1^\bullet) \cap a_1 M_1 D_1^\bullet\ .$$

Note that $\bar{d}(K_1, D_1) \leq a_1^2 M_1 M_1^* \leq c a_1^2 \log n$. Also note that

$$K \subseteq \bigcup_{i \in I} K_1^i \ , \qquad K^\bullet \subseteq \bigcup_{i \in I} H_1^i \ , \qquad \# I \leq e^{Cn/a_1^2} \tag{A.1}$$

where $K_1^i = (x_i + a_1 M_1^* D_1) \cap K,\ H_1^i = (y_i + a_1 M_1 D_1^\bullet) \cap K^\bullet,\ \{x_i\}_{i \in I} \subseteq K,\ \{y_i\}_{i \in I} \subseteq K^\bullet$.

Note that, since

$$\frac{(x_i + a_1 M_1^* D_1) \cap K + (-x_i + a_1 M_1^* D_1) \cap K}{2} \subseteq K_1$$

and

$$\frac{(y_i + a_1 M_1 D_1^\bullet) \cap K^\bullet + (-y_i + a_1 M_1 D_1^\bullet) \cap K^\bullet}{2} \subseteq K_1^\bullet \ ,$$

$$M_D^*(K_1^i) \leq M_D^*(K_1) \ \ and \ \ M_D(H_1^i) \leq M_D(K_1^\bullet)$$

for any ellipsoid D.

Step 2: Now let D_2 be an ℓ-ellipsoid of K_1 and denote

$$M_{D_2}^*(K_1) = M_2^* \ , \qquad M_{D_2}(K_1) = M_2 = 1 \ .$$

($M_{D_2}(K_1) = M_{D_2}^*(K_1^\circ)$ where the polarity is with respect to D_2. Without loss of generality, we may assume $M_2 = 1$.) Then $M_2 M_2^* \leq C \log \bar{d}(K_1, D_1) \leq C \log(C a_1^2 \log n)$ and for each $i \in I$ and $\lambda > 0$, $N(K_1^i, \lambda D_2) \leq e^{Cn(M_2^*/\lambda)^2}$ and $N(H_1^i, \lambda D_2^\bullet) \leq e^{Cn(1/\lambda)^2}$.

Here $L^\bullet$ denotes duality w.r.t. the same yet unspecified E. It follows from (A.1) that, for $\lambda > 0$,)

$$N(K, \lambda D_2) \leq e^{Cn\left[\frac{1}{a_1^2} + \left(\frac{M_2^*}{\lambda}\right)^2\right]}$$

and

$$N(K^\bullet, \lambda D_2^\bullet) \leq e^{Cn\left[\frac{1}{a_1^2} + \left(\frac{1}{\lambda}\right)^2\right]} \ .$$

Let $a_2 > 1$ and put

$$K_2 = conv\left((K_1 \cap a_2 M_2^* D_2) \cup \frac{1}{a_2 M_2} D_2\right)$$

then

$$K_2^\bullet = conv\left(K_1^\bullet \cup \frac{1}{a_2 M_2^*} D_2^\bullet\right) \cap a_2 M_2 D_2^\bullet$$

(duality w.r.t. E) and $\bar{d}(K_2, D_2) \le a_2^2 M_2^* M_2 \le a_2^2 C \log(C a_1^2 \log n)$.

Now, for each i,

$$K_1^i \subseteq \bigcup_{j\in I_i} K_2^{i,j}\ , \qquad H_1^i \subseteq \bigcup_{j\in I_i} H_2^{i,j}\ , \qquad \#I_i \le e^{Cn/a_2^2}$$

where $K_2^{i,j} = (x_{ij} + a_2 M_2^* D_2) \cap K_1^i$, $H_2^{i,j} = (y_{i,j} + a_2 M_2 D_2^\bullet) \cap K_1^\bullet$, $x_{ij} \in K_1^i$, $y_{ij} \in H_1^i$.

Again

$$M_D^*(K_2^{i,j}) \le M_D^*(K_2),\ M_D(H_2^{i,j} \le M_D(K_2^\circ)$$

for every ellipsoid D.

Step 3: Letting D_3 be an ℓ-ellipsoid of K_2 and denoting

$$M_3^* = M_{D_3}^*(K_2), \qquad M_3 = M_{D_3}(K_2^\circ) = 1$$

(duality w.r.t. D_3) we get $M_3 M_3^* \le C \log\left(a_2^2 C \log(C a_1^2 \log n)\right)$ and for each $j \in I_i$ and $\lambda > 0$

$$N(K_2^{i,j}, D_3) \le e^{Cn(M_3^*/\lambda)^2} \quad (resp.\ \ N(H_2^{i,j}, D_3) \le e^{Cn(1/\lambda)^2}).$$

It follows that, for $\lambda > 0$,

$$N(K, \lambda D_3) \le e^{Cn\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \left(\frac{M_3^*}{\lambda}\right)^2\right)}$$

and

$$N(K^\bullet, \lambda D_3^\bullet) \le e^{Cn\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \left(\frac{1}{\lambda}\right)^2\right)}\ .$$

Step k: Continuing in the same way we get, for any sequence $a_1, \cdots, a_{k-1} > 1$, a sequence of convex symmetric bodies $K = K_0, K_1, \cdots, K_{k-1}$ and ellipsoids $D_1, \cdots, D_k$ with $M_{D_i}(K_{i-1}) = 1,\ i = 1, \cdots, k-1,\ D_i,\ K_{i-1}$ depending only on $a_1 \cdots a_{i-1}$, satisfying

$$\begin{aligned} \bar{d}(K_i, D_i) &\le a_i^2 M_{D_i}^*(K_{i-1}) \\ &\le a_i^2 C \log\left(\bar{d}(K_{i-1}, D_{i-1})\right), \quad i = 1, \cdots, k-1\ . \end{aligned} \tag{A.2}$$

$\left(\bar{d}(K, D_0) \overset{d}{=} n\right)$. Also $M_{D_k}^*(K_{k-1}) \le C \log\ \bar{d}(K_{i-1}, D_{i-1})$ and, for all $\lambda > 0$,

$$N(K, \lambda D_k) \le e^{Cn\left(\frac{1}{a_1^2} + \cdots + \frac{1}{a_{i-1}^2} + \left(\frac{M_{D_k}^*(K_{k-1})}{\lambda}\right)^2\right)} \tag{A.3}$$

and

$$N(K^\bullet, \lambda D_K^\bullet) \le e^{Cn\left(\frac{1}{a_1^2}+\cdots+\frac{1}{a_{k-1}^2}+\frac{1}{\lambda^2}\right)}. \qquad (A.4)$$

Conclusion: Fix $\lambda_0 > 1$ and consider the process above with $a_i = M^*_{D_i}(K_{i-1})$ as long as $a_i > \lambda_0$ and $a_{k-1} = \lambda_0$ for the first k for which $M^*_{D_{k-1}}(K_{k-2}) \le \lambda_0$. Note that $\sum_{i=1}^{k-1} \frac{1}{a_i^2} \le C \frac{1}{a_{k-1}^2} = \frac{C}{\lambda_0^2}$. Also $M^*_{D_k}(K_{k-1}) \le C \log \lambda_0$ so that we get

$$N(K, \lambda D_k) \le e^{Cn\left(\frac{1}{\lambda_0^2}+\frac{\log \lambda_0}{\lambda^2}\right)}$$

resp. $N(K^\bullet, \lambda D_k^\bullet) \le e^{Cn\left(\frac{1}{\lambda_0^2}+\frac{1}{\lambda^2}\right)}$ for all $\lambda > 0$. In particular,

$$N(K, \lambda D_k) \le e^{Cn\frac{\log \lambda_0}{\lambda^2}}$$

for all $\log \lambda_0 < \lambda \le \lambda_0$ and

$$N(K^\bullet, \lambda D_k^\bullet) \le e^{Cn\frac{1}{\lambda^2}}$$

for all $\lambda \le \lambda_0$.

This is true where $\bullet$ denotes duality w.r.t. any ellipsoid E. Take $E = D_k$ to finish the proof.

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