AN "ISOMORPHIC" VERSION OF DVORETZKY'S THEOREM

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Abstract - For each $1 \leq k < n$ we give an upper bound on the minimal distance of a k-dimensional subspace of an arbitrary n-dimensional normed space to the Hilbert space of dimension k. The result is best possible up to a multiplicative universal constant.

UNE VERSION "ISOMORPHE" DU THÉORÈME DE DVORETZKY

Résumé - Pour chaque $1 \leq k < n$, nous donnons une borne supérieure pour la distance minimale de Banach-Mazur entre un sous-espace de dimension k d'un espace normé arbitraire de dimension n et l'espace de Hilbert de dimension k. Le résultat est le meilleur possible, à une constante multiplicative près.

Version française abrégée - Le résultat principal est le théorème 2, formalisant le résumé ci-dessùs.

Théorème 2. *Il existe une constante $K > 0$ telle que, pour chaque n et pour chaque $k \geq K \log n$, tout espace normé X de dimension n contient un sous-espace Y de dimension k satisfaisant $d(Y, \ell_2^k) \leq K\sqrt{\frac{k}{\log(n/k)}}$. En particulier, si $k \leq n^{1-K\varepsilon^2}$, il existe un sous-espace Y de dimension k (d'un espace normé arbitraire X de dimension n) avec $d(Y, \ell_2^k) \leq \frac{K}{\varepsilon}\sqrt{\frac{k}{\log n}}$.*

Le résultat est le meilleur possible, à une constante multiplicative près. Pour $k = L \log n$ nous obtenons une version isomorphe (par opposition à une version presque isométrique) et quantitative du théorème de Dvoretzky figurant dans le titre: *Chaque espace normé de dimension n contient, pour chaque $L \geq 1$, un sous-espace de dimension $[L \log n]$ dont la distance à un espace de Hilbert dépend seulement de L. Plus précisément, la distance est bornée par le produit d'une constante universelle et de $\sqrt{L}$.*

Le preuve suit le schéma général de [M]. Il y a cependant trois nouveaux éléments: 1. Nous utilisons deux méthodes différentes pour mesurer la probabilité de grande déviation de la norme dans X de sa moyenne sur une sphère euclidienne, l'une pour le déviation supérieure (qui est l'estimation ordinaire), l'autre pour la déviation inférieure. La deuxième est étonnamment bonne. 2. Nous utilisons les résultats précédents pour obtenir deux réseaux de points dans un certain sous-espace de dimension k dans lequel la norme a un "bon comportement". L'un sur lequel on obtient une bonne estimation supérieure (c'est de nouveau la méthode habituelle) et l'autre, beaucoup plus fin, sur lequel on obtient l'estimation inférieure désirée. Ces deux éléments 1 et 2 peuvent être utilisés pour prouver qu'un sous-espace aléatoire de dimension k de ℓ_∞^n satisfait l'estimation du théorème 2. 3. Le troisième nouvel élément est l'utilisation d'un raffinement du lemme de Dvoretzky-Rogers, prouvé par Bourgain et Szarek [BS], pour réduire la preuve du cas général à celle du cas ℓ_∞^n.

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Our main result is Theorem 2 below which formalizes the statement in the abstract above.

The proof follows the general scheme of [M]. There are however three new features: 1. One uses two different ways to measure the probability of large deviation of the norm in X from its average over an Euclidean sphere, one for the upper deviation (which is the usual estimate) and another one for the lower deviation. The second one is surprisingly good. 2. One uses the deviation results above to get **two** nets of points in some k-dimensional subspace on which the norm behaves nicely. One net on which one gets a good upper estimate (again, this is the usual method) and another, much finer, net on which one gets the desired lower estimate. Items 1 and 2 above can be used to prove that a random k-dimensional subspace of ℓ_∞^n satisfies the distance estimate in Theorem 2. 3. The third new feature is a use of a refinement of the Dvoretzky-Rogers Lemma, proved by Bourgain and Szarek [BS], to reduce the proof in the general case to the proof of the ℓ_∞^n case.

We use [MS] as our standard reference. Let $\mu_{S^{m-1}}$ denote the normalized Haar measure on the Euclidean sphere in $\mathbf{R}^m$. We start with a standard lemma.

Lemma 1. *There exist constants $0<\eta, K<\infty$ such that for all $\varepsilon>0$ and all $m\in\mathbf{N}$ satisfying $\varepsilon\sqrt{\log m}\ge K$,*

$$\mu_{S^{m-1}}\left(\left\{a=(a_1,\dots,a_m);\ \max_{1\le i\le m}|a_i|\le\varepsilon\sqrt{\frac{\log m}{m}}\right\}\right)\le e^{-\eta m^{1-K\varepsilon^2}}$$

Proof. Let $g_1,\dots,g_m$ be independent standard Gaussian random variables. Then

$$\begin{aligned}
&\mu_{S^{m-1}}\left(\left\{a=(a_1,\dots,a_m);\ \max_{1\le i\le n}|a_i|\le\varepsilon\sqrt{\frac{\log m}{m}}\right\}\right)\\
&=Prob\left(\frac{\max_{1\le i\le m}|g_i|}{(\sum g_i^2)^{1/2}}\le\varepsilon\sqrt{\frac{\log m}{m}}\right)\\
&\le Prob\left(\max_{1\le i\le m}|g_i|\le 2\varepsilon\sqrt{\log m}\ \text{ or }\ (\sum g_i^2)^{1/2}>2\sqrt{m}\right)\\
&\le Prob^m(|g_1|\le 2\varepsilon\sqrt{\log m})+e^{-\eta m}\ \le\ \left(1-e^{-K\varepsilon^2\log m}\right)^m+e^{-\eta m}\\
&\le e^{-\eta m^{1-K\varepsilon^2}}+e^{-\eta m}
\end{aligned}$$

Here η and K are universal constants, not necessarily the same in every instance. ■

Theorem 2. *There exists a $K>0$ such that, for every n and every $k\ge K\log n$, any n-dimensional normed space, X, contains a k-dimensional subspace, Y, satisfying $d(Y,\ell_2^k)\le K\sqrt{\frac{k}{\log(n/k)}}$. In particular, if $k\le n^{1-K\varepsilon^2}$, there exists a k-dimensional subspace Y (of an arbitrary n-dimensional normed space X) with $d(Y,\ell_2^k)\le\frac{K}{\varepsilon}\sqrt{\frac{k}{\log n}}$.*

Proof. In what follows $0<\eta,K<\infty$ denote absolute constants, not necessarily the same in each instance. By Theorem 2 of [BS] (see remark 4 below) we may assume without loss of generality that there exists a subspace, $Z \subseteq X$, with $m = dim Z > n/2$ and $\alpha\|x\|_{\ell_\infty^m} \le \|x\|_Z \le \|x\|_{\ell_2^m}$ for all $x \in Z$ for some absolute constant $\alpha > 0$.

Let M denote the median of $\|x\|_Z$ over $S^{m-1} = S^{n-1} \cap Z$. Fix k and put $C = \sqrt{\frac{k}{\eta \log n}}$. If $M > C\sqrt{\frac{\log n}{n}}$ then, by [M] (see also [FLM] or [MS], Theorem 4.2), Z and thus X contains a $k = \eta C^2 \log n$ - dimensional subspace, Y, satisfying $d(Y, \ell_2^k) \le 2$.

If $M \le C\sqrt{\frac{\log n}{n}}$ then, by Lemma 1,

$$\mu_{S^{m-1}}\left(\|x\|_Z \le \varepsilon\alpha\sqrt{\frac{\log n}{n}}\right) \le \mu_{S^{m-1}}\left(\|x\|_{\ell_\infty^m} \le \varepsilon\sqrt{\frac{\log m}{m}}\right) \le \exp(-\eta n^{1-K\varepsilon^2}).$$

On the other hand (see chapters 2 and 6 in [MS]),

$$\mu_{S^{m-1}}\left(\|x\|_Z \ge 2C\sqrt{\frac{\log n}{n}}\right) \le \mu_{S^{m-1}}\left(\|x\|_Z - M \ge C\sqrt{\frac{\log n}{n}}\right) \le \exp(-\eta C^2 \log n).$$

Fix a k-dimensional subspace, Y_0, of Z and fix two nets (with respect to the Euclidean norm) in $S^{k-1} = S^{m-1} \cap Y_0$; an $\frac{\varepsilon\alpha}{10C}$-net $\mathcal{N}$ and a $\frac{1}{2}$-net $\mathcal{M}$. Note that we may assume $|\mathcal{N}| \le \left(\frac{30C}{\varepsilon\alpha}\right)^k$ and $|\mathcal{M}| \le 6^k$ (see [MS], Lemma 2.6).

Denoting by ν the Haar measure on the orthogonal group $O(m)$, we get that,

$$\nu\left(U \ ; \ \|Ux\|_Z \le \varepsilon\alpha\sqrt{\frac{\log n}{n}} \ \text{ for some } x \in \mathcal{N}\right) \le \exp(k\log(30C/\varepsilon\alpha) - \eta n^{1-K\varepsilon^2}).$$

and

$$\nu\left(U \ ; \ \|Ux\|_Z \ge 2C\sqrt{\frac{\log n}{n}} \ \text{ for some } x \in \mathcal{M}\right) \le \exp(k\log 6 - \eta C^2 \log n).$$

Thus, if

$$(1) \qquad k \le \eta n^{1-K\varepsilon^2}/\log(30C/\varepsilon\alpha) \ \text{ and } \ k \le \eta C^2 \log n$$

then there exists a $U \in O(m)$ such that

$$\|Ux\|_Z \ge \varepsilon\alpha\sqrt{\frac{\log n}{n}} \ \text{ for all } x \in \mathcal{N} \ \text{ and } \ \|Ux\|_Z \le 2C\sqrt{\frac{\log n}{n}} \ \text{ for all } x \in \mathcal{M}.$$

Using the usual successive approximation procedure (see Lemma 4.1 in [MS]) we get that, for all $x \in Y_0$,

$$\|Ux\|_Z \le 4C\sqrt{\frac{\log n}{n}}\|x\|_2.$$

Now, given an $x \in S^{k-1}$, pick a $y \in \mathcal{N}$ with $\|x-y\|_Z \le \frac{\varepsilon\alpha}{10C}$. Then,

$$\|Ux\|_Z \ge \|Uy\|_Z - \|Ux - Uy\|_Z \ge \varepsilon\alpha\sqrt{\frac{\log n}{n}} - \frac{\varepsilon\alpha}{10C}4C\sqrt{\frac{\log n}{n}} \ge \frac{\varepsilon\alpha}{2}\sqrt{\frac{\log n}{n}}.$$

The desired subspace is $Y = UY_0$. It remains to estimate the relation between its dimension, k, and its distance from a Hilbert space, bounded from above by $8C/\varepsilon\alpha$.

Fix an $0 < \varepsilon < 1$, and k such that $k\log k \le \eta n^{1-K\varepsilon^2}$ and $k \ge K\log n$. Put $C = \sqrt{\frac{k}{\eta\log n}}$. Then k satisfies both requirements of (1) and thus

$$d(Y, \ell_2^k) \le 8C/\varepsilon\alpha \le \frac{K}{\varepsilon}\sqrt{\frac{k}{\log n}},$$

which proves the "in particular" statement of the theorem.

Next we want to investigate what happens when we allow ε to depend on n. Note that the estimate above makes sense only for $\varepsilon >> (\log n)^{-1/2}$. (Otherwise the "trivial" estimate $\sqrt{k}$ is better). Going back to (1) we see that, fixing k, with the same choice of C and with the choice of ε which solves $k\log k = \eta n^{1-K\varepsilon^2}$ ($\varepsilon = \sqrt{\frac{1}{K'}(1-\frac{\log k}{\log n})}$), we get that

$$d(Y, \ell_2^k) \le \frac{K}{\sqrt{1-\frac{\log k}{\log n}}}\sqrt{\frac{k}{\log n}} = K\sqrt{\frac{k}{\log(n/k)}} \quad .$$

■

Remarks. 1. For $k = K\log n$ we get the isomorphic (as oppose to almost isometric) quantitative version of Dvoretzky's theorem pertain to in the title: *Every n-dimensional normed space contains, for every $K \ge 1$ a subspace of dimension $[K\log n]$ whose distance from a Hilbert space depends only on K. More precisely the distance is bounded by a universal constant times $\sqrt{K}$.*

2. The result of Theorem 2 is best possible, except for the choise of the constant K. Indeed, for any k, any k-dimensional subspace of ℓ_∞^n has distance larger than $\eta\sqrt{k/\log(1+n/k)}$ from a Hilbert space, for some absolute $\eta > 0$ (see [FJ] for a somewhat weaker result and [CP] and [G] for the full one).

3. In the case the space X is ℓ_∞^n, the theorem above was proved in [FJ] (see the Example on page 96 and also [CP] and [G]). Note however that our proof gives more: In the case of ℓ_∞^n there is no need to refer to [BS] and the proof of Theorem 2 shows that the subspace we get is "random". This solves the Question on page 97 of [FJ] in the negative: For $k \ge K\log n$ a random k-dimensional subspace of ℓ_∞^n has distance from ℓ_2^k bounded by $K\sqrt{\frac{k}{\log(n/k)}}$.

4. Note that only a simpler version of Theorem 2 of [BS] (with "given $\delta \in (0,1)$" replaced by "there exists $\delta \in (0,1)$") is needed for the proof of Theorem 2 above. As indicated in Remark (6) in [BS], this version is much simpler to prove. [ST] contains a simpler proof

of the main results of [BS]. Again if one is only interested in the "there exists $\delta \in (0,1)$" version, the proof there takes even a simpler form. (One needs to use only the "classical" version of the Sauer-Shelah lemma rather than the "isomorphic" version.)

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