An editorial comment on the preceding paper

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I would like to present a more direct proof of Theorem 1 of the preceding paper [AV] of Arias-de-Renya and Villa. I shall give the details of the proof for the most interesting case of $p = 1$ and remark at the end how to prove in a similar way the case $1 < p < 2$. I follow the notations of [AV]. Recall first a theorem of Talagrand [Tal], an equivalent form of which is also used in [AV].

Theorem *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function satisfying*

$$|f(x) - f(y)| \le \alpha \|x - y\|_2 \quad \textit{and} \quad |f(x) - f(y)| \le \beta \|x - y\|_1$$

Then

$$\gamma_1^n \left(|f(x) - \mathbf{E}f| > r \right) \le C \exp(-\delta \min(r/\beta, r^2/\alpha^2)).$$

In particular,

$$\gamma_1^n \left(\left| \frac{\sum |x_i|}{n} - 1 \right| > r \right) \le C \exp(-\delta n \min(r, r^2)).$$

Now if $f : \partial B_1^n :\to R$ satisfy $|f(x) - f(y)| \le \|x - y\|_1$ extend it to a function F on $\mathbb{R}^n$ with the same Lip constant with respect to $\|\cdot\|_1$ and note that $|F(x) - F(y)| \le \sqrt{n}\|x - y\|_2$. Put $S = \sum |x_i|, T = \sum |y_i|$. Then, considering (x, y) as an element of $\mathbb{R}^{2n}$,

$$\gamma_1^{2n}\Big(\Big|F\Big(\frac{x}{S}\Big) - F\Big(\frac{y}{T}\Big)\Big| > 3r\Big) \le 2\gamma_1^n\Big(\Big|F\Big(\frac{x}{S}\Big) - F\Big(\frac{x}{n}\Big)\Big| > r\Big) + \gamma_1^{2n}\Big(\Big|F\Big(\frac{x}{n}\Big) - F\Big(\frac{y}{n}\Big)\Big| > r\Big).$$

By the $\|\cdot\|_1$-Lipschitsity of F, we get from the Theorem above that, for all $0 < r < 1$,

$$\gamma_1^n\Big(\Big|F\Big(\frac{x}{S}\Big) - F\Big(\frac{x}{n}\Big)\Big| > r\Big) \le \gamma_1^n\Big(\Big|1 - \frac{S}{n}\Big| > r\Big) \le C \exp(-\delta n r^2).$$

While

$$\gamma_1^{2n}\Big(\Big|F\Big(\frac{x}{n}\Big) - F\Big(\frac{y}{n}\Big)\Big| > r\Big) \le C \exp(-\delta n r^2)$$

since for $F(\frac{\cdot}{n})$ $\beta = 1/n$ and $\alpha = 1/\sqrt{n}$.

Now, if x is distributed according to γ_1^n then x/S is distributed according to the normalized surface measure on the sphere of ℓ_1^n. This is easy and known fact. The papers [MP] and [SZ] contain this and also a similar fact for γ_p^n (in this case the relevant measure is not the surface

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measure but the one induced fron the Lebegue measure on the full ball - the measure of a set A on the sphere is the normalized Lebegue measure of $[0,1] \times A$). In [SZ] this fact is used in a similar way to the one here. It follows that if X and Y are independent random variables distributed uniformly on the sphere of ℓ_1^n then for all $r \le 2$,

$$\text{Prob}(|f(X) - f(Y)| > r) \le C' e^{-\delta' r^2 n}$$

from which the analog of Theorem 1 of [AV] for the sphere of ℓ_1^n easily follows. Going from the sphere to the ball is again easy. The proof for $1 < p < 2$ is very similar: use the relation, mentioned above, between γ_p^n and the normalized Lebesgue measure on the ball of ℓ_p^n and replace the use of the Theorem abovewith another theorem of Talagrand also used in [AV] (see (1) there). Again it is more convenient to state this theorem in its concentration form: *There are positive constants C and δ such that if $f : I\!R^n \to I\!R$ has Lipschits constant 1 with respect to $\|\cdot\|_p$, $1 \le p \le 2$, then*

$$\gamma_p^n(|f(x) - \mathbf{E}f| > r) \le C \exp(-\delta \min(r^p, r^2 n^{1-2/p})).$$

References

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