Extremal configurations for moments of sums of independent positive random variables

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Abstract

We find the extremal configuration for the p-moment of sums of independent positive random variables while constraining the sum of the expectations of the random variables and the sum of their p-moments.

1 Introduction

If $X_1, X_2, \ldots, X_n$ are non-negative independent random variables and $p > 1$ then

$$\max\{\sum_{i=1}^{n} \mathbb{E}X_i, (\sum_{i=1}^{n} \mathbb{E}X_i^p)^{1/p}\} \le (\mathbb{E}(\sum_{i=1}^{n} X_i)^p)^{1/p} \le K_p \max\{\sum_{i=1}^{n} \mathbb{E}X_i, (\sum_{i=1}^{n} \mathbb{E}X_i^p)^{1/p}\} \quad (1)$$

where K_p depends only on p. This and a similar and better known inequality for mean zero random variables were proved by Rosenthal in [Ro]. These inequalities and their variants were quite useful in both Probability and Functional Analysis. The left inequality is easy and the constant one in it is easily seen to be best possible. The constant in the right hand side inequality has been shown in[JSZ] to be of order $p/\log p$. The latest result in this direction is contained in [La] and gives a version of the inequality above (and more importantly of its version for symmetric variables) with constants independent of p. Of course the term $\max\{\sum_{i=1}^{n} \mathbb{E}X_i,\ \sum_{i=1}^{n} \mathbb{E}X_i^p\}$ is replaced in [La] with another (equivalent) quantity still depending on the individual X_i-s only. The trigger to the present note is the recent paper [BT] in which a related extremal problem is treated.

In this paper we find, for each $0 < A, B < \infty$, the supremum of $\mathbb{E}(\sum X_i)^p$ subject to: $\{X_i\}$ are non-negative, independent random variables with $\sum \mathbb{E}X_i = A$ and $\sum \mathbb{E}X_i^p = B$. We also find the random variables which asymptotically achieve this supremum. As a corollary one finds the actual best possible constant K_p in (1). The determination of this constant has previously been done in [IS] (see also [dlPIS] for a more general result).

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The similar problem for symmetric random variables was treated and solved in [Ut] and [FHJSZ]. The proofs here are based on ideas we developed in collaboration with Figiel, Hitczenko, Johnson and Zinn in the preliminary stages of [FHJSZ] (where we treated the symmetric case). The proofs that actually appear in [FHJSZ] are quite different from these preliminary ideas. Now seem to us to be a good opportunity to expose these ideas which didn't appear in print. We see this as the main purpose of this note. Moreover, dealing with the positive rather than the symmetric case turn out to be substantially simpler. We find the proof quite interesting and we hope these ideas will find more applications.

Our main result is

Theorem 1 *The supremum of $\mathbb{E}(\sum_{i=1}^n X_i)^p$ over all n and all independent non-negative random variables X_i satisfying*

$$\sum_{i=1}^n \mathbb{E}X_i = A \quad \textit{and} \quad \sum_{i=1}^n \mathbb{E}X_i^p = B$$

is

$$\mathbb{E}\Big(\Big(\frac{B}{A}\Big)^{1/(p-1)} P_{A^{p/(p-1)}/B^{1/(p-1)}}\Big)^p \quad \textit{for} \quad p \geq 2$$

where P_μ denotes a Poisson random variable with parameter μ, and

$$A^p + B \quad \textit{for} \quad 1 < p \leq 2.$$

Next we show how to (asymptotically) achieve the supremum. The proof of the theorem will show that this is basically the only way the supremum can be achieved. For $p \geq 2$ and large enough k let X_i, $i = 1, \dots, k$, be independent identically distributed random variables each taking the values

$$(B/A)^{1/(p-1)} \quad \text{with probability} \quad A^{p/(p-1)}/kB^{1/(p-1)}$$

and

$$0 \quad \text{with probability} \quad 1 - A^{p/(p-1)}/kB^{1/(p-1)}.$$

Then

$$\sum_{i=1}^k \mathbb{E}X_i = A, \quad \sum_{i=1}^k \mathbb{E}X_i^p = B$$

and

$$\mathbb{E}(\sum_{i=1}^k X_i)^p \overset{k\to\infty}{\longrightarrow} \mathbb{E}\Big(\Big(\frac{B}{A}\Big)^{1/(p-1)} P_{A^{p/(p-1)}/B^{1/(p-1)}}\Big)^p.$$

For $1 < p \leq 2$, let $X_1, \dots, X_k$ denote independent random variables each taking the value $1/k$ with probability A (assuming as we may that $A \leq 1$) and the value 0 with probability $1 - A$, and let Y be a random variable independent of the X_i-s and taking the value $k^{1/p}$

with probability B/k (assuming $k \geq B$) and the value 0 with probability $1 - B/k$. Then $\sum_{i=1}^{k} \mathbb{E}X_i + \mathbb{E}Y \stackrel{k\to\infty}{\longrightarrow} A$, $\sum_{i=1}^{k} \mathbb{E}X_i^p + \mathbb{E}Y^p \stackrel{k\to\infty}{\longrightarrow} B$ and $\mathbb{E}(\sum_{i=1}^{k} X_i + Y)^p \stackrel{k\to\infty}{\longrightarrow} A^p + B$.

By taking $A^p = B$ in Theorem 1 it is easy to get the following corollary, previously obtained in [dlPIS],

Corollary 1 *The best value of K_p is $(\mathbb{E}(P_1)^p)^{1/p}$ for $p \geq 2$ and $2^{1/p}$ for $1 < p \leq 2$.*

2 proof of the theorem

The proof of the theorem has two main ingredients: In proposition 1 and Corollary 2 we reduce the problem to that of a maximization of the p-th moment of linear combination of independent Poisson random variables subject to certain constrains on the coefficients and the parameters involved. This reduces the problem to what looks like a sophisticated Calculus problem. After calculating the needed partial derivatives in proposition 2 we solve a related and somewhat stronger maximization problem in the Main Proposition 3. The proof involves a certain trick - adding a seemingly superfluous constrain.

Proposition 1 *Let $p \geq 1$, let A and B be two disjoint events in a probability space of equal probability and let $\tilde{A}$ and $\tilde{B}$ be two independent events of the same probability as A. Let a, b, c be non-negative numbers then*

$$\mathbb{E}(c + a1_A + b1_B)^p \leq \mathbb{E}(c + a1_{\tilde{A}} + b1_{\tilde{B}})^p.$$

We shall present two proofs.

Proof 1: Put $\alpha = P(A)$. Then,

$$\mathbb{E}(c + a1_A + b1_B)^p = (1 - 2\alpha)c^p + \alpha((a + c)^p + (b + c)^p)$$

and

$$\mathbb{E}(c + a1_{\tilde{A}} + b1_{\tilde{B}})^p = (1 - \alpha)^2 c^p + \alpha(1 - \alpha)((a + c)^p + (b + c)^p) + \alpha^2(a + b + c)^p.$$

Subtracting, cancelling α and dividing by c (assuming as we may it is positive) we see that it is enough to show that

$$f(a, b) = 1 + (1 + a + b)^p - (1 + a)^p - (1 + b)^p \geq 0$$

for all $a, b \geq 0$. For all $b \geq 0$ $g_b(a) = (1 + a + b)^p - (1 + a)^p$ is increasing in a so

$$f(a, b) = g_b(a) - g_b(0) \geq 0.$$

■

Proof 2: This proof proves a more general statement: $\tilde{A}$ and $\tilde{B}$ can be *any* two events of that same probability as A, not necessarily independent. Also, one can replace the power p function with any convex function.

By [JS, Lemma 4], $a1_A + b1_B$ is in the convex hull of all random variables on the same probability space which have the same distribution as $a1_{\tilde{A}} + b1_{\tilde{B}}$. Clearly this implies that $c + a1_A + b1_B$ is in the convex hull of all random variables on the same probability space which have the same distribution as $c + a1_{\tilde{A}} + b1_{\tilde{B}}$. The result now follows by convexity. ■

Corollary 2 *For $p \geq 1$ and $A, B > 0$ put $K_p(A, B) = \sup(\mathbb{E}(\sum X_i)^p)^{1/p}$ where the* sup *is taken over all independent random variables $X_i \geq 0$ with $\sum \mathbb{E}(X_i) \leq A$ and $\sum \mathbb{E}(X_i)^p \leq B^p$. Then*

$$K_p(A, B) = \sup(\mathbb{E}(\sum a_i P_{\mu_i})^p)^{1/p}$$

where the sup *is taken over all $a_i, \mu_i \geq 0$ with $\sum a_i \mu_i \leq A$, $\sum a_i^p \mu_i \leq B^p$ and independent random variables P_{μ_i} where P_{μ_i} has Poisson distribution with parameter μ_i.*

Proof: Let $X_i = \sum_{j=1}^{m_i} \alpha_{i,j} 1_{A_{i,j}}$, $i = 1, \ldots, n$ be a finite sequence of simple non-negative random variables (i.e., for each i, $\{A_{i,j}\}_{j=1}^{m_i}$ are disjoint) satisfying the constrains:

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} \alpha_{i,j} P(A_{i,j}) = A \quad \text{and} \quad \sum_{i=1}^{n} \sum_{j=1}^{m_i} \alpha_{i,j}^p P(A_{i,j}) = B.$$

Let $Y_{i,j}$ have the same distribution as $\alpha_{i,j} 1_{A_{i,j}}$ for each i, j while $Y_{i,j}$, $i = 1, \ldots, n, j = 1, \ldots, m_i$ are independent. Then $\sum_{i,j} \mathbb{E} Y_{i,j} = A$, $\sum_{i,j} \mathbb{E} Y_{i,j}^p = B$ and, by Proposition 1,

$$\mathbb{E}(\sum_{i,j} \alpha_{i,j} Y_{i,j})^p \geq \mathbb{E}(\sum_i X_i)^p.$$

Using Proposition 1 again we can replace each $Y_{i,j}$ with a Poisson random variable $P_{\mu_{i,j}}$, with $\mu_{i,j} = P(A_{i,j})$ while $\mathbb{E}(\sum_{i,j} \alpha_{i,j} P_{\mu_{i,j}})^p \geq \mathbb{E}(\sum_{i,j} \alpha_{i,j} Y_{i,j})^p$. Indeed, divide each $A_{i,j}$ into k disjoint sets of equal measure $A_{i,j,\ell}$, $\ell = 1, \ldots, k$. Let $\tilde{A}_{i,j,\ell}$, $i = 1, \ldots, n, j = 1, \ldots, m_i, \ell = 1, \ldots, k$ be independent with $P(\tilde{A}_{i,j,\ell}) = P(A_{i,j,\ell})$. Then

$$\mathbb{E}(\sum_{i,j} \alpha_{i,j} P_{\mu_{i,j}})^p = \lim_{k \to \infty} \mathbb{E}(\sum_{i,j,\ell} \alpha_{i,j,\ell} 1_{\tilde{A}i,j,\ell})^p \geq \mathbb{E}(\sum_{i,j} \alpha_{i,j} Y_{i,j})^p.$$

This proves that $K_p(A, B) \leq \sup(\mathbb{E}(\sum a_i P_{\mu_i})^p)^{1/p}$. The other inequality follows easily by presenting each P_{μ_i} as a limit of appropriate sum of independent indicator functions. Since we are going to use only the inequality proved above, we leave the details to the reader. ■

In the next proposition we compute some partial derivatives needed in the proof of the following Main Proposition.

Proposition 2 *Let X be a positive random variable which have finite p-th moment, $p > 1$. Let P_μ denote an independent Poisson variable with parameter μ. Then, for all $a > 0$, $\mu > 0$,*

$$\frac{\partial}{\partial a}\mathbb{E}(aP_\mu + X)^p = p\mu\mathbb{E}(a + aP_\mu + X)^{p-1} \tag{2}$$

and

$$\frac{\partial}{\partial \mu}\mathbb{E}(aP_\mu + X)^p = pa\mathbb{E}(aU + aP_\mu + X)^{p-1} \tag{3}$$

where U is a random variable uniformly distributed over $[0,1]$ and independent of (P_μ, X).

Proof: Assume first that $1 < p < 2$. Recall that for every positive variable Y with p-th moment

$$\mathbb{E}Y^p = c_p \int_0^\infty \frac{\varphi_Y(t) - 1 - it\mathbb{E}Y}{t^{p+1}} dt$$

and

$$\mathbb{E}Y^{p-1} = c_{p-1} \int_0^\infty \frac{1 - \varphi_Y(t)}{t^p} dt$$

where $c_p = (\int_0^\infty \frac{e^{it}-1-it}{t^{p+1}} dt)^{-1}$, $c_{p-1} = (\int_0^\infty \frac{1-e^{it}}{t^p} dt)^{-1}$ and $\varphi_Y(t) = \mathbb{E}e^{itY}$. Note that integration by parts gives $\frac{-ic_p}{c_{p-1}} = p$.

Differentiating formally,

$$\begin{aligned} \frac{\partial}{\partial a}\mathbb{E}(aP_\mu + X)^p &= c_p \frac{\partial}{\partial a} \int_0^\infty \frac{e^{\mu(e^{ita}-1)}\varphi_X(t) - 1 - it(a\mu + \mathbb{E}X)}{t^{p+1}} dt \\ &= c_p \int_0^\infty \frac{it\mu e^{ita} e^{\mu(e^{ita}-1)}\varphi_X(t) - it\mu}{t^{p+1}} dt \\ &= -i\mu c_p \int_0^\infty \frac{1 - e^{ita} e^{\mu(e^{ita}-1)}\varphi_X(t)}{t^p} dt \\ &= \frac{-i\mu c_p}{c_{p-1}} \mathbb{E}(a + aP_\mu + X)^{p-1}. \end{aligned} \tag{4}$$

It is easy to justify the differentiation under the integral sign.

Similarly,

$$\begin{aligned} \frac{\partial}{\partial \mu}\mathbb{E}(aP_\mu + X)^p &= c_p \frac{\partial}{\partial \mu} \int_0^\infty \frac{e^{\mu(e^{ita}-1)}\varphi_X(t) - 1 - it(a\mu + \mathbb{E}X)}{t^{p+1}} dt \\ &= c_p \int_0^\infty \frac{(e^{ita}-1) e^{\mu(e^{ita}-1)}\varphi_X(t) - ita}{t^{p+1}} dt \\ &= -iac_p \int_0^\infty \frac{1 - \frac{e^{ita}-1}{ia} e^{\mu(e^{ita}-1)}\varphi_X(t)}{t^p} dt \\ &= \frac{-iac_p}{c_{p-1}} \mathbb{E}(aU + aP_\mu + X)^{p-1}. \end{aligned} \tag{5}$$

This concludes the proof for $1 < p < 2$. For the other values of p one can use a similar proof, using

$$\mathbb{E}Y^p = c_p \int_0^\infty \frac{\varphi_Y(t) - \sum_{j=0}^k \frac{\mathbb{E}(itY)^j}{j!}}{t^{p+1}} dt \quad \text{for} \quad k < p < k+1, \quad k \text{ an integer}$$

and

$$\mathbb{E}Y^k = (-i)^k \frac{\partial^k \mathbb{E}\varphi_Y(t)}{\partial t^k}_{|t=0}, \quad k \text{ an integer.}$$

Alternatively, one can notice first that the four quantities:

$$\mathbb{E}\frac{\partial}{\partial a}(aP_\mu + X)^q, \quad aq\mathbb{E}(a + aP_\mu + X)^{q-1}$$

and

$$\mathbb{E}\frac{\partial}{\partial \mu}(aP_\mu + X)^q, \quad \mu q\mathbb{E}(aU + aP_\mu + X)^{q-1}$$

are well defined for all complex q in an open strip containing $1 < Real(q) \le p$ and are analytic functions of q there (we omit the justification). Since the first two coincide on the interval $(0,1)$ they coincide in the whole strip. The same holds for the last two quantities. Finally, it is easy to justify changing the order of $\mathbb{E}$ and the differentiation. ■

The next proposition is the main one.

Proposition 3 (Main Proposition) *Given a positive random variable X with p-th moment $p > 1$, the supremum of $\mathbb{E}(aP_\mu + bP_\nu + X)^p$ subject to $a, b, \mu, \nu > 0$, $a\mu + b\nu = A$, $a^p\mu + b^p\nu = B$ and where P_μ, P_ν are Poisson variables with the indicated parameters and P_μ, P_ν and X are independent, is given by*

$$\mathbb{E}\Big(\Big(\frac{B}{A}\Big)^{1/(p-1)} P_{A^{p/(p-1)}/B^{1/(p-1)}} + X\Big)^p \quad \textit{for} \quad p \ge 2$$

and

$$\mathbb{E}(A + X)^p + B \quad \textit{for} \quad 1 < p \le 2.$$

Proof: It is tempting to use Proposition2 to find the supremum by the method of Lagrange multipliers. We were not successful trying to do it. Moreover, as we shall see below, in the case $1 < p < 2$ the supremum is not attained at an inner point of the domain. Instead, we first add another constrain, $a\mu = C$ $(0 < C < A)$. Solving for b, μ and ν in terms of a we get:

$$\mu = \frac{C}{a}, \; b = \Big(\frac{B - Ca^{p-1}}{A - C}\Big)^{1/(p-1)}, \; \nu = \frac{(A-C)^{p/(p-1)}}{(B - Ca^{p-1})^{1/(p-1)}}. \tag{6}$$

These quantities are well defined and positive as long as $a < (B/C)^{1/(p-1)}$. We thus need to find the supremum of

$$g(a) = \mathbb{E}(aP_\mu + bP_\nu + X)^p$$

(where μ, b, ν are given by (6)) in the range $0 < a < (B/C)^{1/(p-1)}$. For that we compute the derivatives of μ, b, ν with respect to a:

$$\frac{d\mu}{da} = \frac{-C}{a^2}, \ \frac{db}{da} = \frac{-Ca^{p-2}}{(B - Ca^{p-1})^{(p-2)/(p-1)}(A - C)^{1/(p-1)}}, \ \frac{d\nu}{da} = \frac{C(A - C)^{p/(p-1)}a^{p-2}}{(B - Ca^{p-1})^{p/(p-1)}}, \tag{7}$$

and use them and the formulas (2),(3) to find $g'(a)$. Denote $Y = aP_\mu + bP_\nu + X$ then, by the chain rule,

$$\begin{aligned} g'(a) &= \frac{pC}{a}\mathbb{E}(a + Y)^{p-1} - \frac{pC}{a}\mathbb{E}(aU + Y)^{p-1} \\ &\quad - \frac{pC\nu a^{p-2}}{(B - Ca^{p-1})^{(p-2)/(p-1)}(A - C)^{1/(p-1)}}\mathbb{E}(b + Y)^{p-1} \\ &\quad + \frac{pC(A - C)^{p/(p-1)}ba^{p-2}}{(B - Ca^{p-1})^{p/(p-1)}}\mathbb{E}(bU + Y)^{p-1} \end{aligned} \tag{8}$$

where b and ν are given by (6) and U is a standard uniform random variable independent of Y. Using (6) again we can write (8) as

$$\begin{aligned} g'(a) &= \frac{pC}{a}(\mathbb{E}(a + Y)^{p-1} - \mathbb{E}(aU + Y)^{p-1}) \\ &\quad - \frac{pCa^{p-2}}{b^{p-1}}(\mathbb{E}(b + Y)^{p-1} - \mathbb{E}(bU + Y)^{p-1}) \end{aligned}$$

or

$$g'(a) = pCa^{p-2}\Big(\mathbb{E}\Big(1 + \frac{Y}{a}\Big)^{p-1} - \mathbb{E}\Big(U + \frac{Y}{a}\Big)^{p-1} - \mathbb{E}\Big(1 + \frac{Y}{b}\Big)^{p-1} + \mathbb{E}\Big(U + \frac{Y}{b}\Big)^{p-1}\Big). \tag{9}$$

We now separate between different ranges of p. First, For $p = 2$ it is easy to see that $\mathbb{E}(aP_\mu + bP_\nu + X)^2 = A^2 + B + \mathbb{E}X^2 + 2A\mathbb{E}X$ for all a, b, μ, ν satisfying the two original constrains. It is also easy to check that this coincides with the (two) conclusion(s) of the Proposition for $p = 2$.

If $p > 2$ then we claim that $g'(a) > 0$ if and only if $a < b$ (equivalently, if and only if $a < (B/A)^{1/(p-1)}$). Indeed, by (9), it is enough to show that

$$h(x) = (1 + x)^{p-1} - \mathbb{E}(U + x)^{p-1} = (1 + x)^{p-1} - \int_0^1 (t + x)^{p-1}dt$$

is an increasing function on $(0, \infty)$. Now

$$h'(x) = (p - 1)\int_0^1 ((1 + x)^{p-2} - (t + x)^{p-2})dt > 0$$

since $p > 2$. (For later use note that, if $1 < p < 2$, h is decreasing on $(0, \infty)$.) This proves that the maximum of g is attained for $a = b = (B/A)^{1/(p-1)}$. The corresponding values of μ and ν are

$$\mu = C(A/B)^{1/(p-1)} \quad \text{and} \quad \nu = (A - C)(A/B)^{1/(p-1)}.$$

For these values, the distribution of $aP_\mu + bP_\nu$ is the same as that of

$$\Big(\frac{B}{A}\Big)^{1/(p-1)} P_{A^{p/(p-1)}/B^{1/(p-1)}}$$

(using the fact that the sum of two independent Poisson random variables is a Poisson variable whose parameter is the sum of the parameters of the original variables). Thus, the supremum of $\mathbb{E}(aP_\mu + bP_\nu + X)^p$ given the three constarains is

$$\mathbb{E}\Big(\Big(\frac{B}{A}\Big)^{1/(p-1)} P_{A^{p/(p-1)}/B^{1/(p-1)}} + X\Big)^p.$$

Since C does not appear in this last quantity the same holds under the two original constrains.

The case $1 < p < 2$ is a bit more complicated. As was remarked above, h is decreasing on $(0, \infty)$ and thus the supremum of g is given at one of the end points; i.e., as a tends to 0 or to $(B/C)^{1/(p-1)}$. In the first case the distribution of aP_μ tends to that of the constant C, as is easy to see from the behavior of the characteristic function: $\mathbb{E}e^{itaP_\mu} = e^{\mu(e^{ita}-1)} \to e^{itC}$ (since $a\mu = C$ and $a \to 0$). In the second case, since $b \to 0$ we get similarly that bP_ν tends to the constant $A - C$. We thus get that the supremum of $g(a)$ is the maximum between

$$\mathbb{E}\Big(C + \Big(\frac{B}{A-C}\Big)^{1/(p-1)} P_{(A-C)^{p/(p-1)}/B^{1/(p-1)}} + X\Big)^p$$

and

$$\mathbb{E}\Big(A - C + \Big(\frac{B}{C}\Big)^{1/(p-1)} P_{C^{p/(p-1)}/B^{1/(p-1)}} + X\Big)^p.$$

It follows that the supremum of $\mathbb{E}(aP_\mu + bP_\nu + X)^p$ subject to $a, b, \mu, \nu > 0$, $a\mu + b\nu = A$, $a^p\mu + b^p\nu = B$ is

$$\sup_{0<C<A} \mathbb{E}\Big(A - C + \Big(\frac{B}{C}\Big)^{1/(p-1)} P_{C^{p/(p-1)}/B^{1/(p-1)}} + X\Big)^p.$$

To find this supremum denote

$$\varphi(C) = \mathbb{E}\Big(A - C + \Big(\frac{B}{C}\Big)^{1/(p-1)} P_{C^{p/(p-1)}/B^{1/(p-1)}} + X\Big)^p.$$

By Proposition 2,

$$\begin{aligned}
\varphi'(C) &= -p\mathbb{E}\Big(A-C+\Big(\frac{B}{C}\Big)^{1/(p-1)}P_{C^{p/(p-1)}/B^{1/(p-1)}}+X\Big)^{p-1}\\
&\quad -\frac{p}{p-1}\mathbb{E}\Big(\Big(\frac{B}{C}\Big)^{1/(p-1)}+A-C+\Big(\frac{B}{C}\Big)^{1/(p-1)}P_{C^{p/(p-1)}/B^{1/(p-1)}}+X\Big)^{p-1}\\
&\quad +\frac{p^2}{p-1}\mathbb{E}\Big(\Big(\frac{B}{C}\Big)^{1/(p-1)}U+A-C+\Big(\frac{B}{C}\Big)^{1/(p-1)}P_{C^{p/(p-1)}/B^{1/(p-1)}}+X\Big)^{p-1}
\end{aligned}$$

where U is a standard uniform random variable independent of the other ones. We would like to show that φ is a decreasing function. For that it would be enough to show that for any $s,t>0$

$$\psi(s,t)=-t^{p-1}-\frac{1}{p-1}(s+t)^{p-1}+\frac{p}{p-1}\mathbb{E}(sU+t)^{p-1}<0.$$

(Take $s=(\frac{B}{C})^{1/(p-1)}$ and $t=A-C+(\frac{B}{C})^{1/(p-1)}P_{C^{p/(p-1)}/B^{1/(p-1)}}+X$ in the previous equation.) This is simple:

$$\psi(s,t)=-t^{p-1}-\frac{1}{p-1}(s+t)^{p-1}+\frac{1}{p-1}\frac{(s+t)^p-t^p}{s}=-t^{p-1}+\frac{t}{p-1}\frac{(s+t)^{p-1}-t^{p-1}}{s}.$$

The concavity of the function x^{p-1} implies that, for each $t>0$, ψ in a decreasing function of s. Since $\lim_{s\to 0^+}\psi(s,t)=0$ we get that ψ is negative. We conclude that

$$\sup\mathbb{E}(aP_\mu+bP_\nu+X)^p=\lim_{C\to 0}\mathbb{E}\Big(A-C+\Big(\frac{B}{C}\Big)^{1/(p-1)}P_{C^{p/(p-1)}/B^{1/(p-1)}}+X\Big)^p.$$

When $C\to 0$ the support of $P_{C^{p/(p-1)}/B^{1/(p-1)}}$ tends to zero. Also, using to the definition of the Poisson distribution, it is easy to see that $\lim_{C\to 0}\mathbb{E}\Big(\Big(\frac{B}{C}\Big)^{1/(p-1)}P_{C^{p/(p-1)}/B^{1/(p-1)}}\Big)^p=B$. Consequently,

$$\sup\mathbb{E}(aP_\mu+bP_\nu+X)^p=\mathbb{E}(A+X)^p+B.$$

■

The proof of Theorem 1 is a simple consequence of Corollary 2 and Proposition 3.

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