

# The random version of Dvoretzky's theorem in $\ell_\infty^n$

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## Abstract

We show that with “high probability” a section of the  $\ell_\infty^n$  ball of dimension  $k \leq c\varepsilon \log n$  ( $c > 0$  a universal constant) is  $\varepsilon$  close to a multiple of the Euclidean ball in this section. We also show that, up to an absolute constant the estimate on  $k$  cannot be improved.

## 1 Introduction

Milman's version of Dvoretzky's theorem states that:

*There is a function  $c(\varepsilon) > 0$  such that for all  $k \leq c(\varepsilon) \log n$ ,  $\ell_2^k$   $(1 + \varepsilon)$ -embeds into any normed space of dimension  $n$ .*

See [Dv] for the original theorem of Dvoretzky (in which the dependence of  $k$  on  $n$  is weaker), [Mi] for Milman's original work, and [MS] and [Pi] for expository outlets of the subject (there are many others). It would be important for us to notice that the proof(s) of the theorem above actually give more: The vast majority of subspaces of the stated dimension are  $(1 + \varepsilon)$ -isomorphic to  $\ell_2^k$ .

The dependence of  $k$  on  $n$  in the theorem above is known to be best possible (for  $\ell_\infty^n$ ) but the dependence on  $\varepsilon$  is far from being understood. The best known estimate is  $c(\varepsilon) \geq \frac{c\varepsilon}{(\log \frac{1}{\varepsilon})^2}$  given in [Sc] (here and elsewhere in this paper  $c$  and  $C$  denote positive universal constants). However, the proof in [Sc] does not give the additional information that *most* subspaces are  $(1 + \varepsilon)$ -isomorphic to  $\ell_2^k$ . If one also want this requirement then the best estimate for  $c(\varepsilon)$  that was known was  $c(\varepsilon) \geq c\varepsilon^2$  ([Go]).

As an upper bound for  $c(\varepsilon)$  one gets  $C/\log \frac{1}{\varepsilon}$  for some universal  $C$ . Indeed, if  $\ell_2^k$   $(1 + \varepsilon)$  embed into  $\ell_\infty^n$  then  $k \leq C \log n / \log \frac{1}{\varepsilon}$ . This is also the right order of  $k$  in the  $\ell_\infty$  case: If  $k \leq c \log n / \log \frac{1}{\varepsilon}$  then  $\ell_2^k$   $(1 + \varepsilon)$  embed into  $\ell_\infty^n$ .

We show here that, in the  $\ell_\infty$  case, if one is interested in the probabilistic statement of Dvoretzky theorem (i.e, that the vast majority of subspaces of  $\ell_\infty^n$  of a certain dimension are  $(1 + \varepsilon)$ -isomorphic to Euclidean spaces) then the right estimate for  $c(\varepsilon)$  is  $c\varepsilon$ .

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**Theorem 1** For  $k < c\varepsilon \log n$ , with probability  $> 1 - e^{-ck}$ , the  $\ell_\infty^n$  norm and a multiple of the  $\ell_2^n$  norm are  $1 + \varepsilon$  equivalent on a  $k$  dimensional subspace. Moreover, this doesn't hold anymore for  $k$  of higher order. i.e., For every  $a$  there is an  $A$  such that if, with probability larger than  $1 - e^{-ak}$ , a  $k$  dimensional subspace satisfies that the ratio between the  $\ell_\infty^n$  norm and a multiple of the  $\ell_2^n$  norm are  $1 + \varepsilon$  equivalent for all vectors in the subspace, then  $k \leq A\varepsilon \log n$ .

## 2 Computation of the concentration of the max norm

Let  $g_1, g_2, \dots$  be a sequence of standard independent Gaussain variables. fix  $n$  and let  $M$  be the median of  $\|(g_1, g_2, \dots, g_n)\|_\infty$ . In this section we compute some fine estimates on the probability of deviation of  $\|(g_1, g_2, \dots, g_n)\|_\infty$  from  $M$ .

**Claim 1**

$$(1 - 2^{-1/n}) \frac{\sqrt{\pi}M}{\sqrt{2}} \leq e^{-M^2/2} \leq (1 - 2^{-1/n}) \frac{\sqrt{\pi}(M+1)}{\sqrt{2}(1 - e^{-\frac{1}{2}}e^{-M})}. \quad (1)$$

**Proof:**

$$\frac{1}{2} = P(\max_{1 \leq i \leq n} |g_i| < M) = \left(1 - \sqrt{\frac{2}{\pi}} \int_M^\infty e^{-s^2/2} ds\right)^n.$$

Consequently,

$$\begin{aligned} 1 - 2^{-1/n} &= \sqrt{\frac{2}{\pi}} \int_M^\infty e^{-s^2/2} ds \geq \sqrt{\frac{2}{\pi}} \frac{1}{M+1} \int_M^{M+1} se^{-s^2/2} ds \\ &\geq \sqrt{\frac{2}{\pi}} \frac{1}{M+1} e^{-M^2/2} (1 - e^{-\frac{1}{2}}e^{-M}), \end{aligned} \quad (2)$$

or

$$e^{-M^2/2} \leq (1 - 2^{-1/n}) \frac{\sqrt{\pi}(M+1)}{\sqrt{2}(1 - e^{-\frac{1}{2}}e^{-M})}. \quad (3)$$

Similarly,

$$1 - 2^{-1/n} = \sqrt{\frac{2}{\pi}} \int_M^\infty e^{-s^2/2} ds \leq \sqrt{\frac{2}{\pi}} \frac{1}{M} \int_M^\infty se^{-s^2/2} ds \leq \sqrt{\frac{2}{\pi}} \frac{e^{-M^2/2}}{M},$$

or

$$e^{-M^2/2} \geq (1 - 2^{-1/n}) \frac{\sqrt{\pi}M}{\sqrt{2}}. \quad (4)$$

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**Claim 2**

$$\frac{\log 2}{4 + \log 2} e^{-3\varepsilon M^2/2} \leq P(\max_{1 \leq i \leq n} |g_i| > (1 + \varepsilon)M) \leq \log 2(1 + o(1))e^{-\varepsilon M^2} \quad (5)$$

where  $o(1)$  means  $a(n)$  with  $a(n) \rightarrow 0$  as  $n \rightarrow \infty$  independently of  $\varepsilon$ .

**Proof:** (3) implies

$$\begin{aligned} P(\max_{1 \leq i \leq n} |g_i| > (1 + \varepsilon)M) &\leq \sqrt{\frac{2}{\pi}} \frac{n}{(1 + \varepsilon)M} e^{-(1 + \varepsilon)^2 M^2 / 2} \\ &\leq \frac{n}{(1 + \varepsilon)M} (1 - 2^{-1/n}) \frac{M+1}{1 - e^{-\frac{1}{2}e^{-M}}} e^{-\varepsilon M^2} e^{-\varepsilon^2 M^2 / 2} \end{aligned} \quad (6)$$

and, since  $M$  is of order  $\sqrt{\log n}$ , we get from this that

$$P(\max_{1 \leq i \leq n} |g_i| > (1 + \varepsilon)M) \leq \log 2(1 + o(1))e^{-\varepsilon M^2} \quad (7)$$

(For a fixed  $\varepsilon$  one can replace  $\log 2(1 + o(1))$  with a quantity tending to 0 with  $n$ .)

We now look for a lower bound on  $P(\max_{1 \leq i \leq n} |g_i| > (1 + \varepsilon)M)$ . Since for iid  $X_i$ -s,

$$\begin{aligned} P(\max_{1 \leq i \leq n} (X_i > t)) &= 1 - (1 - P(X_1 > t))^n \geq 1 - e^{-nP(X_1 > t)} \\ &\geq 1 - \frac{1}{1 + nP(X_1 > t)} = \frac{nP(X_1 > t)}{1 + nP(X_1 > t)}, \end{aligned} \quad (8)$$

$$P(\max_{1 \leq i \leq n} |g_i| > (1 + \varepsilon)M) \geq \frac{nP(|g_1| > (1 + \varepsilon)M)}{1 + nP(|g_1| > (1 + \varepsilon)M)}. \quad (9)$$

The right hand side is an increasing function of  $P(|g_1| > (1 + \varepsilon)M)$  and, by (4),

$$\begin{aligned} P(|g_1| > (1 + \varepsilon)M) &\geq \sqrt{\frac{2}{\pi}} \frac{1}{(1 + \varepsilon)M+1} e^{-(1 + \varepsilon)^2 M^2 / 2} (1 - e^{-\frac{1}{2}e^{-(1 + \varepsilon)M}}) \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{(1 + \varepsilon)M+1} e^{-M^2 / 2} e^{-\varepsilon M^2 - \varepsilon^2 M^2 / 2} (1 - e^{-\frac{1}{2}e^{-(1 + \varepsilon)M}}) \\ &\geq \frac{M(1 - 2^{-1/n})}{(1 + \varepsilon)M+1} e^{-\varepsilon M^2 - \varepsilon^2 M^2 / 2} (1 - e^{-\frac{1}{2}e^{-(1 + \varepsilon)M}}) \\ &\geq \frac{\log 2}{4n} e^{-\varepsilon M^2 - \varepsilon^2 M^2 / 2} \geq \frac{\log 2}{4n} e^{-3\varepsilon M^2 / 2}, \end{aligned} \quad (10)$$

for  $\varepsilon \leq 1$  and  $n$  large enough (independently of  $\varepsilon$ ). Using (9), we get

$$P(\max_{1 \leq i \leq n} |g_i| > (1 + \varepsilon)M) \geq \frac{\frac{\log 2}{4} e^{-3\varepsilon M^2 / 2}}{1 + \frac{\log 2}{4} e^{-3\varepsilon M^2 / 2}} \geq \frac{\log 2}{4 + \log 2} e^{-3\varepsilon M^2 / 2}. \quad (11)$$

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**Claim 3** For some absolute positive constants  $c, C$  and for all  $0 < \varepsilon < 1/2$ ,

$$\exp(-Ce^{\varepsilon M^2}) \leq P(\max_{1 \leq i \leq n} |g_i| < (1 - \varepsilon)M) \leq C \exp(-ce^{3\varepsilon M^2 / 4}) \quad (12)$$

**Proof:**

$$\begin{aligned} P(\max_{1 \leq i \leq n} |g_i| < (1 - \varepsilon)M) &= \left(1 - \sqrt{\frac{2}{\pi}} \int_{(1 - \varepsilon)M}^{\infty} e^{-s^2 / 2} ds\right)^n \\ &\leq \left(1 - \sqrt{\frac{2}{\pi}} \int_{(1 - \varepsilon)M}^M e^{-s^2 / 2} ds\right)^n \end{aligned}$$

$$\begin{aligned}
&\leq \left(1 - \sqrt{\frac{2}{\pi}} \frac{1}{M} \int_{(1-\varepsilon)M}^M s e^{-s^2/2} \right)^n \\
&= \left(1 - \sqrt{\frac{2}{\pi}} \frac{1}{M} (e^{-(1-\varepsilon)^2 M^2/2} - e^{-M^2/2}) \right)^n \\
&= \left(1 - \sqrt{\frac{2}{\pi}} \frac{1}{M} e^{-M^2/2} (e^{\varepsilon M^2 - \varepsilon^2 M^2/2} - 1) \right)^n \\
&\leq \left(1 - (1 - 2^{-1/n}) (e^{\varepsilon M^2 - \varepsilon^2 M^2/2} - 1) \right)^n \quad \text{by (4)} \\
&\leq \exp(-n(1 - 2^{-1/n}) (e^{\varepsilon M^2 - \varepsilon^2 M^2/2} - 1)) \\
&\leq 2(1 + o(1)) \exp(-\log 2(1 + o(1)) e^{3\varepsilon M^2/4})
\end{aligned}$$

Which proves the right hand side inequality in (12). As for the left hand side,

$$\begin{aligned}
P(\max_{1 \leq i \leq n} |g_i| < (1 - \varepsilon)M) &= \left(1 - \sqrt{\frac{2}{\pi}} \int_{(1-\varepsilon)M}^{\infty} e^{-s^2/2} \right)^n \\
&\geq \left(1 - \sqrt{\frac{2}{\pi}} \frac{1}{(1 - \varepsilon)M} \int_{(1-\varepsilon)M}^{\infty} s e^{-s^2/2} \right)^n \\
&= \left(1 - \sqrt{\frac{2}{\pi}} \frac{1}{(1 - \varepsilon)M} (e^{-(1-\varepsilon)^2 M^2/2}) \right)^n \\
&= \left(1 - \sqrt{\frac{2}{\pi}} \frac{e^{-M^2/2}}{(1 - \varepsilon)M} e^{\varepsilon M^2 - \varepsilon^2 M^2/2} \right)^n \\
&\geq \exp\left(-\frac{2n(1 - 2^{-1/n})}{1 - e^{-\frac{1}{2} - M}} \frac{M + 1}{M} e^{\varepsilon M^2 - \varepsilon^2 M^2/2}\right) \quad \text{by (3)} \\
&\geq \exp(-2 \log 2(1 + o(1)) e^{\varepsilon M^2}).
\end{aligned}$$

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We summarize Claims 2 and 3 in a form that will be useful for us later in the following Proposition.

**Proposition 1** *For some positive absolute constants  $c, C$  and for all  $0 < \varepsilon < 1$  and  $n \in \mathbb{N}$ , denoting  $g = (g_1, g_2, \dots, g_n)$ ,*

$$c e^{-C\varepsilon \log n} \leq P(\|g\|_{\infty} < \frac{(1 - \varepsilon)M}{\sqrt{n}} \|g\|_2 \quad \text{or} \quad \|g\|_{\infty} > \frac{(1 + \varepsilon)M}{\sqrt{n}} \|g\|_2) \leq C e^{-c\varepsilon \log n}.$$

**Proof:** This follows easily from Claims 2 and 3 and the facts that  $e^x > x$  for all  $x$ ,  $M$  is of order  $\sqrt{\log n}$  and

$$P(\|g\|_2 < (1 - \varepsilon)\sqrt{n} \quad \text{or} \quad \|g\|_2 > (1 + \varepsilon)\sqrt{n}) < C e^{-\varepsilon^2 n}.$$

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### 3 Proof of the theorem

The first part of the Theorem follows easily from the, by now well exposed, proof of Milman's version of Dvoretzky's theorem (see e.g, [MS] or [Pi]) with the improved concentration estimate in (the right hand side of the inequality in) Proposition 1 replacing the classical estimates. For the proof of the second part we need:

**Lemma 1** *Let  $\mathcal{A}$  be a subset of  $G_{n,k}$  of  $\mu_{n,k}$  measure  $a$ . Put  $U_{\mathcal{A}} = \bigcup_{E \in \mathcal{A}} E$ , then*

$$P((g_1, g_2, \dots, g_n) \in U_{\mathcal{A}}) \geq a^{1/k}.$$

**Proof:** Let  $X_1, X_2, \dots, X_k$  be  $k$  independent random vectors distributed according to  $P$ , the canonical Gaussian measure on  $\mathbb{R}^n$ . Note that, since  $\mu_{n,k}$  is the unique rotational invariant probability measure on  $G_{n,k}$ , the distribution of  $\text{span}\{X_1, \dots, X_k\}$  is  $\mu_{n,k}$ . Accordingly,

$$\begin{aligned} P(U_{\mathcal{A}})^k &= P(X_1, X_2, \dots, X_k \in U_{\mathcal{A}}) \\ &\geq P(\text{span}\{X_1, X_2, \dots, X_k\} \in \mathcal{A}) \\ &= \mu_{n,k}(\mathcal{A}). \end{aligned}$$

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**Remark 1** As we'll see below we use only a weak form of Lemma 1. We actually believe there is a much stronger form of it.

**Proof of the moreover part in Theorem 1:** Let  $\mathcal{A} \subset G_{n,k}$  be such that every  $E \in \mathcal{A}$  there is an  $M_E$  such that

$$M_E \|x\|_2 \leq \|x\|_{\infty} \leq (1 + \varepsilon) M_E \|x\|_2$$

for all  $x \in E$ . Let  $\mathcal{B}$  be the subset of  $\mathcal{A}$  of all  $E$  for which  $\frac{(1-3\varepsilon)M}{\sqrt{n}} \leq M_E \leq \frac{(1+\varepsilon)M}{\sqrt{n}}$ , and let  $\mathcal{C} = \mathcal{A} \setminus \mathcal{B}$ . By Lemma 1,

$$\mu_{n,k}(\mathcal{C})^{1/k} \leq P(\{x; \|x\|_{\infty} < \frac{(1+\varepsilon)(1-3\varepsilon)M}{\sqrt{n}} \|x\|_2 \text{ or } \|x\|_{\infty} > \frac{(1+\varepsilon)M}{\sqrt{n}} \|x\|_2\})$$

and, by Proposition 1, this last quantity is smaller than  $Ce^{-c\varepsilon \log n}$ . It follows that

$$\mu_{n,k}(\mathcal{B}) > 1 - e^{-ak} - Ce^{-c\varepsilon k \log n}.$$

We may assume that  $\varepsilon \log n$  is much larger than  $a$  so that the last term above is dominated by  $e^{-ak}$ . Applying Lemma 1 once more we get

$$P(\{x; (\frac{(1-3\varepsilon)M}{\sqrt{n}} \|x\|_2 \leq \|x\|_{\infty} \leq \frac{(1+\varepsilon)^2 M}{\sqrt{n}} \|x\|_2\}) \geq \mu_{n,k}(\mathcal{B}) > 1 - 2e^{-ak}.$$

Using now the other part of Proposition 1 we get that

$$C\varepsilon \log n > ak.$$

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