MAXIMAL ℓ_p^n-STRUCTURES IN SPACES WITH EXTREMAL PARAMETERS

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ABSTRACT. We prove that every n-dimensional normed space with a type $p < 2$, cotype 2, and (asymptotically) extremal euclidean distance has a quotient of a subspace, which is well isomorphic to ℓ_p^k and with the dimension k almost proportional to n. A structural result of a similar nature is also proved for a sequence of vectors with extremal Rademacher average inside a space of type p. The proofs are based on new results on restricted invertibility of operators from ℓ_r^n into a normed space X with either type r or cotype r.

1. INTRODUCTION

The initial motivation of this paper was the following problem from [J-S]: Let $1 \leq p \leq 2$ and let X be a n-dimensional subspace of L_p whose distance from Euclidean space satisfies the inequality $d(X, \ell_2^n) \geq \alpha n^{1/p-1/2}$. Does X contain a subspace of proportional dimension, which is well isomorphic to ℓ_p^k? For $p = 1$, the answer is positive [J-S], while for $1 < p \leq 2$ the question is still open. The paper [B-T] contains related results and solution to other problems from [J-S], but left this particular problem open as well. Although the present paper leaves this problem open as well, we do show here that X has an almost proportional *quotient of a subspace* which is almost well isomorphic to ℓ_p^k. The term "almost" above refers to factors of order a power of $\log n$. We actually get the same conclusion for a wider class of spaces. This clearly is implied by Theorem 13.

Not surprisingly our approach involves restricted invertibility methods. We have two kinds of such results. The first is for operators from ℓ_q^n into spaces with cotype q. This is the content of Corollary 6. Section 2 in which it is contained is heavily based on a method developed by Gowers in [G1] and [G2]. The second restricted invertibility result is for operators from either ℓ_2^n or ℓ_p^n into spaces with type p. This is contained in Section 3. Section 4 contains the proof of the structural Theorem 13. Finally, Section 5 contains a related result: Under the same conditions as in Theorem 13 one can get a subspace, rather than quotient of a subspace, almost well isomorphic to ℓ_p^k. However, its dimension k is a certain power of n rather than being close to a proportion of n.

Most of the undefined notions here can be found in [TJ]. We only recall here the definition of the Lorentz spaces $L_{p,q}$.

Let (Ω, Σ, μ) be a measure space, $1 \le p \le \infty$, and $1 \le q \le \infty$. The Lorentz space $L_{p,q}(\mu)$ consists of all equivalent classes of μ-measurable functions f such that

$$\|f\|_{p,q} = \Big(\int_0^\infty (t^{1/p} f^*(t))^q \; dt/t \Big)^{1/q} < \infty \qquad \text{if } 1 \le q < \infty,$$
$$\|f\|_{p,\infty} = \sup_t t^{1/p} f^*(t) < \infty,$$

where f^* is the decreasing rearrangement of $|f|$, i.e. $f^*(t) = \inf\big(a : \mu\{|f| > a\} \le t\big)$, $0 < t < \infty$.

If $p = q$ then $L_{p,p}(\mu)$ is $L_p(\mu)$. In general $\|f\|_{p,q}$ is a quasi-norm, which for $p > 1$ is equivalent to a norm, the equivalence constant depending on p and q only. So we consider $L_{p,q}(\mu)$ under this norm.

For a positive integer n, one defines the finite dimensional spaces $\ell_{p,q}^n$ to be $L_{p,q}(\mu)$, where μ is the uniform measure on the interval $I = \{1, \dots, n\}$, $\mu(\{i\}) = 1$.

It can be easily checked for $1 \le p < \infty$, $1 \le q \le \infty$ that $\|x\|_p \le (\log n)^{1/p} \|x\|_{p,\infty}$ for all $x \in \ell_{p,q}^n$, and that $\| \sum_{i=1}^n e_i \|_{p,q} \sim n^{1/p}$, where e_i are the coordinate vectors in $\ell_{p,q}^n$.

Our estimates often involve "constants" that depend on various parameters. So we write, for example, $c = c(p, M)$ to denote a constant depending on p and M only.

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2. Restricted invertibility: spaces with cotype.

Let us start with a general theorem about finite symmetric block bases which is of independent interest. This theorem (and its proof) is a variant of Gowers' results on the subject and in a sense lies in-between [G1] and [G2].

Theorem 1. *Let $1 \le q < \infty$, and let $B \ge 1$. Let X be a Banach space, let $n \ge 1$ and $(x_i)_{i \le n}$ be a sequence of n vectors in X satisfying*

$$\big\| \sum a_i x_i \big\| \le B \|a\|_q \qquad \text{and} \qquad \mathbb{E} \big\| \sum \varepsilon_i a_i x_i \big\| \ge \|a\|_q$$

for all $a = (a_i) \in \mathbb{R}^n$. Then for any $\varepsilon > 0$ there exists a block basis $(y_i)_{i \le m}$ of pwemutation of (x_i), which is $(1+\varepsilon)$-symmetric and has cardinality

$$m \ge (c\varepsilon/B)^{2q+2} n/\log n,$$

where $c > 0$ is an absolute constant.

First recall the definition of a symmetric basis in its natural "localized" form used in the proof. Let $m \ge 1$ and consider the group

$$\Psi = \{-1,1\}^m \times S_m$$

acting on $\mathbb{R}^m$ as follows: for $a \in \mathbb{R}^m$ and $(\eta,\sigma) \in \Psi$, we define $a_{\eta,\sigma} = \sum_{i=1}^m \eta_i e_{\sigma(i)}$.

Definition 2. *Let $C \ge 1$. A set of vectors $(y_i)_{i \le m}$ in X is said to be* C-symmetric at $a \in \mathbb{R}^m$ *if for every $(\eta,\sigma) \in \Psi$ we have*

$$\Big\| \sum_i (a_{\eta,\sigma})_i y_i \Big\|_X \le C \Big\| \sum_i a_i y_i \Big\|_X.$$

A set $(y_i)_{i \le m}$ is C-symmetric *if for every $a \in \mathbb{R}^m$, $(y_i)_{i \le m}$ is C-symmetric at a.*

Proof of Theorem 1. Fix an integer m of the form $m = 2 + (c'\varepsilon/B)^{2q+2} n/\log n$ where $c' > 0$ is an absolute constant to be defined later. As in [G1], we divide the interval of natural numbers $[1,n]$ into m blocks of length h (where $h \sim \log n$), and relabel the indices in $[1,n]$ as follows: the pair (i,j) will be the j-th element in the i-th block, $i = 1,\dots,m$, $j = 1,\dots,h$. This identifies $[1,n]$ with the product $[1,m] \times [1,h]$. Consider the group

$$\Omega = \{-1,1\}^n \times S_n.$$

Here we think of S_n as the group of permutations of the product $[1,m] \times [1,h]$. We write $\pi_{ij} = \pi((i,j))$ for $\pi \in S_n$ and $\theta_{ij} = \theta_{(i,j)}$ for $\theta \in \{-1,1\}^n$. Define the random operator $\phi_{\theta,\pi} : \mathbb{R}^m \to X$ by setting

$$\phi_{\theta,\pi}(e_i) = \sum_{j=1}^h \theta_{ij} x_{\pi_{ij}}, \qquad i = 1,\dots,m.$$

We shall show that with high probability the vectors $y_i = \phi_{\theta,\pi}(e_i)$ for $i = 1,\dots,m$ are $(1+\varepsilon)$-symmetric.

The first ingredient in the proof is a lemma from [G2], which says that in any normed space the symmetry of a sequence can be verified on a set of a polynomial, not exponential, cardinality.

Lemma 3. [G2]. *Let $\varepsilon > 0$, let $(\mathbb{R}^m, \|\cdot\|)$ be a normed space and set $N = m^D$, where $D = \varepsilon^{-1}\log(3\varepsilon^{-1})$. There exist a set $\mathcal{N}$ of cardinality N in $\mathbb{R}^m$ such that if the standard basis of $\mathbb{R}^m$ is $(1+\varepsilon)$-symmetric at each element from $\mathcal{N}$, then it is $(1+\varepsilon)(1-6\varepsilon)^{-1}$-symmetric.*

The next lemma is central. For a real valued random variable Z, by $M(Z)$ we denote its median, that is, the number satisfying $\mathbb{P}\{Z \le M(Z)\} \ge 1/2$ and $\mathbb{P}\{Z \ge M(Z)\} \ge 1/2$.

Lemma 4. *Let $1 < q < \infty$ and $B \ge 1$. Let $(x_j)_{j\le n}$ be a sequence of vectors satisfying $\big\|\sum a_j x_j\big\| \le B\big\|a\big\|_q$ for all $a \in \mathbb{R}^n$. Fix $a \in \mathbb{R}^m$ and $0 < \beta < 1/2$. Then with the notation above we have*

$$\mathbb{P}_\Omega\Big(\max_{\eta,\sigma\in\Psi}\Big|\|\phi_{\theta,\pi}(a_{\eta,\sigma})\| - \overline{M}(\|\phi_{\theta,\pi}(a)\|)\Big| > \beta\|a\|_q h^{1/q}\Big) \le m^{-(c/\beta)\log(c/\beta)},$$

where $\overline{M}$ denotes the expectation if $q = 1$, or the median if $q > 1$, and provided that

$$m \le (c\beta/B)^{2q+2} n/\log n,$$

where $c > 0$ is an absolute constant.

This deviation inequality was proved in [G1] (page 195, (iii)) and the form of $\overline{M}$ follows from the proof. Moreover, the inequality is stated in [G1] for a particular value of m however it is clear from the proof that it is valid for all smaller value of m as well.

To successfully apply this lemma we require the estimate

$$\overline{M}(\|\phi_{\theta,\pi}(a)\|) \ge (1/6)\|a\|_q h^{1/q}. \tag{1}$$

For $\overline{M}$ being the expectation, an estimate follows readily from our lower bound assumption in Theorem 1, even with the constant 1 replacing 1/6. This settles the case $q = 1$. For $q > 1$, we will use the following lemma, a version of which will be also needed in Section 5.

Lemma 5. *Let (x_i) be a finite sequence of vectors in a Banach space, and (a_i) be scalars. Then*

$$\mathbb{P}_\Omega\Big\{\big\|\sum \theta_i a_i x_{\pi(i)}\big\| \ \ge (1/2)\mathbb{E}\big\|\sum \theta_i a_i x_{\pi(i)}\big\|\Big\} \ge \delta,$$

where $\delta > 0$ is an absolute constant.

Proof. Define the random variable $Z = \|\sum \theta_i a_i x_{\pi(i)}\|$, and let $\|Z\|_p = (\mathbb{E}|Z|^p)^{1/p}$. By Kahane's inequality for any $0 < p, r < \infty$ we have $\|Z\|_r \le A\|Z\|_p$, where $A = A(p,r)$ (see [M-S] 9.2). Then

$$\mathbb{P}_\Omega\big\{Z \ge 2^{-1/p}\|Z\|_p\big\} \ge (2A^p)^{r/(p-r)} \tag{2}$$

This estimate follows from the standard argument (see e.g., [Le-Ta], Lemma 4.2) based on Hölder's inequality. For $t > 0$ we have

$$\mathbb{E}Z^p \leq t^p + \int_{Z>t} Z^p d\mathbb{P}_\Omega \leq t^p + \|Z\|_r^p \mathbb{P}_\Omega\{Z > t\}^{1-p/r}.$$

Setting $t = 2^{-1/p}\|Z\|_p$ we get (2). Now the conclusion of the lemma follows from (2) with $p = 1$, $r = 1/2$. ■

We return to the proof of Theorem 1. First, to complete the proof of (1), let $Z = \|\phi_{\theta,\pi}(a)\|$. It is easy to check that our lower bound assumption implies that $\mathbb{E}Z \geq \|a\|_q h^{1/q}$. Let δ be as in Lemma 5 and let $c > 0$ be a constant from Lemma 4. Fix $0 < \beta_1 < 1/3$ such that $2^{-(c/\beta_1)\log(c/\beta_1)} < \delta$. We shall ensure later that m satisfies the upper bound assumption of Lemma 4. Using Lemma 4 together with the above lower bound for $\mathbb{E}Z$ we get, since $m \geq 2$,

$$\begin{aligned} &\mathbb{P}_\Omega\{\,|Z - M(Z)| > (1/3)\mathbb{E}Z\} \\ &\quad\leq \mathbb{P}_\Omega\{|Z - M(Z)| > \beta\mathbb{E}Z\} \leq 2^{-(c/\beta)\log(c/\beta)} < \delta. \end{aligned}$$

On the other hand, by Lemma 5 we have $\mathbb{P}_\Omega\{Z \geq (1/2)\mathbb{E}Z\} \geq \delta$. An easy calculation shows $M(Z) \geq (1/6)\mathbb{E}Z \geq (1/6)\|a\|_q h^{1/q}$, which is (1).

Finally, we can now finish the proof of Theorem 1. Fix any $0 < \varepsilon < 1/6$, let D be as in the Lemma 3, given by $D = \varepsilon^{-1}\log(3\varepsilon^{-1})$ and let $\mathcal{N}$ be the set in the conclusion of this lemma. Let $c > 0$ be the constant from Lemma 4. Set $\beta_2 = c\varepsilon/3$. Then $(c/\beta_2)\log(c/\beta_2) > D$. We may additionally assume that $\beta_2 < \varepsilon/2$. By a suitable choice of the constant c' fixed at the beginning of the proof we may ensure that m satisfies the upper bound assumption in Lemma 4 for $\beta = \min(\beta_1, \beta_2)$. By Lemma 4 together with (1) we observe that the vectors $(y_i)_{i\leq m} = (\phi_{\theta,\pi}(e_i))_{i\leq m}$ are $(1+\varepsilon)$-symmetric at any fixed $a \in \mathcal{N}$ with probability at least $1 - m^{-D}$. It follows that there is a choice of $(y_i)_{i\leq m}$ which is $(1+\varepsilon)$-symmetric at each $a \in \mathcal{N}$. Then Lemma 3 yields that $(y_i)_{i\leq m}$ is $(1+\varepsilon)(1-6\varepsilon)^{-1}$-symmetric. This completes the proof of Theorem 1. ■

As an immediate corollary we get a restricted invertibility result for operators $\ell_q^n \to X$ where X is a Banach space of cotype q.

Corollary 6. *Let $q \geq 2$ and $K, M \geq 1$. Let X be a Banach space with cotype q constant $C_q(X) \leq K$. Let $u : \ell_q^n \to X$ be an operator with $\|u\| \leq M$ and satisfying the non-degeneracy condition $\|ue_i\| \geq 1$ for $i = 1, \ldots, n$. Then there exists a subspace E in $\mathbb{R}^n$ spanned by* *disjointly supported vectors such that*

$$\|ux\| \geq (1/2K)\|x\| \qquad \textit{for} \quad x \in E,$$

and

$$\dim E \geq (c/MK)^{2q+2} n/\log n,$$

where $c > 0$ is an absolute constant.

3. Restricted invertibility: spaces with type.

In this section we prove some restricted invertibility results for operators with values in spaces of type p. The conclusion is slightly weaker than the known results for the more special case of operators between ℓ_p^n spaces ([B-T], Theorem 5.7). In that case the consludion holds with the ℓ_p- rather than $\ell_{p,\infty}$-norm. As we will see later such a stronger conclusion does not hold in general under our assumptions (see Remark 2 after Corollary 12).

Theorem 7. *Let $1 < p \leq 2$ and $K, M \geq 1$. Let X be a Banach space with type p constant $T_p(X) \leq K$. Let $u : \ell_2^n \to X$ be an operator with $\|u\| \leq M$ and satisfying the non-degeneracy condition $\ell(u) \geq \sqrt{n}$. Then there exists a subset $\sigma \subset \{1, \dots, n\}$ of cardinality $|\sigma| \geq cn$ such that*

$$\|ux\|_X \geq (c/K) n^{1/2-1/p} \|x\|_{p,\infty} \qquad \textit{for } x \in \mathbb{R}^\sigma,$$

where $c = c(p, M) > 0$.

Remark. Let $p = 2$, let X be a space with dual of cotype 2, $C_2(X^*) \leq K$ and let u satisfies all the assumptions of Theorem 7. Then the resulting estimate can be improved to the lower ℓ_2 estimate $\|ux\|_X \geq c\|x\|_2$ for all $x \in \mathbb{R}^\sigma$, where $c = c(K, M) > 0$.

The proof of the theorem is based on following two lemmas. The first one is a reformulation of the generalization of Elton's theorem in [B-T], Theorem 5.2.

Lemma 8. *Let $1 < r < \infty$ and $M \geq 1$. Let $(x_i)_1^n$ be a set of vectors in a Banach space satisfying*

(1) $\|\sum_\eta x_i\| \leq M|\eta|^{1/r}$ for any subset $\eta \subset \{1, \dots, n\}$;

(2) $\mathbb{E}\|\sum \varepsilon_i x_i\| \geq n^{1/r}$.

Then there exists a subset $\sigma \subset \{1, \dots, n\}$ of cardinality $|\sigma| \geq cn$ such that

$$\Big\| \sum_\sigma a_i x_i \Big\| \geq c n^{-1/r'} \|a\|_1 \qquad \textit{for} \quad a \in \mathbb{R}^\sigma,$$

where $c = c(r, M) > 0$.

The second lemma is a factorization result of Pisier [P] for $(q,1)$-summing operators. We do not need here the definition of such operators and their norms $\pi_{q,1}$, and the interested reader can find them e.g., in [TJ]. Let us only recall that it is easy to see (e.g., [TJ], the proof of Theorem 21.4) that if Y is a Banach space of cotype $q \geq 2$ and $\mathcal{K}$ is a compact Hausdorff space then every bounded operator $T : C(\mathcal{K}) \to Y$ is $(q,1)$-summing and $\pi_{q,1}(T) \leq C_q(Y)\|T\|$. We shall combine this fact with Pisier's factorization theorem which states [P] (see also [TJ] Theorem 21.2 and (21.6))

Lemma 9. *Let $1 \leq q < \infty$, let Y be a Banach space and let $T : C(\mathcal{K}) \to Y$ be a $(q,1)$-summing operator. There exists a probability measure λ on $\mathcal{K}$ such that T factors as $T = \tilde{T} j$,*

$$T: \ C(\mathcal{K}) \xrightarrow{j} L_{q,1}(\lambda) \xrightarrow{\tilde{T}} Y,$$

where j is the natural inclusion map and $\|\tilde{T}\| \leq c\pi_{q,1}(T)$, where c is an absolute constant.

Corollary 10. *Let $q > 2$ and $K \geq 1$. Let Y be a Banach space with $C_q(Y) \leq K$. Let $T : \ell_\infty^n \to Y$. Then there exists a subset $\sigma \subset \{1, \dots, n\}$ of cardinality $|\sigma| \geq n/2$ such that*

$$\|TR_\sigma : \ell_{q,1}^n \to Y\| \leq c\, K n^{-1/q} \|T\|,$$

where R_σ denotes the coordinate projection in $\mathbb{R}^n$ onto $\mathbb{R}^\sigma$ and c is an absolute constant.

Proof. Observe that $\pi_{q,1}(T) \leq K\|T\|$. Consider Pisier's factorization

$$T: \ \ell_\infty^n \xrightarrow{j} L_{q,1}(\lambda) \xrightarrow{\tilde{T}} Y,$$

where λ is a probability measure on $\{1, \dots, n\}$ and $\|\tilde{T}\| \leq c\pi_{q,1}(T)$. Then the set $\sigma = \{j : \lambda(j) \leq 2/n\}$ has cardinality at least $n/2$. Moreover

$$\|jR_\sigma : \ell_{q,1}^\sigma \to L_{q,1}(\lambda)\| \leq (2/n)^{1/q}.$$

This immediately completes the proof. ∎

In the dual setting, this gives

Corollary 11. *Let $1 < p < 2$ and let $T_p(X) \leq K$. Consider vectors $(y_j)_1^n$ in X such that $\|\sum a_i y_i\| \geq \|a\|_1$ for all $a \in \mathbb{R}^n$. Then there exists a subset $\sigma \subset \{1, \dots, n\}$ of cardinality $|\sigma| \geq n/2$ such that*

$$\Big\| \sum_\sigma a_i y_i \Big\| \geq (c/K) n^{1/p'} \|a\|_{p,\infty} \qquad \text{for } a \in \mathbb{R}^\sigma,$$

where $c > 0$ is an absolute constant.

Proof. Let X_0 be the span of $(y_j)_1^n$, and define $T : X_0 \to \ell_1^n$ by $Ty_j = e_j$ for $j = 1, \dots, n$. Then $\|T\| \le 1$, so $\|T^* : \ell_\infty^n \to X_0^*\| \le 1$. Apply Corollary 10 with $Y = X_0^*$ and $q = p'$. We get a subset σ of cardinality at least $n/2$ such that

$$\|T^* R_\sigma : \ell_{p',1}^n \to X_0^*\| \le cKn^{-1/p'}.$$

Thus

$$\|R_\sigma T : X_0 \to \ell_{p,\infty}^n\| \le cKn^{-1/p'}.$$

Note that $R_\sigma T x_j = e_j$ for $j \in \sigma$. From this the desired estimate follows.

■

Now, Theorem 7 is a combination of Lemma 8 (for $r = 2$) and Corollary 11. One needs only to recall that X has cotype q, where $q < \infty$ and $C_q(X)$ both depend only on p and $T_p(X)$ (see [K-T] for quantitative estimates), and that $\ell(u) \le C\mathbb{E}\|\sum \varepsilon_i u e_i\|$ where C depends on q and $C_q(X)$ only. If $p = 2$, the remark following the theorem is proved by a similar argument, with use of Pisier's factorization in Lemma 9 replaced by Maurey's strengthening of Grothendieck's theorem ([TJ], Theorem 10.4) and Pietsch's factorization for 2-summing operators ([TJ], Theorem 9.2).

As a corollary we have a further invertibility result.

Corollary 12. *Let $1 < p \le 2$, $K, M \ge 1$ and $\alpha > 0$. Let X be a Banach space with type p constant $T_p(X) \le K$. Let $u : \ell_p^n \to X$ be an operator with $\|u\| \le M$ and satisfying the non-degeneracy condition $\ell(u : \ell_2^n \to X) \ge n^{1/p}$. Then there exists a subset $\sigma \subset \{1, \dots, n\}$ of cardinality $|\sigma| \ge cn$ such that*

$$\|ux\|_X \ge (c/K)\|x\|_{p,\infty} \qquad \text{for } x \in \mathbb{R}^\sigma,$$

where $c = c(p, M) > 0$.

The proof is an easy application of Theorem 7 for the operator $w = n^{1/2-1/p}u : \ell_2^n \to X$.

Remarks.1. The proof above show that Theorem 7 remains valid with the same estimates if the norm $\|u : \ell_2^n \to X\|$ is replaced by $M = \|u : \ell_{2,1}^n \to X\|$. An analogous fact is true also for Corollary 12. If $p = 1$, both Theorem 7 and Corollary 12 are true (and follow directly from Lemma 8) if the space X is assumed to have cotype q, for some $q < \infty$.

2. The space $\ell_{p,q}$ (with $1 < p < 2$ and $1 < q < \infty$) has type p. This known fact follows for example from the easy fact that $\ell_{p,q}$ has an upper p-estimate for disjoint vectors, together with Theorems 1.e.16 and 1.f.10 in [L-T]. It follows that one cannot improve the conclusions of Theorem 7 and Corollary 12 by replacing $\|\cdot\|_{p,\infty}$ by $\|\cdot\|_p$.

4. Spaces with extremal Euclidean distance

In this section we concentrate on the structure of finite-dimensional normed spaces which, while satisfying geometric type-cotype conditions, have the distance to a Euclidean space of maximal order. The maximality of the distance is expressed in terms of the lower estimate which for some $1 \le p \le 2$ (depending on of the properties of X) has the form

$$d_X = d(X, \ell_2^n) \ge \alpha n^{1/p-1/2} \tag{3}$$

for some constant $\alpha > 0$.

The main result of this section is

Theorem 13. *Let $1 < p \le 2$, $K \ge 1$ and $\alpha > 0$. Let X be an n-dimensional normed space with cotype 2 constant $C_2(X) \le K$ and type p constant $T_p(X) \le K$, and whose euclidean distance satisfies (3). Then there exists Y, a quotient of a subspace of X, of dimension $k \ge cn(\log n)^{-b}$ such that $d(Y, \ell_p^k) \le C(\log n)^{1/p}$, where $c = c(p, K, \alpha) > 0$, $C = C(p, K, \alpha)$ and $b = b(p) > 0$.*

We do not know whether the log-factor can be removed in either the distance or the dimension estimates. We also do not know whether "quotient of a subspace" can be replaced by "subspace" without an essential change to the estimates.

The proof of this theorem depends on two succesive steps: the first is the lower estimate result for spaces satisfying our assumptions, and the second is a lower estimate for dual spaces. The latter step is based on Corollary 6, while the former one is contained in the following lower $\ell_{p,\infty}$-estimate for spaces with maximal Euclidean distance.

Theorem 14. *Let $1 < p < 2$, $K \ge 1$ and $\alpha > 0$. Let X be an n-dimensional normed space with cotype 2 constant $C_2(X) \le K$ and type p constant $T_p(X) \le K$, and whose euclidean distance satisfies $d(X, \ell_2^n) \ge \alpha n^{1/p-1/2}$. Then there exist $k \ge cn$ norm one vectors $y_1, \dots, y_k$ in X such that*

$$\Big\| \sum_i a_i y_i \Big\|_X \ge c\|a\|_{p,\infty} \qquad \text{for } a \in \mathbb{R}^k,$$

where $c = c(p, K, \alpha) > 0$.

Remark. As often happens in such cases, the proof has the unsatisfactory feature that it yields constants tending to 0 as $p \to 2$. Of course, by Kwapien's theorem (see e.g., [TJ] Theorem 13.15) an even stronger statement holds for $p = 2$.

To prove Theorem 14 we require some preliminaries. First recall the definition which has been often used in a similar context (see [TJ], §27). The *relative Euclidean factorization constant* $e_k(X)$ $(k = 1, 2, \dots)$ of a Banach space X is the smallest C such that for every subspace E of X of dimension k there exists a projection P in X onto E with the ℓ_2 factorable norm satisfying $\gamma_2(P) \le C$.

Note that the Euclidean distance satisfies

$$d(X, \ell_2^n) \le e_n(X).$$

We will work with a relaxation of the parameter $e_k(X)$ which will be shown to be comparable to $e_k(X)$ (up to a logarithm of the dimension).

Definition 15. *For $k = 1, 2, \dots$, we denote by $e'_k(X)$ the smallest C such that for every subspace E of X of dimension k there exists a projection P in X such that $P(X) \subset E$, $\mathrm{rank} P \ge k/2$, and $\gamma_2(P) \le C$.*

Lemma 16. *Let X be a Banach space and n be a natural number. Then*

$$e'_n(X) \le e_n(X) \le \sum_{k=0}^{\infty} e'_{n/2^k}(X).$$

Proof. Assume for simplicity that n is a power of 2, the general case easily follows. It is well known in the theory of 2-factorable operators (see e.g., [TJ], Theorem 27.1) that the right hand side inequality will follow once we prove that for every $v : \ell_2^n \to X$ such that $\pi_2(v^*) = 1$ we have

$$\pi_2(v) \le \sum_{k=0}^{\infty} e'_{n/2^k}(X).$$

To this end fix v as above and without loss of generality assume that v is one-to-one. Let P_0 be a projection on X such that $P_0(X) \subset v(\ell_2^n)$, $\mathrm{rank} P_0 \ge n/2$ and $\gamma_2(P_0) \le e'_n(X)$. Let $H_0 = v^{-1}(P_0 X)$. By passing to a smaller subspace if necessary we may assume that $\dim H_0 = n/2$.

By induction construct $k_0 = \log_2 n$ mutually orthogonal subspaces $H_k \subset \ell_2^n$ with $\dim H_k = n/2^{k+1}$ and projections P_k from X onto $v(H_k)$ such that $\gamma_2(P_k) \le e'_{n/2^k}(X)$ for $k = 0, \dots, k_0 - 1$.

For $k = 0, \dots, k_0 - 1$, denote by $Q_k : \ell_2^n \to H_k$ the orthogonal projection onto H_k. Then

$$\pi_2(vQ_k) = \pi_2(P_k v Q_k) \le \pi_2(P_k v) \le \gamma_2(P_k)\pi_2(v^*) \le e'_{n/2^k}(X).$$

Since $\ell_2^n = H_0 \oplus \dots \oplus H_{k_0-1}$, then

$$\pi_2(v) = \pi_2\Big(\sum_{k=0}^{k_0-1} vQ_k\Big) \le \sum_{k=0}^{k_0-1} e'_{n/2^k}(X),$$

as required. ■

Let us recall a standard set-up for finite-dimensional normed spaces. The Euclidean unit ball on $\mathbb{R}^n$ is denoted by B_2^n (and it corresponds to the Euclidean norm $\|\cdot\|_2$). Let $\|\cdot\|_X$ be a norm on $\mathbb{R}^n$, and X be the corresponding normed space. Let Q be an orthogonal projection in $\mathbb{R}^n$. Then by QX we denote the quotient of X with the canonical norm $\|y\|_{QX} = \inf\{\|x\|_X : \ Qx = y\}$. This way we view QX as the vector space $Q(\mathbb{R}^n)$ with the norm $\|\cdot\|_{QX}$. In particular, QX carries the Euclidean structure inherited from $\mathbb{R}^n$ with the unit ball $Q(B_2^n) = B_2^n \cap Q(\mathbb{R}^n)$.

Lemma 17. *Let X be a normed space, $\dim X = n$, and assume that $\pi_2(id : X \to \ell_2^n) \le A\sqrt{n}$. Let Q be an orthogonal projection in $\mathbb{R}^n$. Let $Y \subset Q(\mathbb{R}^n)$ be an m-dimensional subspace on which we consider two norms: the Euclidean norm $\|\cdot\|_2$ and the norm $\|\cdot\|_{QX}$. Then*

$$\ell\big(id : (Y, \|\cdot\|_2) \to (Y, \|\cdot\|_{QX})\big) \ge (1/AT_2(X^*)^2) m/\sqrt{n}.$$

Proof. To shorten the notation, denote the operator $id : (Y, \|\cdot\|_2) \to (Y, \|\cdot\|_{QX})$ by u. We first estimate $\pi_2(u^{-1})$. Recall that for any operator $w : Z \to Z_1$, the norm $\pi_2(w)$ is equal to the supremum of $(\sum \|wve_j\|^2)^{1/2}$ where the supremum runs over all operators $v : \ell_2^k \to Z$ with $\|v\| \le 1$ and all k (see [TJ] Proposition 9.7).

Thus fix $v : \ell_2^k \to (Y, \|\cdot\|_{QX})$ with $\|v\| \le 1$. Consider v as an operator into QX. Using Maurey's extension theorem for the dual operator (see [TJ] Theorem 13.13), there exists a lifting $v' : \ell_2^k \to X$, $Qv' = v$, with $\|v'\| \le T_2(X^*)$ (note that $(QX)^*$ is a subspace of X^*). Therefore

$$\begin{aligned} \Big(\sum_{i=1}^k \|u^{-1}ve_i\|_2^2\Big)^{1/2} &= \Big(\sum_{i=1}^k \|Qv'e_i\|_2^2\Big)^{1/2} \le \Big(\sum_{i=1}^k \|v'e_i\|_2^2\Big)^{1/2} \\ &\le T_2(X^*)\pi_2(id : X \to \ell_2^n) \le AT_2(X^*)\sqrt{n}. \end{aligned}$$

Thus $\pi_2(u^{-1}) \le AT_2(X^*)\sqrt{n}$.

It is now sufficient to use two well known and easy facts (see [TJ], Proposition 9.10 and Theorem 12.2 (ii)) that $m \le \pi_2(u)\pi_2(u^{-1})$ and $\pi_2(u) \le C_2(Y)\ell(u)$, to get $\ell(u) \ge m/AC_2(Y)T_2(X^*)\sqrt{n}$. Since $C_2(Y) \le T_2(Y^*) \le T_2(X^*)$, this completes the proof. ■

Proof of Theorem 14. It is well known and easy to see from Maurey's extension theorem (see [TJ], Prop. 27.4) that for every $k = 1, 2, \dots$ we have

$$e_k'(X) \le cC_2(X)T_p(X)k^{1/p-1/2} \le cK^2k^{1/p-1/2}, \tag{4}$$

where c is an absolute constant.

Assume again that n is a power of 2, let $A_p = 1 - 2^{1/p-1/2}$ and let k_0 be the smallest k such that $e'_{n/2^k}(X) \geq (A_p\alpha/2)(n/2^k)^{1/p-1/2}$. If no such k exists let $k_0 = \infty$. By the maximality of distance, Lemma 16 and (4) we get

$$\begin{aligned} \alpha n^{1/p-1/2} &\leq e_n(X) \leq \sum_{k=0}^{\infty} e'_{n/2^k}(X) \\ &\leq n^{1/p-1/2}\Big(\frac{A_p\alpha}{2}\sum_{k=0}^{\infty} 2^{-k(1/p-1/2)} + cK^2 \sum_{k=k_0}^{\infty} 2^{-k(1/p-1/2)}\Big) \\ &\leq n^{1/p-1/2}\big((\alpha/2) + cK^2 2^{-k_0(1/p-1/2)} A_p^{-1}\big). \end{aligned}$$

This shows that k_0 is finite and $k_0 \leq C$, where $C = C(p, K, \alpha)$.

Set $m = n/2^{k_0}$ and $d = (A_p\alpha/2)m^{1/p-1/2}$. Then $m \geq \beta n$ and $d \geq \beta d_X$, where $\beta = \beta(p, K, \alpha) > 0$. Moreover,

$$e'_m(X) \geq d. \tag{5}$$

Let $|\cdot|_2$ be a Euclidean norm on X given by a combination of a distance ellipsoid and the maximal volume ellipsoid (see [TJ], Prop. 17.2). Denote the n-dimensional Hilbert space $(X, |\cdot|_2)$ by H and write $\|\cdot\|_X$ for the norm in X. Then we have

$$(\sqrt{2}d_X)^{-1}|x|_2 \leq \|x\|_X \leq \sqrt{2}|x|_2, \qquad \text{for} \quad x \in X,$$
$$\pi_2(id : X \to H) \leq \sqrt{2n}.$$

Using (5), we will prove

Lemma 18. *Under the notation above there exist vectors $x_1, \dots, x_{\beta' m}$ in X with $\|x_i\|_X \leq d^{-1}$ and an orthogonal projection R on H with* $\mathrm{rank} R \geq m/2$ *and such that*

$$\beta'\|a\|_2 \leq \Big|\sum a_i R x_i\Big|_2 \leq B\|a\|_2 \qquad \text{for} \quad a \in \mathbb{R}^{\beta' m},$$

where $\beta' = \beta'(p, K, \alpha) > 0$ and $B = B(p, K, \alpha)$.

Proof. Estimate (5) implies that there exists a subspace E in X with $\dim E = m$ such that for every projection P in X with $P(X) \subset E$, $\mathrm{rank} P \geq m/2$ we have $\|P : X \to H\| \geq d/\|id : H \to X\| \geq d/\sqrt{2}$.

Our vectors x_i will be chosen among a sequence of vectors constructed by induction as follows. Assume that $1 \leq k < m/2$ and that vectors $x_1, \dots, x_{k-1}$ have been already constructed. Let P be the orthogonal projection in H onto $[x_1, \dots, x_{k-1}]^\perp \cap E$. Then P satisfies the assumptions above, so there esists a $x_k \in X$ such that $\|x_k\|_X = 1/d$ and $|Px_k|_2 \geq 1/\sqrt{2}$. Let also $f_k = Px_k/|Px_k|_2$.

This procedure gives us vectors $x_1, \dots, x_{m/2}$ with $\|x_i\|_X = 1/d$ and orthonormal vectors $f_1, \dots, f_{m/2}$ such that

$$\langle x_i, f_i\rangle \geq 1/\sqrt{2} \qquad \text{for} \quad 1 \leq i \leq m/2.$$

Let $\beta = \beta(p, K, \alpha)$ be the constant appearing before (5), and we may clearly assume that $\beta \leq 1$.

Note that

$$|x_i|_2 \leq \sqrt{2} d_X/d \leq \sqrt{2}/\beta \qquad \text{for} \quad i \leq m/2. \tag{6}$$

A known and easy argument shows that for every $0 < \delta < 1/2$ there exists an orthogonal projection R in $[x_i]_{i \leq m/2}$ with $\text{corank} R \leq \delta m$ and such that

$$\Big|\sum_1^{m/2} a_i R x_i\Big|_2 \leq 1/(\beta^2\delta)\|a\|_2 \qquad \text{for} \quad a \in \mathbb{R}^{m/2}. \tag{7}$$

Indeed, denote by H_1 the space $([x_i], |\cdot|_2)$ and consider the operator $T : \ell_2^{m/2} \to H_1$ defined by $Te_i = x_i$ for $i = 1, \dots, m/2$. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ be the s-numbers of T so that $Tf_i = \lambda_i f_i'$ for some orthonormal bases $\{f_i\}$ and $\{f_i'\}$ in $\ell_2^{m/2}$ and H_1, respectively. We have, by (6),

$$\sum_{i=1}^{m/2} \lambda_i^2 = \|T\|_{HS}^2 = \sum_{i=1}^{m/2} |x_i|_2^2 \leq m/\beta^2.$$

This implies that for $i_0 = \delta m$ we have $\lambda_{i_0} \leq 1/(\beta^2\delta)$, and then the projection R onto $[f_{i_0+1}, \dots, f_{m/2}]$ satisfies (7).

Set $\delta = \beta^2/32$ and let R satisfy (7). Extend R to all of H by setting $Rx = x$ for $x \in [x_i]^\perp$. Then for $R' = id - R$ we have $\text{rank} R' \leq \delta m$. Therefore

$$\begin{aligned}
\sum_{i=1}^{m/2} |Rx_i|_2 &\geq \sum_{i=1}^{m/2} \langle Rx_i, f_i\rangle = \sum_{i=1}^{m/2} \langle x_i, f_i\rangle - \sum_{i=1}^{m/2} \langle R'x_i, f_i\rangle \\
&\geq 2^{-3/2} m - \sum_{i=1}^{m/2} \langle x_i, R'f_i\rangle \geq 2^{-3/2} m - (\sqrt{2}/\beta) \sum_{i=1}^{m/2} |R'f_i|_2 \\
&\geq 2^{-3/2} m - (\sqrt{2}/\beta)\sqrt{m/2}\, \|R'\|_{\text{HS}} \\
&\geq 2^{-3/2} m - (\sqrt{\delta}/\beta) m = 2^{-5/2} m.
\end{aligned}$$

From this inequality and (6) it easily follows that the set $\sigma = \{i : |Rx_i|_2 \geq 1/4\sqrt{2}\}$ has cardinality $|\sigma| \geq \beta m/16$. Applying Theorem 1.2 from [B-T] for the operator $T : \ell_2^\sigma \to H$ defined by $Te_i = Rx_i$ for $i \in \sigma$, we get, by (7) and the definition of σ that there exists a subset $\sigma' \subset \sigma$ such that

$$\Big|\sum_{i\in\sigma'} a_i Rx_i\Big|_2 \geq \beta' \|a\|_2 \qquad \text{for} \quad a \in \mathbb{R}^{\sigma'}.$$

Moreover, $|\sigma'| \geq \beta' m$, where $\beta' = \beta'(p, K, \alpha) > 0$. This together with (7) completes the proof by relabeling the vectors from σ'. ■

Returning to the proof of Theorem 14, identify X with $\mathbb{R}^n$ in such a way that $|\cdot|_2$ coincides with the usual ℓ_2^n-norm $\|\cdot\|_2$. Let $x_1, \dots, x_{\beta' n}$ be vectors constructed in Lemma 18. If $(RX, \|\cdot\|_{RX})$ denotes the quotient of X given by R then first note that

$$\Big\|\sum a_i Rx_i\Big\|_{RX} \leq \sqrt{2}\Big|\sum a_i Rx_i\Big|_2 \leq \sqrt{2} B \|a\|_2 \qquad \text{for} \quad a \in \mathbb{R}^{\beta' n}.$$

Consider the subspace $Y = [Rx_i]_{i=1}^{\beta' m}$ of RX, (i.e., with the norm $\|\cdot\|_{RX}$ inherited from RX), and consider also the norm $\|\cdot\|_2$ on Y inherited from ℓ_2^n. To apply Lemma 17 note that since X has control of the cotype 2 constant and the K-convexity constant (having non-trivial type) then $T_2(X^*)$ is bounded above by a function of K. Thus by the lemma, the ℓ-norm of the identity operator satisfies $\ell(id : (Y, \|\cdot\|_2) \to Y) \geq c\sqrt{n}$, where $c = c(p, K, \alpha) > 0$. On the other hand, since the set of vectors (Rx_i) admits a lower ℓ_2-estimate, then by the ideal property of the ℓ-norm $\ell(id)$ can be estimated using that set, namely,

$$\ell(id : (Y, \|\cdot\|_2) \to Y) \leq (1/\beta')\mathbb{E}\Big\|\sum_{i=1}^{\beta' m} g_i Rx_i\Big\|_{RX}.$$

Thus

$$\mathbb{E}\Big\|\sum_{i=1}^{\beta' n} g_i Rx_i\Big\|_{RX} \geq c_1\sqrt{n},$$

where $c_1 = c_1(p, K, \alpha) > 0$. Then by Theorem 7 there exists a subset η of $\{1, \dots, \beta' m\}$ of cardinality $|\eta| > c_2 n$ and such that

$$\Big\|\sum_{\eta} a_i Rx_i\Big\|_{RX} \geq c_2 n^{1/2-1/p}\|a\|_{p,\infty}, \qquad \text{for} \quad a \in \mathbb{R}^{\eta},$$

where $c_2 = c_2(p, K, \alpha) > 0$. Recall that $\|x_i\|_X = d^{-1}$. Then for $i \in \eta$ let $y_i = dx_i$. Clearly, y_i's are unit vectors in X and for $a \in \mathbb{R}^\eta$ we have

$$\Big\|\sum a_i y_i\Big\|_X \geq d\Big\|\sum a_i Rx_i\Big\|_{RX} \geq c_2(dn^{1/2-1/p})\|a\|_{p,\infty} \geq c_3\|a\|_{p,\infty},$$

where $c_3 = c_3(p, K, \alpha) > 0$. This completes the proof of Theorem 14. ■

Now we are ready to prove Theorem 13, as a combination of Theorem 14 and Corollary 6.

Proof of Theorem 13. We can clearly assume that $1 < p < 2$, because for $p = 2$ the whole space X is K^2-isomorphic to ℓ_2^n by Kwapien's Theorem (see [TJ] Theorem 13.15).

We apply Theorem 14, and let $(y_i)_{i \le k}$ be the vectors from its conclusion, $k \ge cn$ with $c = c(p, K, \alpha)$. Consider the space $X_1 = [y_i]_{i \le k}$ as a subspace of X. Since the vectors y_i are necessarily linearly independent, we may define the operator $v : X_1 \to \ell_p^k$ by

$$vy_i = e_i, \quad \text{for } i \le k.$$

Then by the conclusion of Theorem 14

$$\|v\| \le \|v : X_1 \to \ell_{p,\infty}^k\| \|id : \ell_{p,\infty}^k \to \ell_p\| \le C_1 (\log n)^{1/p},$$

where $C_1 = C_1(p, K, \alpha)$. Consider the adjoint operator $v^* : \ell_q^k \to X_1^*$, where $1/q + 1/p = 1$. Then

$$\|v^*\| \le C_1 (\log n)^{1/p} \tag{8}$$

and for all $i \le k$,

$$\|v^* e_i\| \ge \langle v^* e_i, y_i \rangle = \langle e_i, e_i \rangle = 1.$$

Applying Corollary 6 we get norm one vectors $(h_i)_{i \le m}$ in ℓ_q^k with disjoint supports satisfying for all $a \in \mathbb{R}^k$,

$$\Big\| v^* \Big(\sum_{i=1}^m a_i h_i \Big) \Big\|_{X_1^*} \ge \frac{1}{2K} \Big\| \sum_{i=1}^m a_i h_i \Big\|_q = \frac{1}{2K} \|a\|_q.$$

Moreover, $m \ge c_1 (\log n)^{-2(q+1)/p} n / \log n$, where $c_1 = c_1(p, K, \alpha) > 0$.

Also from (8),

$$\Big\| v^* \Big(\sum_{i=1}^m a_i h_i \Big) \Big\|_{X_1^*} \le C_1 (\log n)^{1/p} \|a\|_q.$$

Thus the sequence of vectors $z_i = v^* h_i$, $i \le m$, spans in X_1^* a subspace Z, which is $C(\log n)^{1/p}$-isomorphic to ℓ_q^m, with $C = C(p, K, \alpha)$. Since Z is a subspace of a quotient X_1^* of X^*, the space Z^* is a quotient of a subspace of X and is $C(\log n)^{1/p}$-isomorphic to ℓ_p^m. This completes the proof of Theorem 13. ∎

5. ℓ_p^n SUBSPACES IN SPACES WITH EXTREMAL TYPE p

We show another interesting application of methods discussed here to the structure of subspaces of spaces which attain their best type. More precisely, if a Banach space of type p contains a sequence of vectors with extremal Rademacher average, then it contains a relatively large subspace close to ℓ_p^k.

Proposition 19. *Let $1 < p \le 2$, $K \ge 1$ and $\alpha > 0$. Let X be a Banach space with type p constant $T_p(X) \le K$. Assume that there exist norm one vectors $x_1, \dots, x_n$ in X such that $\mathbb{E}\|\sum_1^n \varepsilon_i x_i\| \ge \alpha n^{1/p}$. Then there is a block basis of permutation of $(x_i)_{i \le n}$ of cardinality m which is $C(\log n)^{1/p}$-equivalent to the unit vector basis in ℓ_p^m, and $m \ge c(\log n)^{-1} n^{2/p-1}$, where $C = C(p, K, \alpha)$ and $c = c(p, K, \alpha) > 0$.*

The proof combines the main result of [G2] and Corollary 12.

Proof. Fix an $\varepsilon > 0$. By [G2], our assumption on the Rademacher average of (x_i) implies that there exists a block basis $(y_i)_1^m$ of permutation of $(x_i)_1^n$, with blocks of random ± 1 coefficients and equal lengths, which is 2-symmetric with probability larger than $1 - \varepsilon$. Moreover $m \ge c(\log n)^{-1} n^{2/p-1}$, where $c = c(p, \alpha, \varepsilon)$ and we may assume that m is an even number. The precise definition of the random vectors (y_i) is given in the proof of Theorem 1, the underlying probability space being denoted by $\mathbb{P}_\Omega$.

Then with probability larger than $1 - \varepsilon$ the following holds for all subsets σ of $\{1, \dots, m\}$ of cardinality $|\sigma| = m/2$:

$$2\Big\|\sum_{i\in\sigma} y_i\Big\| \ge \Big\|\sum_{i\in\sigma} y_i\Big\| + \frac{1}{2}\Big\|\sum_{i\in\sigma^c} y_i\Big\| \ge \frac{1}{2}\Big\|\sum_{i=1}^m y_i\Big\|.$$

On the other hand, by Lemma 5 we have, with probability larger than $\delta > 0$,

$$\begin{aligned}\Big\|\sum_{i=1}^m y_i\Big\| &= \Big\|\sum_{i=1}^n \varepsilon_i x_{\pi(i)}\Big\| \ge \frac{1}{2}\mathbb{E}_\Omega\Big\|\sum_{i=1}^n \varepsilon_i x_{\pi(i)}\Big\| \\ &= \frac{1}{2}\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\| \ge \frac{1}{2}\alpha n^{1/p}\end{aligned}$$

(where π denotes a random permutation of $\{1, \dots, n\}$, and $\mathbb{E}$ is the expectation over random signs ε_i). Therefore with probability larger than $\delta - \varepsilon$

$$\Big\|\sum_{i\in\sigma} y_i\Big\| \ge \frac{1}{8}\alpha n^{1/p} \quad \text{holds for all subsets } \sigma,\ |\sigma| = m/2. \tag{9}$$

Let $n = mh$ and we can assume that h is integer. By the type assumption and the definition of y_i, for every $i \le m$ we have $\mathbb{E}_\Omega \|y_i\| \le Kh^{1/p}$. Then $\mathbb{E}_\Omega \big(\sum_{i=1}^m \|y_i\| \big) \le Kmh^{1/p}$. So, with probability larger than $1-\varepsilon$ we have $\big(\sum_{i=1}^m \|y_i\| \big) \le (1/\varepsilon) Kmh^{1/p}$. this clearly implies that

$$\exists \text{ a subset } \sigma,\ |\sigma| = m/2, \text{ such that } \|y_i\| \le \frac{2}{\varepsilon} K h^{1/p} \qquad \text{for} \quad i \in \sigma. \tag{10}$$

With probability at least $\delta - 2\varepsilon$, events (9) and (10) hold simultaneously. For $\varepsilon = \delta/3$ this probability is positive, so we can consider a realization of (y_i) for which both events occur. Let $z_i = h^{-1/p} y_i$, $i \in \sigma$. Then $\|z_i\| \le (6/\delta)K$, so by the type p and symmetry

$$\Big\| \sum_{i \in \sigma} a_i z_i \Big\| \le (12/\delta) K^2 \|a\|_p \qquad \text{for all} \quad (a_i)_{i \in \sigma}.$$

Next, by (9) and symmetry

$$\Big\| \sum_{i \in \sigma} \varepsilon_i z_i \Big\| \ge \frac{1}{2} h^{-1/p} \Big\| \sum_{i \in \sigma} y_i \Big\| \ge \frac{1}{16} \alpha m^{1/p}.$$

Corollary 12 yields then that there exists a subset $\sigma_1 \subset \sigma$ with cardinality $|\sigma_1| \ge cm$ and such that

$$\Big\| \sum_{i \in \sigma_1} a_i z_i \Big\| \ge c \|a\|_{p,\infty} \qquad \text{for all} \quad (a_i)_{i \in \sigma},$$

where $c = c(p, K, \alpha) > 0$. This completes the proof. ∎

Remark. It is not clear whether the exponent $2/p - 1$ in Proposition 19 is optimal. However, for $p = 2$ the optimal exponent must be 0, because the identical vectors $x_i = 1$ in $X = \mathbb{R}^1$ satisfy the assumptions of Proposition 19.

As a corollary we get a variant of Theorem 13 where the conclusion is improved by getting a subspace rather than quotient of a subspace, at the price of a worse estimate on the dimension.

Proposition 20. *Under the assumptions of Theorem 13, there exists a subspace Y of X of dimension $k \ge c(\log n)^{-1} n^{2/p-1}$, with $d(Y, \ell_p^k) \le C(\log n)^{1/p}$, where $C = C(p, K, \alpha)$ and $c = c(p, K, \alpha) > 0$.*

Proof. By (3) and Kwapien's theorem we get $T_2(X) \ge c_1 K^{-1} \alpha n^{1/p - 1/2}$, where $c_1 > 0$ is an absolute constant. By Tomczak-Jaegermann's result on few vectors, (cf. [TJ], Theorems 25.6 and 25.1), the type 2 constant can be essentially computed on n vectors, i.e. there exist vector $(x_i)_{i\le n}$ in X such that

$$\mathbb{E}\Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\| \ge c_2 K^{-1} \alpha n^{1/p-1/2} \Big(\sum_{i=1}^{n} \|x_i\|^2 \Big)^{1/2} \tag{11}$$

for some absolute $c_2 > 0$.

Now we employ a known argument to show that the vectors x_i can be essentially chosen of norm one. We can assume that $\sum_{i=1}^{n} \|x_i\|^2 = n$, so that the right side in (11) is $c_2 K^{-1} \alpha n^{1/p}$. Fix a positive number M and let $\sigma = \{i \in [1,n] : \ \|x_i\| \le M\}$. Then $|\sigma^c| \le (\sum_{i=1}^{n} \|x_i\|^2)/M^2 = M^{-2} n$. Therefore, using the type p of X we see that

$$\begin{aligned} \mathbb{E}\Big\| \sum_{i\in\sigma^c} \varepsilon_i x_i \Big\| &\le K \Big(\sum_{i\in\sigma^c} \|x_i\|^p \Big)^{1/p} \le K |\sigma^c|^{1/p-1/2} \Big(\sum_{i\in\sigma^c} \|x_i\|^2 \Big)^{1/2} \\ &\le K M^{1-2/p} n^{1/p}. \end{aligned}$$

Define the vectors $y_i = x_i/\|x_i\|$, $i \in \sigma$. By the standard comparison principle it follows that

$$\begin{aligned} \mathbb{E}\Big\| \sum_{i\in\sigma} \varepsilon_i y_i \Big\| &\ge M^{-1} \mathbb{E}\Big\| \sum_{i\in\sigma} \varepsilon_i x_i \Big\| \ge M^{-1} \Big(\mathbb{E}\Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\| - \mathbb{E}\Big\| \sum_{i\in\sigma^c} \varepsilon_i x_i \Big\| \Big) \\ &\ge M^{-1} (c_2 K^{-1} \alpha n^{1/p} - K M^{1-2/p} n^{1/p}). \end{aligned} \tag{12}$$

Choosing M so that $KM^{1-2/p} = (c_2/2) K^{-1} \alpha$, we make the right hand side in (12) bounded below by $((c_2\alpha/2) K^{2+p})^{2-p}$. This clearly implies that there exist norm one vectors $(z_i)_{i\le n}$ in X for which

$$\mathbb{E}\Big\| \sum_{i=1}^{n} \varepsilon_i z_i \Big\| \ge c(K, \alpha) n^{1/p}.$$

An application of Proposition 19 for the vectors (z_i) finishes the proof. ■

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