Concentration on the ℓ_p^n Ball

G. Schechtman[1] and J. Zinn[2]

[1] Department of Mathematics, Weizmann Institute of Science, Rehovot, Israel
[2] Department of Mathematics, Texas A&M University, College Station, Texas, USA

Abstract. We prove a concentration inequality for functions, Lipschitz with respect to the Euclidean metric, on the ball of ℓ_p^n, $1 \le p < 2$ equipped with the normalized Lebesgue measure.

1 Introduction

In [SZ] the authors proved an inequality which can be interpreted as giving the right order of the tail distribution of the ℓ_q^n norm on the ℓ_p^n ball equipped with the normalized Lebesgue measure. More precisely and specializing to the case of $q = 2$ and $p = 1$, we proved there that $\mu(\{x : \|x\|_2 > t\})$ is bounded above by $C \exp(-ctn)$ for $t > T/\sqrt{n}$ (where C, c and T are absolute constants) and bounded from below by a similar quantity (with different absolute constants). The measure μ can be either the normalized Lebesgue measure on B_1^n - the ball of ℓ_1^n or the normalized Lebesgue measure on ∂B_1^n - the sphere of ℓ_1^n. (For other p's the relevant measure on the sphere is a different one than the usual surface measure.)

Following a question of M. Gromov, we generalize here this inequality so as to give the right deviation inequality for a general Lipschitz function with respect to the Euclidean metric and for any deviation, i.e., not only for large enough t.

More precisely we prove in Theorem 3.1 that

$$\mu\left(\left\{x : |f(x) - \int f d\mu| > t\right\}\right) \le C \exp(-ctn)$$

For some absolute positive constants C, c and for all $t > 0$.

We also prove a similar result for μ replaced with the normalized Lebesgue measure on the ℓ_p^n ball, $1 < p < 2$. Here the right hand side of the inequality above takes the form $C \exp(-ct^p n)$. This is treated in Section 4. Somewhat surprisingly, for the function $f(x) = \|x\|_2$ we get better concentration results than for a general Lipschitz function. This is done in Section 5.

The first author was supported in part by BSF.
The second author was supported in part by BSF and NSF.

2 Preliminaries

Let λ denote the normalized Lebesgue measure on ∂B_1^n - the unit sphere of ℓ_1^n.

We recall the following known lemma (an equivalent version of which was also used and proved in [SZ]).

Lemma 2.1 *Let $X_1, X_2, \ldots, X_n$ be independent random variables each with density function $\frac{1}{2}e^{-|t|}$ and put $S = (\sum_{i=1}^n |X_i|)$. Then $\left(\frac{X_1}{S}, \frac{X_2}{S}, \ldots, \frac{X_n}{S}\right)$ induces the measure λ on ∂B_1^n. Moreover, $\left(\frac{X_1}{S}, \frac{X_2}{S}, \ldots, \frac{X_n}{S}\right)$ is independent of S.*

We also recall the following fact which is essentially Theorem 3.1 in [SZ] (combine the statement of (1) of Theorem 3 with the last line on page 220).

Theorem 2.2 ([SZ]) *There are absolute positive constants T, c such that for all $t > T/\sqrt{n}$, putting $X = (X_1, X_2, \ldots, X_n)$,*

$$\Pr\left(\|X\|_2/S > t\right) \leq \exp(-ctn).$$

We shall also make essential use of a theorem of Talagrand [Tal] giving a fine deviation inequality, with respect to the probability measure induced by X on R^n, for functions which have controlled Lipschitz constants with respect to both the ℓ_2^n and the ℓ_1^n norms. Maurey [Mau] discovered a relatively simple proof of this inequality while Bobkov and Ledoux [BL] found another simple proof and a far reaching generalization of this inequality. We also refer to the lecture notes [Led] by Ledoux which gives a very nice treatment of this and related inequalities.

Theorem 2.3 ([Tal],[Mau],[BL]) *Let $F : R^n \to R$ be a function satisfying*

$$|F(x) - F(y)| \leq \alpha\|x - y\|_2 \quad \text{and} \quad |F(x) - F(y)| \leq \beta\|x - y\|_1.$$

Then

$$\Pr\left(|F(x) - EF| > r\right) \leq C\exp(-c\min(r/\beta, r^2/\alpha^2))$$

for some absolute positive constants C, c and all $r > 0$. In particular,

$$\Pr\left(\left|\frac{S}{n} - 1\right| > r\right) \leq C\exp(-cn\min(r, r^2)).$$

We refer to [Led], (4.3) on p. 53 from which a similar inequality, with EF replaced by the median of F follows immediately. Replacing the median with the mean is standard (see e.g. [MS] Prop V.4, page 142). The "In particular" part follows from the fact that the function $F(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^n |x_i|$ satisfies

$$|F(x) - F(y)| \leq \frac{1}{\sqrt{n}}\|x - y\|_2 \quad \text{and} \quad |F(x) - F(y)| \leq \frac{1}{n}\|x - y\|_1.$$

We shall also use the following simple corollary to Theorem 2.3.

Corollary 2.4 *For some absolute positive constants C, c and all $r > 0$,*

$$\Pr\left(\left|\frac{S}{n} - 1\right| > r\right) \le C\exp(-c\sqrt{n}r).$$

3 The Main Result

Theorem 3.1 *There exist positive constants C, c such that if $f : \partial B_1^n \to R$ satisfies $|f(x) - f(y)| \le \|x - y\|_2$ for all $x, y \in \partial B_1^n$ then, for all $t > 0$,*

$$\lambda(\{x : \ |f(x) - E(f)| > t\}) \le C\exp(-ctn).$$

Remark. Considering functions that depend only on the first $n-1$ variables, we easily get a similar statement for the full ball B_1^{n-1}.

The main technical part of the proof of Theorem 3.1 is contained in the following lemma. The probability distribution in its statement is the one introduced in the previous section.

Lemma 3.2 *Put $X = (X_1, X_2, \dots, X_n)$. Then for some absolute positive constants C, c and for every Lipschitz function with Lipschitz constant* 1

$$\Pr\left(\left|f\left(\frac{X}{S}\right) - f\left(\frac{X}{n}\right)\right| > t\right) \le Ce^{-ctn}, \text{ for all } 0 < t \le 2.$$

.

Proof. Since f has Lipschitz constant 1,

$$\Pr\left(\left|f\left(\frac{X}{S}\right) - f\left(\frac{X}{n}\right)\right| > t\right) \le \Pr\left(\frac{\|X\|_2}{S}\left|\frac{S}{n} - 1\right| > t\right).$$

We distinguish between two cases.

Case 1. $t \le \frac{T}{\sqrt{n}}$, where T is from Theorem 2.2.

Using first Corollary 2.4 and then Theorem 2.2

$$\begin{aligned}
\Pr\left(\frac{\|X\|_2}{S}\left|\frac{S}{n} - 1\right| > t\right) &= E_{\frac{\|X\|_2}{S}} \Pr\left(\left|\frac{S}{n} - 1\right| > \frac{tS}{\|X\|_2}\right) \\
&\le CE_{\frac{\|X\|_2}{S}} \exp\Big(-\frac{ct\sqrt{n}}{\|X\|_2/S}\Big) \\
&\le C\int_0^{T/\sqrt{n}} \frac{ct\sqrt{n}}{u^2}\exp\Big(-\frac{ct\sqrt{n}}{u}\Big)du \qquad (1) \\
&\quad + C\int_{T/\sqrt{n}}^1 \frac{ct\sqrt{n}}{u^2}\exp\Big(-\frac{ct\sqrt{n}}{u}\Big)\exp(-cun)du. \qquad (2)
\end{aligned}$$

Here and elsewhere in this note C and c denote absolute constants, not necessarily the same in each instance.

The first summand, (1), is equal to $C\exp(-\frac{ct\sqrt{n}}{u})\Big|_0^{T/\sqrt{n}} = C\exp(-c'tn)$. For the second summand, (2), observe that the maximum of $\exp(\frac{-ct\sqrt{n}}{u} - cun)$ occurs when $u^2 = t/\sqrt{n}$ and at this point the maximum is: $\exp(-2c\sqrt{t}n^{3/4})$. Hence (2) is dominated by

$$C\exp(-2c\sqrt{t}n^{3/4})\int_{T/\sqrt{n}}^{1}\frac{ct\sqrt{n}}{u^2} \le C'tn\exp(-2c\sqrt{t}n^{3/4}).$$

Since, for $t \le \frac{T}{\sqrt{n}}$, $t^{1/2}n^{3/4} \ge T^{-1/2}tn$, the last quantity is bounded by $Ctn\exp(-ctn)$. Since we may assume that $tn \ge 1$, this quantity is bounded by $C\exp(-ctn)$.

Case 2. $\frac{T}{\sqrt{n}} \le t \le 2$.

We write

$$\Pr\left(\frac{\|X\|_2}{S}\left|\frac{S}{n}-1\right| > t\right) = \Pr\left(\frac{\|X\|_2}{S}\left|\frac{S}{n}-1\right| > t \text{ and } \frac{\|X\|_2}{S} \le t\right) + \Pr\left(\frac{\|X\|_2}{S}\left|\frac{S}{n}-1\right| > t \text{ and } \frac{\|X\|_2}{S} > t\right). \quad (3)$$

Using Lemma 2.1 and Theorem 2.3,

$$\begin{aligned}\Pr\left(\frac{\|X\|_2}{S}\left|\frac{S}{n}-1\right| > t \text{ and } \frac{\|X\|_2}{S} \le t\right) &\le E_{\frac{\|X\|_2}{S}}\left(\exp\left(\frac{-cntS}{||X||_2}\right) I_{\left(\frac{||X||_2}{S} \le t\right)}\right)\\ &\le C\int_0^t \frac{tn}{u^2}\exp\left(\frac{-ctn}{u}\right)\Pr\left(u \le \frac{\|X\|_2}{S} \le t\right)\,du\\ &\le C\int_0^t \frac{tn}{u^2}\exp\left(\frac{-ctn}{u}\right)\,du.\end{aligned}$$

Now,

$$\int_0^{1/\sqrt{n}} \frac{tn}{u^2}\exp(-ctn/u)\,du = C\exp(-ctn^{3/2}).$$

While if T is large enough,

$$\begin{aligned}\int_{1/\sqrt{n}}^{t} \frac{tn}{u^2}\exp(-ctn/u)\,du &\le \int_{1/\sqrt{n}}^{t} \frac{tn}{u^2}\exp(-cn)\,du\\ &= nte^{-cn}[\sqrt{n} - 1/t] \le n^{3/2}te^{-cn} \le Ce^{-c'n}.\end{aligned}$$

This takes care of the first summand in (3). For the second summand we use a similar line of inequalities to that of Case 1, using Corollary 2.4 and Theorem 2.2.

$$\begin{aligned}\Pr\left(\frac{\|X\|_2}{S}\left|\frac{S}{n}-1\right|>t \text{ and } \frac{\|X\|_2}{S}>t\right)&\\ &\le C\int_t^1 \frac{t\sqrt{n}}{u^2}\exp\left(\frac{-ct\sqrt{n}}{u}-cun\right)du\\ &\le C\exp(-ctn)\int_t^1 \frac{t\sqrt{n}}{u^2}\exp\left(\frac{-ct\sqrt{n}}{u}\right)du\\ &\le C\exp(-ctn).\end{aligned}$$

Proof of Theorem 3.1. Given a function on ∂B_1^n which is Lipschitz with constant one, extend it to a function on all of R^n satisfying the same Lipschitz condition. (There are many ways to do it, for example $\bar{f}(x)=\inf_{y\in\partial B_1^n}(f(y)+\|x-y\|_2)$.) We shall continue to call the extended function f.

Theorem 2.3 implies easily that it Y is an independent copy of $X=(X_1,\ldots,X_n)$ then for all $t>0$

$$\Pr\left(\left|f\left(\frac{X}{n}\right)-f\left(\frac{Y}{n}\right)\right|>t\right)\le Ce^{-ctn}.$$

Combining this with Lemma 3.2 and Lemma 2.1 we get that

$$\lambda\times\lambda\left(\{(x,y):\ |f(x)-f(y)|>t\right)\le Ce^{-ctn}$$

from which the statement of the theorem follows by standard arguments (see e.g. [MS] Prop V.4, page 142).

Next we show in Proposition 3.4 below that, for each fixed $t\le 1/4$, the result of Theorem 3.1 is best possible, except for the choice of the universal constants c,C. Note that the function $f(x)=\|x\|_\infty=\max\{|x_1|,\ldots,|x_n|\}$ is Lipschitz with constant one. We first prove a somewhat weaker result whose proof is much simpler.

Proposition 3.3 *For each* $2\log n/n\le t\le 1/4$*,*

$$\lambda\Big(\Big\{x:\ \Big|\|x\|_\infty-\int\|\cdot\|_\infty d\lambda\Big|>t\Big\}\Big)\ge e^{-4tn}.$$

Proof. Note first that by Lemma 2.1, $\int\|\cdot\|_\infty d\lambda=E(\max\{|X_1|,\ldots,|X_n|\})/ES$. It is now easy to deduce that $\int\|\cdot\|_\infty d\lambda<2\log n/n$. Thus,

$$\begin{aligned}\lambda\Big(\Big\{x:\ \Big|\|x\|_\infty-\int\|\cdot\|_\infty d\lambda\Big|>t\Big\}\Big)&\\ &\ge\Pr\left(\max_{i\le n}|X_i|\ge(2\log n/n+t)S\right)\end{aligned}$$

$$\geq \Pr(|X_1| \geq 2tS) \geq \Pr\left(|X_1| \geq \frac{2t}{1-2t}\sum_{i=2}^{n}|X_i|\right)$$
$$= Ee^{\frac{-2t}{1-2t}\sum_{i=2}^{n}|X_i|} = (1-2t)^{n-1}$$
$$\geq e^{-\frac{2t(n-1)}{1-2t}} \geq e^{-4tn}.$$

Proposition 3.4 *For each $t \leq 1/4$,*

$$\lambda\left(\left\{x: \left|\|x\|_\infty - \int \|\cdot\|_\infty d\lambda\right| > t\right\}\right) \geq ce^{-4tn}$$

for some absolute constant $c > 0$.

Proof. By Proposition 3.3 it is enough to assume $t < 2\log n/n$. Let $M = \|X\|_\infty$, $\alpha = t + \frac{EM}{n} < 1, \beta = \frac{\alpha}{1-\alpha}$ and let $S_k = \sum_{i=1}^{k}|X_i|$.

$$P\left(\frac{M}{S} > \alpha\right) \geq nP(X_n > \alpha S) - \binom{n}{2}P(X_{n-1}, X_n > \alpha S) = I - II.$$

But,

$$I = nP(X_n > \frac{\alpha}{1-\alpha}S_{n-1}) = nE\exp(-\beta S_{n-1})$$
$$= n(E\exp(-\beta X))^{n-1} = n\left(\frac{1}{1+\beta}\right)^{n-1}$$

while,

$$II = \binom{n}{2}P(X_{n-1} - \beta X_n > \beta S_{n-2}, X_n - \beta X_{n-1} > \beta S_{n-2})$$
$$\leq \binom{n}{2}P((1-\beta^2)X_{n-1}, (1-\beta^2)X_n > (\beta^2+\beta)S_{n-2})$$
$$= \binom{n}{2}\left(E\exp\left(-2\Big(\frac{\beta^2+\beta}{1-\beta^2}\Big)X\right)\right)^{n-2} = \binom{n}{2}\left(\frac{1}{1+\frac{2(\beta^2+\beta)}{1-\beta^2}}\right)^{n-2}$$
$$= \binom{n}{2}\left(\frac{1-\beta}{1+\beta}\right)^{n-2} \leq \frac{n^2}{2}\left(\frac{1}{1+\beta}\right)^{2(n-2)}.$$

Factoring,

$$I - II \geq \frac{n}{(1+\beta)^{n-1}}\left[1 - \frac{n}{2}(1-\alpha)^{n-3}\right]. \tag{4}$$

Notice that

$$\frac{n}{(1+\beta)^{n-1}} = n(1-\alpha)^{n-1} \geq n\exp\Big(-(n-1)\frac{\alpha}{1-\alpha}\Big)$$

$$
\begin{aligned}
&= n \exp\left(\frac{\alpha}{1-\alpha}\right) \exp(-(nt/(1-\alpha)) \exp(-EM/(1-\alpha)) \\
&\geq n^{\frac{-\alpha}{1-\alpha}} \exp\left(\frac{\alpha}{1-\alpha}\right) \exp(-(nt/(1-\alpha)) \exp\left(-\frac{(EM - \log n)}{1-\alpha}\right).
\end{aligned}
$$

But,

$$
\begin{aligned}
EM &= \int_0^\infty P(M > t)\, dt = \int_0^\infty \left[1 - (1 - \exp(-t))^n\right] dt \\
&= \int_0^1 \frac{1-u^n}{1-u}\, du = \int_0^1 \sum_{j=0}^{n-1} u^j\, du = \sum_{j=0}^{n-1} \frac{1}{j+1},
\end{aligned}
$$

so $EM - \log n$ converges to a positive limit (Euler's constant). Since both $n^{\frac{-\alpha}{1-\alpha}}$ and $\exp(\frac{\alpha}{1-\alpha})$ tend to 1, we get from (4) that

$$
I - II \geq c\left(1 - \frac{n}{2}(1-\alpha)^{n-3}\right) e^{-4tn} \geq c\left(1 - \frac{n}{2}\exp(-\alpha(n-3))\right) e^{-4tn}.
$$

Hence, it suffices to have $(n-3)\alpha \geq \log n$ or $nt \geq \log n - EM + 3\alpha$. This last quantity is asymptotically negative, since $\log n - EM$, is asymptotically negative and $\alpha \to 0$.

4 Concentration on the ℓ_p^n Ball, $1 < p < 2$

Theorem 3.1 and the well known concentration estimate for Lipschitz function on the Euclidean sphere or ball (which is of the form $C\exp(-ct^2 n)$) suggest that a similar result with estimate $C\exp(-ct^p n)$ holds on the ℓ_p^n ball, $1 < p < 2$. This is indeed the case but the proof seems to require a very new result due to Latała and Oleszkiewicz [LO]. Actually [LO] was motivated in part by a question of the authors whose motivation was Theorem 4.1 below. Since the proof is very similar to that of the case $p = 1$ we only sketch it.

Theorem 4.1 *There exist positive constants C, c such that if $1 < p < 2$ and $f : \partial B_p^n \to R$ satisfies $|f(x) - f(y)| \leq \|x - y\|_2$ for all $x, y \in B_p^n$ then, for all $t > 0$,*

$$
\lambda(\{x :\ |f(x) - E(f)| > t\}) \leq C\exp(-ct^p n).
$$

Here λ denotes the measure on the ℓ_p^n sphere which assign to each set A the normalized Lebesgue measure, on the ℓ_p^n ball, of the set $\{tA\ :\ 0 \leq t \leq 1\}$. It is then easy to deduce the same result (with different constants) for the normalized Lebesgue measure on the ℓ_p^n ball.

As we indicated above, the role of Theorem 2.3 will be replaced here with the following result which is a combination of Theorems 1 and 2 of [LO].

Theorem 4.2 ([LO]) *Let $F : R^n \to R$ be a function satisfying $|F(x) - F(y)| \le \alpha \|x-y\|_2$, let* Pr *denotes the probability distribution on R^n with density $c_p^n \exp(-|x_1|^p - \cdots - |x_n|^p)$ and denote $S = (|X_1|^p + \cdots + |X_n|^p)^{1/p}$. Then*

$$\Pr(|F(X) - EF| > r) \le C\exp(-c(r/\alpha)^p)$$

for some absolute positive constants C, c and all $r > 0$. In particular,

$$\Pr\left(\left|\frac{S}{n^{1/p}} - \frac{ES}{n^{1/p}}\right| > r\right) \le C\exp(-cn^{p/2}r^p).$$

The analogues of Lemma 2.1 and Theorem 2.2 also hold, i.e. with the new interpretation of Pr, λ and S, $\left(\frac{X_1}{S}, \frac{X_2}{S}, \ldots, \frac{X_n}{S}\right)$ induces the measure λ on ∂B_p^n, and $\left(\frac{X_1}{S}, \frac{X_2}{S}, \ldots, \frac{X_n}{S}\right)$ is independent of S. Also

Theorem 4.3 ([SZ]) *There are absolute positive constants T, c such that for all $t > T/n^{1/2-1/p}$, putting $X = (X_1, X_2, \ldots, X_n)$,*

$$\Pr(\|X\|_2/S > t) \le \exp(-ct^p n).$$

Denote $\alpha = \alpha(n,p) = \frac{ES}{n^{1/p}}$ and note that $\alpha(n,p)$ is bounded away from zero and ∞ by universal constants, for $1 < p < 2$. As for the case $p = 1$, Theorem 4 will follow (using Theorem 4.2) once we establish

Lemma 4.4 *For every Lipschitz function f with constant* 1

$$\Pr\left(\left|f\left(\frac{X}{S}\right) - f\left(\frac{\alpha X}{n^{1/p}}\right)\right| > t\right) \le Ce^{-ct^p n}.$$

For some absolute positive constants C, c and for all $0 < t \le 2$.

Sketch of proof. As in the proof of Lemma 3.2, the proof reduces to estimating

$$\Pr\left(\frac{\|X\|_2}{S}\left|\frac{S}{n^{1/p}} - \alpha\right| > t\right) = E_{\frac{\|X\|_2}{S}} \Pr\left(\left|\frac{S}{n} - \alpha\right| > \frac{tS}{\|X\|_2}\right)$$

which by Theorem 4.2 is dominated by

$$CE_{\frac{\|X\|_2}{S}} \exp\left(-\frac{ct^p n^{p/2}}{(\|X\|_2/S)^p}\right)$$
$$= C\int_0^\infty \frac{pct^p n^{p/2}}{u^{p+1}} \exp\left(-\frac{ct^p n^{p/2}}{u^p}\right) \Pr\left(\frac{\|X\|_2}{S} > u\right) du. \qquad (5)$$

Case 1. $t \le Tn^{1/2-1/p}$.

The right hand side of (5) is dominated by

$$\begin{aligned}\int_0^{Tn^{1/2-1/p}} \frac{pct^p n^{p/2}}{u^{p+1}} \exp\Big(-\frac{ct^p n^{p/2}}{u^p}\Big) du \\ + \int_{Tn^{1/2-1/p}}^1 \frac{pct^p n^{p/2}}{u^{p+1}} \exp\left(-c\Big(\frac{t^p n^{p/2}}{u^p} + u^p n\Big)\right) du \\ = I + II.\end{aligned}$$

I is equal to $\exp(-c't^p n)$ while in II the integrand is dominated by $\frac{ct^p n^{p/2}}{u^{p+1}}$ $\exp(-c't^{p/2} n^{p/4+1/2})$ and thus II is dominated by $t^p n \exp(-ct^{p/2} np/4 + 1/2)$. The bound $t \le Tn^{1/2-1/p}$ implies now that II is also dominated by $\exp(-c't^p n)$.

Case 2. $t > Tn^{1/2-1/p}$.
Write the right hand side of (5) as

$$C\left(\int_0^{Tn^{1/2-1/p}} + \int_{Tn^{1/2-1/p}}^t + \int_t^1\right) \frac{pct^p n^{p/2}}{u^{p+1}} \exp\Big(-\frac{ct^p n^{p/2}}{u^p}\Big) \Pr\left(\frac{\|X\|_2}{S} > u\right) du$$
$$= I + II + III. \tag{6}$$

Estimating $\Pr\left(\frac{\|X\|_2}{S} > u\right)$ by 1, I is bounded by $\exp(-c't^p n)$. II is bounded by

$$\exp(-cn^{p/2}) \int_{Tn^{1/2-1/p}}^t \frac{pct^p n^{p/2}}{u^{p+1}} du \le ct^p n \exp(-cn^{p/2}) \le \exp(-c'n^{p/2}).$$

Finally III of (6) is dominated by

$$\begin{aligned}&\int_t^1 \frac{pct^p n^{p/2}}{u^{p+1}} \exp\left(-c\Big(\frac{t^p n^{p/2}}{u^p} + u^p n\Big)\right) du \\ &\le \exp(-ct^p n) \int_t^1 \frac{pct^p n^{p/2}}{u^{p+1}} \exp\Big(-c\frac{t^p n^{p/2}}{u^p}\Big) du \\ &< \exp(-ct^p n).\end{aligned}$$

5 The Function $f(x) = \|x\|_2$

As we remarked in the introduction, for t larger than an absolute constant divided by $\sqrt{n}$, the conclusion of Theorem 3.1 is best possible for the function $f(x) = \|x\|_2$. It turns out that for smaller values of t a stronger inequality holds.

Proposition 5.1 *Let*

$$\alpha(n,t) = \begin{cases} nt, & \text{for } t > n^{-1/2} \\ n^{3/4} t^{1/2}, & \text{for } n^{-5/6} < t \le n^{-1/2} \\ n^2 t^2, & \text{for } 0 < t \le n^{-5/6}. \end{cases}$$

Then, for some absolute constants $0 < c, C < \infty$,

$$\lambda\left(\left\{x:\ \left|\|x\|_2 - \int \|\cdot\|_2 d\lambda\right| > t\right\}\right) \le Ce^{-c\alpha(n,t)}.$$

For $t > T/\sqrt{n}$ the proposition was proved in [SZ]. For the lower values of t, as in the proof of the main result, it is enough to prove

$$\Pr\left(\frac{\|X\|_2}{S}\left|\frac{S}{n} - 1\right| > t\right) \le Ce^{-c\alpha(n,t)} \tag{7}$$

and, for Y independent of X,

$$\Pr\left(|\|X\|_2 - \|Y\|_2| > nt\right) \le Ce^{-c\alpha(n,t)}. \tag{8}$$

Proof of (7). As in Case 1 in the proof of Lemma 3.2 (using Theorem 2.3 instead of Corollary 2.4),

$$\begin{aligned}\Pr\left(\frac{\|X\|_2}{S}\left|\frac{S}{n} - 1\right| > t\right) &= E_{\frac{\|X\|_2}{S}} \Pr\left(\left|\frac{S}{n} - 1\right| > \frac{tS}{\|X\|_2}\right) \\ &\le CE_{\frac{\|X\|_2}{S}} \exp\left(-cn\left(\left(\frac{tS}{\|X\|_2}\right) \wedge \left(\frac{tS}{\|X\|_2}\right)^2\right)\right).\end{aligned}$$

Now,

$$E_{\frac{\|X\|_2}{S}}\left[\exp\left(-cn\left(\left(\frac{tS}{\|X\|_2}\right) \wedge \left(\frac{tS}{\|X\|_2}\right)^2\right)\right) 1_{\|X\|_2 \le tS}\right] \le e^{-cn}$$

while by Theorem 2.2,

$$\begin{aligned}&E_{\frac{\|X\|_2}{S}}\left[\exp\left(-cn\left(\left(\frac{tS}{\|X\|_2}\right) \wedge \left(\frac{tS}{\|X\|_2}\right)^2\right)\right) 1_{\|X\|_2 > tS}\right] \\ &\quad = \int_0^1 \frac{2cnt^2}{u^3} e^{-cnt^2/u^2} \Pr\left(\frac{\|X\|_2}{S} > u \vee t\right) \\ &\quad \le \int_0^{T/\sqrt{n}} \frac{2cnt^2}{u^3} e^{-cnt^2/u^2} + \int_{T/\sqrt{n}}^{\infty} \frac{2cnt^2}{u^3} e^{-cnt^2/u^2} e^{-cnu} du. \qquad (9)\end{aligned}$$

The first summand in (9) is equal to $e^{-cn^2t^2/T^2}$. Noticing that the maximum of $e^{-cnt^2/u^2}e^{-cnu}$ occurs at $u = 2^{1/3}t^{2/3}$ and is equal to $e^{-c'nt^{2/3}}$, we get that the second summand in (9) is dominated by

$$e^{-c'nt^{2/3}} \int_{T/\sqrt{n}}^{\infty} \frac{2cnt^2}{u^3} du = \frac{cn^2t^2}{T^2} e^{-c'nt^{2/3}} \le C'e^{-c''nt^{2/3}}$$

for the relevant range of t (i.e., t larger than an absolute constant times $1/n$). Summarizing, we get that (with different absolute constants c, C)

$$\Pr\left(\frac{\|X\|_2}{S}\left|\frac{S}{n}-1\right|>t\right)\le Ce^{-cn^2t^2}+Ce^{-cnt^{2/3}} \tag{10}$$

and it is easy to see that this is dominated by $Ce^{-c\alpha(n,t)}$.

Proof of (8). First note that by Theorem 2.3, $\Pr(\|Y\|_2<c\sqrt{n})<Ce^{-c\sqrt{n}}$ for an appropriate C and that $\sqrt{n}$ is larger than a constant times $\alpha(n,t)$ in the range in question. Thus,

$$\begin{aligned}\Pr\left(\|X\|_2-\|Y\|_2>nt\right)&=\Pr\left(\|X\|_2^2>\|Y\|_2^2+2\|Y\|_2nt+n^2t^2\right)\\&\le Ce^{-c\alpha(n,t)}+\Pr\left(\|X\|_2^2>\|Y\|_2^2+cn^{3/2}t\right)\end{aligned} \tag{11}$$

and it is enough to prove that

$$\Pr\left(\sum_{i=1}^n(X_i^2-Y_i^2)>cn^{3/2}t\right)\le Ce^{-c\alpha(n,t)}. \tag{12}$$

To prove (12) note that for each p the left hand side of (12) is dominated by an absolute constant times $\frac{E|\sum\epsilon_iX_i|^p}{(n^{3/2}t)^p}$. Apply now the result of Hitczenko, Montgomery-Smith and Oleszkiewicz [HMO, Th. 4.2] to the variables X_i^2 and get that, for each p, the left hand side of (12) is dominated by

$$\left(\frac{C(n^{1/p}(2p)^2\wedge p^{1/2}n^{1/2})}{n^{3/2}t}\right)^p. \tag{13}$$

For t larger than $n^{-5/6}$ take $p=c'n^{3/4}t^{1/2}$ for c' small with respect to C. Then (13) is dominated by $C'e^{-c''n^{3/4}t^{1/2}}$. For t smaller than $n^{-5/6}$ but larger than a constant times $1/n$ take $p=c'n^2t^2$ and get that (13) is dominated by $C'e^{-c''n^2t^2}$.

Remarks. 1. The use of [HMO] was suggested to us by S. Kwapien. The result of [HMO] was considerably generalized by Latała [Lat]. Originally we had another (more direct but also more special to the variables in question) proof of (12): After appropriately truncating the X_i's evaluate $\exp(t\sum\epsilon_iX_i^2\mathbb{1}_{|X_i|<a})$ by carefully evaluating the terms in the Taylor expansion. This follows closely [Bou, Lemma 1].

2. Is the statement of Lemma 5.1 best possible? for $t>Cn^{-1/2}$ this was proved in [SZ]. For $n^{-5/6}<t<Cn^{-1/2}$ this is still the case. This easily follows from the following three facts: a. The proof of (7) in this case gives an estimate, (10) which is of better order of magnitude than $e^{-c\alpha(n,t)}$.
b. The inequalities in (11) can be inverted (with different constants of course).
c.

$$2\Pr\left(|\sum_{i=1}^n\epsilon_iX_i^2|>cn^{3/2}t\right)\ge\Pr\left(X_1^2>cn^{3/2}t\right)=e^{-n^{3/4}t^{1/2}}.$$

For $n^{-1} < t < n^{-5/6}$ one can show the right lower bound for the quantity in the left end sides of (11) but it doesn't seem to combine nicely with upper bound for (7).

References

[BL] Bobkov S., Ledoux M. (1997) Poincaré's inequalities and Talagrand's concentration phenomenon for the exponential distribution. Probab. Theory Related Fields 107:383–400

[Bou] Bourgain J. (1999) Random points in isotropic convex sets. In: Convex Geometric Analysis (Berkeley, CA, 1996), Math. Sci. Res. Inst. Publ., 34, Cambridge Univ. Press, Cambridge, 53–58

[GM] Gromov M., Milman V.D. (1983) A topological application of the isoperimetric inequality. Amer. J. Math. 105:843–854

[HMO] Hitczenko P., Montgomery-Smith S.J., Oleszkiewicz K. (1997) Moment inequalities for sums of certain independent symmetric random variables. Studia Math. 123:15–42

[Lat] Latała R. (1997) Estimation of moments of sums of independent real random variables. Ann. Probab. 25:1502–1513

[LO] Latała R., Oleszkiewicz K. Between Sobolev and Poincare. This volume

[Led] Ledoux M. (1997) Concentration of measure and logarithmic Sobolev inequalities. Preprint

[Mau] Maurey, B. (1991) Some deviation inequalities. Geom. Funct. Anal. 1:188–197

[MS] Milman V.D., Schechtman G. (1986) Asymptotic theory of finite–dimensional normed spaces. Lecture Notes in Mathematics, 1200, Springer-Verlag, Berlin-New York

[SZ] Schechtman G., Zinn J. (1990) On the volume of the intersection of two L_p^n balls. Proc. A.M.S. 110:217–224

[ScSc] Schechtman G., Schmuckenschläger M. (1991) Another remark on the volume of the intersection of two L_p^n balls. Geometric Aspects of Functional Analysis (1989–90), Lecture Notes in Math., Springer, 1469:174–178

[Tal] Talagrand M. (1991) A new isoperimetric inequality and the concentration of measure phenomenon. Geometric Aspects of Functional Analysis (1989–90), Lecture Notes in Math., Springer, 1469:94–124