

Isotropic constants and Mahler volumes

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Abstract

This paper contains a number of results related to volumes of projective perturbations of convex bodies and the Laplace transform on convex cones. First, it is shown that a sharp version of Bourgain's slicing conjecture implies the Mahler conjecture for convex bodies that are not necessarily centrally-symmetric. Second, we find that by slightly translating the polar of a centered convex body, we may obtain another body with a bounded isotropic constant. Third, we provide a counter-example to a conjecture by Kuperberg on the distribution of volume in a body and in its polar.

1 Introduction

This paper describes interrelations between duality and distribution of volume in convex bodies. A convex body is a compact, convex subset $K \subseteq \mathbb{R}^n$ whose interior $\text{Int}(K)$ is non-empty. If $0 \in \text{Int}(K)$, then the polar body is defined by

$$K^\circ = \{y \in \mathbb{R}^n ; \forall x \in K, \langle x, y \rangle \leq 1\}.$$

The polar body K° is itself a convex body with the origin in its interior, and moreover $(K^\circ)^\circ = K$. The *Mahler volume* of a convex body $K \subseteq \mathbb{R}^n$ with the origin in its interior is defined as

$$s(K) = \text{Vol}_n(K) \cdot \text{Vol}_n(K^\circ),$$

where Vol_n is n -dimensional volume. In the class of convex bodies with barycenter at the origin, the Mahler volume is maximized for ellipsoids, as proven by Santaló [29], see also Meyer and Pajor [20]. The Mahler conjecture suggests that for any convex body $K \subseteq \mathbb{R}^n$ containing the origin in its interior,

$$s(K) \geq s(\Delta^n) = \frac{(n+1)^{n+1}}{(n!)^2}, \quad (1)$$

where $\Delta^n \subseteq \mathbb{R}^n$ is any simplex whose vertices span \mathbb{R}^n and add up to zero. The conjecture was verified for convex bodies with certain symmetries in the works of Barthe and Fradelizi [3], Kuperberg [14] and Saint Raymond [28]. In two dimensions the conjecture was proven by Mahler [18], see also Meyer [19]. There is also a well-known version of the Mahler conjecture for centrally-symmetric convex bodies (i.e., when $K = -K$) that will not be

discussed much here. It was proven by Bourgain and Milman [8] that for any convex body $K \subseteq \mathbb{R}^n$ with the origin in its interior,

$$s(K) \geq c^n \cdot s(\Delta^n) \quad (2)$$

for some universal constant $c > 0$. There are several beautiful, completely different proofs of the Bourgain-Milman inequality in addition to the original argument, including proofs by Kuperberg [14], by Nazarov [23] and by Giannopoulos, Paouris and Vritsiou [11]. The covariance matrix of a convex body $K \subseteq \mathbb{R}^n$ is the matrix $\text{Cov}(K) = (\text{Cov}_{ij})_{i,j=1,\dots,n}$ where

$$\text{Cov}_{ij} = \frac{\int_K x_i x_j dx}{\text{Vol}_n(K)} - \frac{\int_K x_i}{\text{Vol}_n(K)} \cdot \frac{\int_K x_j}{\text{Vol}_n(K)}.$$

The isotropic constant of a convex body $K \subseteq \mathbb{R}^n$ is the parameter $L_K > 0$ defined via

$$L_K^{2n} = \frac{\det \text{Cov}(K)}{\text{Vol}_n(K)^2}. \quad (3)$$

We equip the space of convex bodies in \mathbb{R}^n with the usual Hausdorff topology. The Mahler volume is a continuous function defined on the class of convex bodies in \mathbb{R}^n containing the origin in their interior. A standard compactness argument shows that the minimum of the Mahler volume in this class is indeed attained. Below we present a variational argument in the class of projective images of K that yields the following:

Theorem 1.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body which is a local minimizer of the Mahler volume. Then $\text{Cov}(K^\circ) \geq (n+2)^{-2} \cdot \text{Cov}(K)^{-1}$ in the sense of symmetric matrices, and*

$$L_K \cdot L_{K^\circ} \cdot s(K)^{1/n} \geq \frac{1}{n+2}. \quad (4)$$

Consequently any global minimizer must satisfy $L_K \geq L_{\Delta^n}$ or $L_{K^\circ} \geq L_{\Delta^n}$.

It is well-known that $L_K > c$ for any convex body $K \subseteq \mathbb{R}^n$, where $c > 0$ is a universal constant. In fact, the minimal isotropic constant is attained for ellipsoids. Bourgain's slicing problem [4, 5] asks whether $L_K < C$ for a universal constant $C > 0$. The slicing conjecture has several equivalent formulations, and it is related to quite a few asymptotic questions about the distribution of volume in high-dimensional convex bodies. Currently the best known estimate is $L_K \leq Cn^{1/4}$ for a convex body $K \subseteq \mathbb{R}^n$. This was shown in [12], slightly improving upon an earlier estimate by Bourgain [6, 7]. Two sources of information on the slicing problem are the recent book by Brazitikos, Giannopoulos, Valettas and Vritsiou [9] and the survey paper by Milman and Pajor [21]. A strong version of the slicing conjecture proposes that for any convex body $K \subseteq \mathbb{R}^n$,

$$L_K \leq L_{\Delta^n} = \frac{(n!)^{\frac{1}{n}}}{(n+1)^{\frac{n+1}{2n}} \cdot \sqrt{n+2}}. \quad (5)$$

This conjecture holds true in two dimensions. See also Rademacher [25] for supporting evidence. Theorem 1.1 admits the following:

Corollary 1.2. *The strong version (5) of Bourgain’s slicing conjecture implies Mahler’s conjecture (1).*

In order to see why inequality (4) implies Corollary 1.2, observe that by (4) and (5), for any local minimizer $K \subseteq \mathbb{R}^n$ of the Mahler volume,

$$\frac{1}{(n+2)^n} \leq L_K^n \cdot L_{K^\circ}^n \cdot s(K) \leq L_{\Delta^n}^{2n} \cdot s(K) = \frac{(n!)^2}{(n+1)^{n+1} \cdot (n+2)^n} \cdot s(K),$$

which clearly yields (1). We are aware of two more conditional statements in the spirit of Corollary 1.2. Artstein-Avidan, Karasëv and Ostrover [2] proved that the Mahler conjecture for centrally-symmetric bodies would follow from the Viterbo conjecture in symplectic geometry. It was recently shown that the Minkowski conjecture would follow from a strong version of the centrally-symmetric slicing conjecture, see Magazinov [17].

For the proof of Theorem 1.1 we use the Laplace transform in order to analyze the Mahler volume in the space of projective images of K . The Laplace transform was also used in [12] in order to prove the isomorphic version of the slicing problem. Here we observe the following variant of the result from [12]:

Theorem 1.3. *Let $K \subseteq \mathbb{R}^n$ be a convex body with barycenter at the origin and let $0 < \varepsilon < 1/2$. Then there exists a convex body $T \subseteq \mathbb{R}^n$ such that the following hold:*

- (i) $(1 - \varepsilon)K \subseteq T \subseteq (1 + \varepsilon)K$.
- (ii) The polar body T° is a translate of K° , i.e., $T^\circ = K^\circ - y$ for some $y \in \text{Int}(K^\circ)$.
- (iii) $L_T < C/\sqrt{\varepsilon}$, where $C > 0$ is a universal constant.

We say that two convex sets $K_1 \subseteq \mathbb{R}^{n_1}$ and $K_2 \subseteq \mathbb{R}^{n_2}$ are *affinely equivalent* if there exists an affine map $T : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}$, one-to-one on the set K_1 , with $K_2 = T(K_1)$. It is a curious fact that the quantity

$$L_K^2 \cdot s(K)^{1/n}$$

attains the same value $1/(n+2)$ when $K \subseteq \mathbb{R}^n$ is an ellipsoid and when $K \subseteq \mathbb{R}^n$ is a simplex, see Alonso–Gutiérrez [1], where in these two examples and also in the next one we assume that the barycenter of K lies at the origin. Intriguingly,

$$L_K^2 \cdot s(K)^{1/n} = 1/(n+2) \tag{6}$$

also when $n = \ell(\ell+1)/2 - 1$ and $K \subseteq \mathbb{R}^n$ is affinely equivalent to the collection of all symmetric, positive semi-definite $\ell \times \ell$ matrices of trace one. This is not a mere coincidence. A common feature amongst these three examples is that they are hyperplane sections of *convex homogeneous cones*.

A convex cone in \mathbb{R}^{n+1} is a convex subset V such that $\lambda x \in V$ for any $x \in V$ and $\lambda \geq 0$. We say that a convex cone $V \subseteq \mathbb{R}^{n+1}$ is proper if it is closed, has a non-empty

interior, and does not contain a full line. A convex cone $V \subseteq \mathbb{R}^{n+1}$ is *homogeneous* if for any $x, y \in \text{Int}(V)$ there is an invertible, linear transformation $T : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ with $T(x) = y$ and

$$T(V) = V.$$

Recall that the Santaló point of a convex body $K \subseteq \mathbb{R}^n$ is the unique point x_0 in the interior of K such that the barycenter of $(K - x_0)^\circ$ lies at the origin. See, e.g., Schneider [31, Section 10.5] for information about the Santaló point.

Theorem 1.4. *Let $K \subseteq \mathbb{R}^n$ be a convex body that is affinely equivalent to a hyperplane section of a convex, homogeneous cone in \mathbb{R}^{n+1} . Then,*

(i) *The barycenter of K is its Santaló point.*

(ii) *If the barycenter of K lies at the origin, then*

$$\text{Cov}(K^\circ) = (n + 2)^{-2} \cdot \text{Cov}(K)^{-1}, \quad (7)$$

and consequently $L_K \cdot L_{K^\circ} \cdot s(K)^{1/n} = 1/(n + 2)$.

Thus, for instance, (6) also holds true for the convex set that consists of all positive-definite Hermitian or quaternionic-Hermitian matrices of trace one. In all of the examples above (ball, simplex and convex collections of matrices) the convex body K° has turned out to be a linear image of K as the cone is self-dual, thus $L_K = L_{K^\circ}$. There exist homogeneous cones that are not self-dual (see, e.g., Vinberg [33] and references therein), yet we see from Theorem 1.4 that the barycenters and covariance matrices of hyperplane sections of homogeneous cones automatically satisfy a certain duality property. Similar duality properties apply for higher moments as well.

The relation (7) between the covariance of a convex body and the covariance of the polar body reminds us of the quantity

$$\phi(K) = \mathbb{E}\langle X, Y \rangle^2 = \frac{1}{s(K)} \int_K \int_{K^\circ} \langle x, y \rangle^2 dx dy \quad (8)$$

considered by Kuperberg [14]. Here, X and Y are independent random vectors, X is uniformly distributed in the convex body K , while Y is uniformly distributed in K° . Assume that $K \subseteq \mathbb{R}^n$ satisfies the assumptions of Theorem 1.4(ii). Then by the conclusion of Theorem 1.4,

$$\phi(K) = \text{Tr}[\text{Cov}(K) \cdot \text{Cov}(K^\circ)] = n/(n + 2)^2.$$

It was shown by Kuperberg [16] that the Euclidean ball is a local maximizer of the functional $K \mapsto \phi(K)$ in the class of centrally-symmetric convex bodies in \mathbb{R}^n with a C^2 -smooth boundary endowed with the natural topology. Conjecture 5.1 in [14] suggests that this local maximum is in fact a global one, i.e., that

$$\phi(K) \leq n/(n + 2)^2$$

for any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$. This conjecture was verified in the case where K is the unit ball of ℓ_p^n for $1 \leq p \leq \infty$, see Alonso-Gutiérrez [1]. Relations of $\phi(K)$ to the slicing problem were observed by Giannopoulos in [10], where it was shown that $\phi(K) \leq C/n$ when $K \subseteq \mathbb{R}^n$ is a body of revolution and $C > 0$ is a universal constant.

Nevertheless, Kuperberg described Conjecture 5.1 in [14] as “perhaps less likely”. Our next proposition shows that this conjecture is indeed false in a sufficiently high dimension. A convex body $K \subseteq \mathbb{R}^n$ is *unconditional* if for any $x \in \mathbb{R}^n$,

$$x = (x_1, \dots, x_n) \in K \quad \iff \quad (|x_1|, \dots, |x_n|) \in K.$$

Proposition 1.5. *For any $n \geq 1$ there exists an unconditional, convex body $K \subseteq \mathbb{R}^n$ with*

$$\frac{1}{s(K)} \cdot \int_K x_1^2 dx \cdot \int_{K^\circ} x_1^2 dx \geq c, \quad (9)$$

where $c > 0$ is a universal constant. In particular, $\phi(K) \geq c$.

The example of Proposition 1.5 is optimal up to a universal constant as clearly $\phi(K) \leq 1$ for any centrally-symmetric convex body K in any dimension. We say that $K \subseteq \mathbb{R}^n$ is 1-symmetric or that it has the symmetries of the cube if K is unconditional and furthermore for any permutation $\sigma \in S_n$ and a point $x \in \mathbb{R}^n$,

$$x = (x_1, \dots, x_n) \in K \quad \iff \quad (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \in K.$$

When the convex body $K \subseteq \mathbb{R}^n$ has the symmetries of the cube, not only do we know that $\mathbb{E}\langle X, Y \rangle^2 \leq C/n$, but we may also prove that the random variable $\langle X, Y \rangle$ is approximately Gaussian when the dimension n is large. This is a corollary of the results of [13]. Thus, perhaps Kuperberg’s conjecture or even the stronger versions from [14] hold true in the case of 1-symmetric convex bodies.

Theorem 1.1 and Theorem 1.4 are proven in Sections 2 and 3. Section 4 discusses some examples, while Proposition 1.5 is proven in Section 5. In Sections 6 and 7 we prove Theorem 1.3. We continue with some notation and conventions that will be used below. The relative interior of a convex set $K \subseteq X$ is its interior relative to the affine subspace spanned by K . We abbreviate $A+B = \{a+b; a \in A, b \in B\}$ and $x+A = \{x+a; x \in A\}$. We write $\langle x, y \rangle$ or $x \cdot y$ for the standard scalar product of $x, y \in \mathbb{R}^n$, and $|x| = \sqrt{\langle x, x \rangle}$. We denote by $A^* \in \mathbb{R}^{m \times n}$ the transpose of a matrix $A \in \mathbb{R}^{n \times m}$. A smooth function is C^∞ -smooth and we write \log for the natural logarithm.

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2 Mahler volumes through the Laplace transform

In this section we discuss basic properties of Mahler volumes, the logarithmic Laplace transform and its Legendre transform. Let $K \subseteq \mathbb{R}^n$ be a compact, convex set, and let p be a point belonging to the relative interior of K . The Mahler volume of K with respect to the point p , denoted by

$$s_p(K) \in (0, \infty),$$

is defined as follows: There exists a convex body $K_1 \subseteq \mathbb{R}^n$ and an affine map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which is a bijection from K to K_1 with $T(p) = 0$. We may now set $s_p(K) := s(K_1)$, and observe that this definition does not depend on the choice of K_1 and the map T . Clearly when the origin belongs to the interior of the convex body K , we have $s(K) = s_0(K)$. For a compact, convex set $K \subseteq \mathbb{R}^n$ we define

$$\bar{s}(K) = s_{p_K}(K),$$

where p_K is the Santaló point of K . It is well-known (see, e.g., Schneider [31, Section 10.5]) that

$$\bar{s}(K) = \inf_p s_p(K) \tag{10}$$

where the infimum runs over all points p in the relative interior of K . Moreover, the infimum in (10) is attained at a unique point, the Santaló point of K , which is in fact the only local minimum of the functional $p \mapsto s_p(K)$. When $V \subset \mathbb{R}^{n+1}$ is a proper, convex cone, its dual cone is defined via

$$V^* = \{y \in \mathbb{R}^{n+1}; \forall x \in V, \langle x, y \rangle \leq 0\}.$$

The dual cone V^* is again a proper, convex cone, and additionally $(V^*)^* = V$.

Lemma 2.1. *Let $V \subset \mathbb{R}^{n+1}$ be a proper, convex cone. Let $x_0 \in \text{Int}(V)$ and $y_0 \in \text{Int}(V^*)$ satisfy $\langle x_0, y_0 \rangle = -1$. Then,*

$$s_{x_0}(K) = s_{y_0}(T) = \frac{1}{(n!)^2} \int_V e^{\langle x, y_0 \rangle} dx \cdot \int_{V^*} e^{\langle x_0, y \rangle} dy \tag{11}$$

where $K = \{x \in V; \langle x, y_0 \rangle = -1\}$ and $T = \{y \in V^*; \langle x_0, y \rangle = -1\}$.

Proof. Assume first that there exists a unit vector $e \in S^{n-1}$ with $x_0 = e = -y_0$. By Fubini's theorem,

$$\int_V e^{-\langle x, e \rangle} dx = \int_0^\infty \int_{\{z \in V; \langle z, e \rangle = t\}} e^{-\langle z, e \rangle} dz dt = \int_0^\infty \text{Vol}_n(tK) \cdot e^{-t} dt = n! \cdot \text{Vol}_n(K). \tag{12}$$

By performing a similar computation for V^* , we see that

$$\int_V e^{-\langle x, e \rangle} dx \cdot \int_{V^*} e^{\langle e, y \rangle} dy = (n!)^2 \cdot \text{Vol}_n(K) \cdot \text{Vol}_n(T) = (n!)^2 \cdot \text{Vol}_n(K_1) \cdot \text{Vol}_n(T_1), \tag{13}$$

where we have set $K_1 = K - e$ and $T_1 = T + e$. Note that K_1 and T_1 are convex bodies in the n -dimensional linear space $E = e^\perp = \{x \in \mathbb{R}^n; \langle x, e \rangle = 0\}$. Both convex bodies contain the origin in their interior. Let us show that as subsets of the n -dimensional Euclidean space E , the two sets K_1 and T_1 satisfy

$$K_1 = T_1^\circ. \quad (14)$$

Indeed, a given point $y \in E$ lies in T_1 if and only if $\langle y - e, x \rangle \leq 0$ for all points $x \in V$. By homogeneity, it suffices to look only at points $x \in K$, since any $x \in V$ takes the form $x = tz$ for some $z \in K, t \geq 0$. Thus, a given point $y \in E$ belongs to T_1 if and only if for all $x \in K_1$,

$$0 \geq \langle y - e, x + e \rangle = \langle x, y \rangle - \langle e, e \rangle = \langle x, y \rangle - 1.$$

This proves (14). From (13) and (14) we obtain the conclusion of the proposition for the case where $x_0 = -y_0$ is a unit vector.

We move on to discuss the general case, which will be reduced to the case analyzed above using linear algebra. Since $\langle x_0, -y_0 \rangle > 0$, there exists a positive-definite, symmetric matrix $P \in \mathbb{R}^{n \times n}$ with $Px_0 = -y_0$. We may decompose $P = S^*S$ for some invertible matrix $S \in \mathbb{R}^{n \times n}$ and set

$$e = Sx_0.$$

Note that $S^*e = S^*Sx_0 = Px_0 = -y_0$, and hence $(S^*)^{-1}y_0 = -e$. The vector $e \in \mathbb{R}^{n+1}$ is a unit vector, as

$$-1 = \langle x_0, y_0 \rangle = \langle Sx_0, (S^*)^{-1}y_0 \rangle = -\langle e, e \rangle.$$

Denoting $V_1 = S(V)$ we see that V_1 is a proper, convex cone with $V_1^* = (S^*)^{-1}V^*$. By changing variables $\tilde{x} = Sx$ and $\tilde{y} = (S^*)^{-1}y$ we obtain

$$\int_V e^{\langle x, y_0 \rangle} dx \cdot \int_{V^*} e^{\langle x_0, y \rangle} dy = \int_{V_1} e^{-\langle \tilde{x}, e \rangle} d\tilde{x} \cdot \int_{V_1^*} e^{\langle e, \tilde{y} \rangle} d\tilde{y}. \quad (15)$$

Denote $K_2 = \{\tilde{x} \in V_1; \langle \tilde{x}, -e \rangle = -1\}$ and $T_2 = \{\tilde{y} \in V_1^*; \langle e, \tilde{y} \rangle = -1\}$. We use (15) and the case of the proposition that was already proven and deduce that

$$\frac{1}{(n!)^2} \int_V e^{\langle x, y_0 \rangle} dx \cdot \int_{V^*} e^{\langle x_0, y \rangle} dy = s_e(K_2) = s_{-e}(T_2). \quad (16)$$

However, $S(K) = K_2$ with $S(x_0) = e$ while $(S^*)^{-1}(T) = T_2$ with $(S^*)^{-1}(y_0) = -e$. Therefore $s_e(K_2) = s_{x_0}(K)$ and $s_{-e}(T_2) = s_{y_0}(T)$. Thus (11) follows from (16). \square

Let $V \subset \mathbb{R}^{n+1}$ be a proper, convex cone. For $x \in \text{Int}(V)$ and $y \in \text{Int}(V^*)$ we define

$$K_y = \{z \in V; \langle z, y_0 \rangle = -1\} \quad \text{and} \quad T_x = \{z \in V^*; \langle x, z \rangle = -1\}. \quad (17)$$

The notation (17) will accompany us throughout this paper. Observe that for any $t > 0$ and $y \in V^*$,

$$\int_V e^{\langle ty, x \rangle} dx = \frac{1}{t^{n+1}} \int_V e^{\langle y, x \rangle} dx. \quad (18)$$

By scaling, we obtain the following from Lemma 2.1:

Proposition 2.2. *Let $V \subset \mathbb{R}^{n+1}$ be a proper, convex cone. Let $x_0 \in \text{Int}(V)$ and $y_0 \in \text{Int}(V^*)$ and set $r = -1/\langle x_0, y_0 \rangle$. Then $K_{y_0}, T_{x_0} \subseteq \mathbb{R}^{n+1}$ are n -dimensional, compact, convex sets with*

$$s_{rx_0}(K_{y_0}) = s_{ry_0}(T_{x_0}) = \frac{(-\langle x_0, y_0 \rangle)^{n+1}}{(n!)^2} \int_V e^{\langle x, y_0 \rangle} dx \cdot \int_{V^*} e^{\langle x_0, y \rangle} dy. \quad (19)$$

Remark 2.3. It is possible to interpret the sets K_y and T_x from (17) as polar to each other in an appropriate sense: Set $\tilde{K} = K_y - x$ and $\tilde{T} = T_x - y$. Then \tilde{K} is a convex body in the n -dimensional subspace $X = y^\perp$ while \tilde{T} is a convex body in the n -dimensional subspace $Y = x^\perp$. Moreover,

$$\tilde{T} = \left\{ u \in Y; \forall v \in \tilde{K}, \langle u, v \rangle \leq 1 \right\}, \quad \tilde{K} = \left\{ v \in X; \forall u \in \tilde{T}, \langle u, v \rangle \leq 1 \right\}. \quad (20)$$

Relation (20) is the precise duality relation that \tilde{K} and \tilde{T} satisfy. In particular, we conclude that the set K_y is centrally-symmetric around the point x if and only if the set T_x is centrally-symmetric around the point y . Similarly, x is the barycenter of K_y if and only if y is the Santaló point of T_x .

The logarithmic Laplace transform of the proper, convex cone $V \subset \mathbb{R}^{n+1}$ is the function

$$\Phi_V(y) = \log \int_V e^{\langle y, x \rangle} dx \quad (y \in \mathbb{R}^{n+1}). \quad (21)$$

The function Φ_V is a continuous function from \mathbb{R}^{n+1} to $\mathbb{R} \cup \{+\infty\}$. It attains a finite value at a point $x \in \mathbb{R}^{n+1}$ if and only if $x \in \text{Int}(V^*)$. This may be verified, for example, by using formula (12). The function Φ_V is strictly-convex in $\text{Int}(V^*)$, as follows from the Cauchy-Schwartz inequality (see, e.g., [12]). It follows from (18) that Φ_V has the following homogeneity property: For any $t > 0$,

$$\Phi_V(ty) = \Phi_V(y) - (n+1) \log t. \quad (22)$$

Differentiating (22) with respect to t we obtain the useful relation

$$\langle \nabla \Phi_V(y), y \rangle = -(n+1).$$

Proposition 2.2 tells us that for any $x \in \text{Int}(V)$ and $y \in \text{Int}(V^*)$,

$$\Phi_{V^*}(x) + \Phi_V(y) + (n+1) \log(-\langle x, y \rangle) - 2 \log(n!) = \log s_{rx}(K_y) = \log s_{ry}(T_x), \quad (23)$$

for $r = -1/\langle x, y \rangle$. Thus, any local minimum of the Mahler volume corresponds to a local minimum of the expression on the left-hand side of (23). We would like to compute the first and second variations at a local minimum. We could have proceeded by differentiating the expression in (23) with respect to x and with respect to y , but we find it convenient to eliminate one variable by introducing the Legendre transform. The Legendre transform of a convex function $\Phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \cup \{+\infty\}$ is

$$\Phi^*(x) = \sup_{y \in \mathbb{R}^{n+1}, \Phi(y) < \infty} [\langle x, y \rangle - \Phi(y)] \quad (x \in \mathbb{R}^{n+1}). \quad (24)$$

A standard reference for the Legendre transform and convex analysis is Rockafellar [26]. The function $\Phi^* : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \cup \{+\infty\}$ is again convex. If Φ is finite in a neighborhood of a point $x \in \mathbb{R}^{n+1}$ and differentiable at the point x itself, then

$$\Phi^*(\nabla\Phi(x)) + \Phi(x) = \langle x, \nabla\Phi(x) \rangle. \quad (25)$$

In the case where $\Phi = \Phi_V$, the supremum in (24) runs over $y \in \text{Int}(V^*)$. Moreover, in this case we deduce from (22) that for any $x \in \text{Int}(V)$,

$$\begin{aligned} \Phi_V^*(x) &= \sup_{s>0, y \in \text{Int}(V^*)} [(n+1) \log s + s\langle x, y \rangle - \Phi_V(y)] \\ &= (n+1) \log \left(\frac{n+1}{e} \right) + \sup_{y \in \text{Int}(V^*)} [-(n+1) \cdot \log(-\langle x, y \rangle) - \Phi_V(y)]. \end{aligned} \quad (26)$$

Note the difference between the function Φ_V^* , which is the Legendre transform of Φ_V , and the function Φ_{V^*} , which is the logarithmic Laplace transform of the dual cone V^* .

Corollary 2.4 (“Commuting the Laplace transform with convex duality”). *Let $V \subset \mathbb{R}^{n+1}$ be a proper, convex cone. Then the functions Φ_{V^*} and Φ_V^* attain finite values in $\text{Int}(V)$, and their difference $J := \Phi_{V^*} - \Phi_V^*$ satisfies*

$$J(x) = \kappa_n + \log \bar{s}(T_x) \quad (27)$$

for $x \in \text{Int}(V)$ where $\kappa_n = 2 \log(n!) - (n+1) \log \left(\frac{n+1}{e} \right)$.

Proof. We already know that Φ_{V^*} attains finite values in $\text{Int}(V)$. Fix $x \in \text{Int}(V)$. From (23) and (26),

$$\Phi_{V^*}(x) - \left[\Phi_V^*(x) - (n+1) \log \left(\frac{n+1}{e} \right) \right] - 2 \log(n!) = \inf_{y \in \text{Int}(V^*)} \log s_{ry}(T_x) \quad (28)$$

where $r = -1/\langle x, y \rangle$. When y ranges over the set $\text{Int}(V^*)$, the point $ry = -y/\langle x, y \rangle$ ranges over the entire relative interior of T_x . From (10), the right-hand side of (28) equals $\log \bar{s}(T_x)$, and (27) follows. The function $\bar{s}(T_x)$ attains finite values in $\text{Int}(V)$, as well as the function Φ_{V^*} . By (27) the function Φ_V^* is also finite in $\text{Int}(V)$. \square

Observe that for any homogeneous polynomial $p_k(x)$ of degree k and for any $y \in \text{Int}(V^*)$,

$$\begin{aligned} \int_V p_k(x) e^{\langle y, x \rangle} dx &= \int_0^\infty \int_{\{x \in V; \langle x, y \rangle = -t|y|\}} p_k(x) e^{\langle y, x \rangle} dx dt \\ &= \int_0^\infty (t|y|)^{k+n} e^{-t|y|} dt \cdot \int_{K_y} p_k(x) dx = \frac{(n+k)!}{|y|} \cdot \int_{K_y} p_k(x) dx. \end{aligned} \quad (29)$$

We write $b(K)$ for the barycenter of a convex body K . Recall that $\text{Cov}(K_y)$ is the covariance matrix of a random vector that is distributed uniformly in K_y .

Lemma 2.5 (“Derivatives of Φ_V ”). *Let $V \subset \mathbb{R}^{n+1}$ be a proper, convex cone. Then for any $y \in \text{Int}(V^*)$,*

$$\nabla \Phi_V(y) = (n+1) \cdot b(K_y) \quad (30)$$

and the Hessian matrix is given by

$$\nabla^2 \Phi_V(y) = (n+2)(n+1) \cdot \text{Cov}(K_y) + (n+1) \cdot b(K_y) b^*(K_y), \quad (31)$$

where we view $z \in \mathbb{R}^{n+1}$ as a column vector while z^* is the corresponding row vector.

Proof. The function Φ_V is clearly smooth in $\text{Int}(V^*)$. By differentiating (21) we see that for any $i = 1, \dots, n+1$,

$$\frac{\partial \Phi_V(y)}{\partial y_i} = \frac{\int_V x_i e^{\langle y, x \rangle} dx}{\int_V e^{\langle y, x \rangle} dx} = \frac{(n+1)!}{n!} \cdot \frac{\int_{K_y} x_i dx}{\text{Vol}_n(K_y)},$$

where we used (29) twice in the last passage. This proves (30). By differentiating (21) one more time we see that for $i, j = 1, \dots, n+1$,

$$\begin{aligned} \frac{\partial^2 \Phi_V(y)}{\partial y_i \partial y_j} &= \frac{\int_V x_i x_j e^{\langle y, x \rangle} dx}{\int_V e^{\langle y, x \rangle} dx} - \frac{\int_V x_i e^{\langle y, x \rangle} dx}{\int_V e^{\langle y, x \rangle} dx} \cdot \frac{\int_V x_j e^{\langle y, x \rangle} dx}{\int_V e^{\langle y, x \rangle} dx} \\ &= (n+2)(n+1) \frac{\int_{K_y} x_i x_j dx}{\text{Vol}_n(K_y)} - (n+1)^2 \frac{\int_{K_y} x_i dx}{\text{Vol}_n(K_y)} \cdot \frac{\int_{K_y} x_j dx}{\text{Vol}_n(K_y)}, \end{aligned}$$

which is equivalent to (31). □

The function Φ_V is smooth and strictly-convex in $\text{Int}(V^*)$, and it equals $+\infty$ outside $\text{Int}(V^*)$. Moreover, $\nabla \Phi_V(y) \in \text{Int}(V)$ for any $y \in \text{Int}(V^*)$, according to (30). The function Φ_V^* is finite in $\text{Int}(V)$, and from the standard theory of the Legendre transform (e.g., Rockafellar [26, Section 23]), for any $x \in \text{Int}(V)$ there exists $y \in \text{Int}(V^*)$ with $\nabla \Phi_V(y) = x$.

Corollary 2.6. *For any proper, convex cone $V \subset \mathbb{R}^{n+1}$, the map $\nabla \Phi_V : \text{Int}(V^*) \rightarrow \text{Int}(V)$ is a diffeomorphism.*

Proof. We have just explained that the map $\nabla\Phi_V : \text{Int}(V^*) \rightarrow \text{Int}(V)$ is onto. Since Φ_V is strictly-convex, this map is one-to-one. The derivative of this smooth map is the matrix $\nabla^2\Phi_V(y)$, which is positive-definite and hence invertible by Lemma 2.5. Therefore the map $\nabla\Phi_V : \text{Int}(V^*) \rightarrow \text{Int}(V)$ is a diffeomorphism. \square

It follows from Corollary 2.6 and formula (25) that the function Φ_V^* is differentiable in $\text{Int}(V)$ and moreover, for any $x \in \text{Int}(V)$ and $y \in \text{Int}(V^*)$,

$$\nabla\Phi_V^*(x) = y \quad \iff \quad \nabla\Phi_V(y) = x. \quad (32)$$

In other words, the map $\nabla\Phi_V^*$ is the inverse to the map $\nabla\Phi_V$. Consequently the Hessian matrices are inverse to each other, that is, for any $x \in \text{Int}(V)$,

$$\nabla^2\Phi_V^*(x) = [\nabla^2\Phi_V(y)]^{-1}, \quad (33)$$

where $y = \nabla\Phi_V^*(x)$. From Lemma 2.5 and Corollary 2.6 we also learn that two hyperplane sections of V coincide if and only if their barycenters coincide.

3 Projective perturbations and homogeneous cones

In this section we prove Theorem 1.1 and Theorem 1.4. We say that two convex bodies $K, T \subseteq \mathbb{R}^n$ are *projectively equivalent* or that they are *projective images* of one another if T is affinely-equivalent to a hyperplane section of the cone

$$V = \{(t, tx) \in \mathbb{R} \times \mathbb{R}^n; t \geq 0, x \in K\}. \quad (34)$$

In other words, for a certain $y \in \text{Int}(V^*)$ the set T is affinely equivalent to the convex set K_y from (17) that is associated with the cone V .

The family of projective images of a convex body $K \subseteq \mathbb{R}^n$ with a smooth boundary always contains bodies arbitrarily close to a Euclidean unit ball. In the case where $K \subseteq \mathbb{R}^n$ is a simplicial polytope, there are projective images of K that are arbitrarily close to a simplex. A projective image of a polytope is itself a polytope with the same number of vertices and faces. A projective image of an ellipsoid is always an ellipsoid, and that of a simplex is always a simplex. Our next lemma specializes the results of the previous section to the cone defined in (34). We use $x = (t, y) \in \mathbb{R} \times \mathbb{R}^n$ as coordinates in \mathbb{R}^{n+1} .

Lemma 3.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body with $b(K) = 0$ and let $V \subset \mathbb{R}^{n+1}$ be the proper, convex cone defined in (34). Denote $J = \Phi_{V^*} - \Phi_V^*$ and $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^n \cong \mathbb{R}^{n+1}$. Then,*

$$\nabla J(e) = (n+1) \cdot (0, b(K^\circ)) \in \mathbb{R} \times \mathbb{R}^n \cong \mathbb{R}^{n+1}. \quad (35)$$

In the case where $\nabla J(e) = 0$, the Hessian matrix satisfies

$$\frac{1}{n+1} \cdot \nabla^2 J(e) = (n+2) \cdot \text{Cov}(K^\circ) - \frac{1}{n+2} \cdot \text{Cov}(K)^{-1}. \quad (36)$$

A remark concerning formula (36): The left-hand side is a certain $(n+1) \times (n+1)$ matrix A , while the right-hand side is an $n \times n$ matrix B . What we actually mean, is that $A = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & B \end{array} \right)$. This is consistent with our coordinates notation, which corresponds to the decomposition $\mathbb{R}^{n+1} \cong \mathbb{R} \times \mathbb{R}^n$.

Proof of Lemma 3.1. From (34) we deduce that

$$V^* = \{(-t, tx) \in \mathbb{R} \times \mathbb{R}^n; t \geq 0, x \in K^\circ\}.$$

Recalling the notation of (17) from the previous section, we see that $T_e = \{-1\} \times K^\circ$. By applying Lemma 2.5 with the dual cone V^* , we obtain

$$\nabla\Phi_{V^*}(e) = (n+1) \cdot \mathfrak{b}(T_e) = (n+1) \cdot (-1, \mathfrak{b}(K^\circ)) \in \mathbb{R} \times \mathbb{R}^n.$$

Since $K_{-e} = \{1\} \times K$, when applying Lemma 2.5 with the cone V we get

$$\nabla\Phi_V(-e) = (n+1) \cdot \mathfrak{b}(K_{-e}) = (n+1) \cdot (1, \mathfrak{b}(K)) = (n+1) \cdot e. \quad (37)$$

Recall that $\Phi_V(ty) = \Phi_V(y) - (n+1) \log t$ according to the homogeneity relation (22). By differentiation, we obtain $\nabla\Phi_V(ty) = \nabla\Phi_V(y)/t$, i.e., $\nabla\Phi_V$ is (-1) -homogeneous. Therefore, from (37),

$$\nabla\Phi_V(-(n+1) \cdot e) = e \quad \text{and hence} \quad \nabla\Phi_V^*(e) = -(n+1)e \quad (38)$$

where we used (32) in the last passage. Consequently,

$$\nabla J(e) = \nabla\Phi_{V^*}(e) - \nabla\Phi_V^*(e) = (n+1) \cdot (-1, \mathfrak{b}(K^\circ)) + (n+1)e = (n+1) \cdot (0, \mathfrak{b}(K^\circ)) \in \mathbb{R} \times \mathbb{R}^n,$$

proving (35). We move on to the proof of (36). Since $\nabla J(e) = 0$, then $\mathfrak{b}(K^\circ) = 0$ and $\mathfrak{b}(T_e) = -e$. According to Lemma 2.5,

$$\nabla^2\Phi_V(-e) = (n+2)(n+1)\text{Cov}(K_{-e}) + (n+1)ee^* = (n+2)(n+1)\text{Cov}(K) + (n+1)ee^* \quad (39)$$

and

$$\nabla^2\Phi_{V^*}(e) = (n+2)(n+1)\text{Cov}(T_e) + (n+1)ee^* = (n+2)(n+1)\text{Cov}(K^\circ) + (n+1)ee^*. \quad (40)$$

Since $\nabla\Phi_V$ is (-1) -homogeneous, the Hessian $\nabla^2\Phi_V$ is (-2) -homogeneous, and $\nabla^2\Phi_V(ty) = \nabla^2\Phi_V(y)/t^2$. From (39) we obtain

$$\nabla^2\Phi_V(-(n+1) \cdot e) = \frac{n+2}{n+1} \cdot \text{Cov}(K) + \frac{1}{n+1} \cdot ee^*.$$

Thanks to (33) and (38) we know that

$$\nabla^2\Phi_V^*(e) = [\nabla^2\Phi_V(-(n+1) \cdot e)]^{-1} = \frac{n+1}{n+2} \cdot \text{Cov}(K)^{-1} + (n+1) \cdot ee^*. \quad (41)$$

Now (36) follows from (40), (41) and the fact that $J = \Phi_{V^*} - \Phi_V^*$. \square

We proceed with a discussion and a proof of Theorem 1.4. Let $V \subset \mathbb{R}^{n+1}$ be a proper, convex cone. Denote by $\text{Aut}(V)$ the group of all invertible, linear transformations T with $T(V) = V$. Clearly $T \in \text{Aut}(V)$ implies that $T^* \in \text{Aut}(V^*)$ and vice versa, as for any $x \in \mathbb{R}^{n+1}$,

$$\sup_{y \in V} \langle x, y \rangle = \sup_{y \in V} \langle x, Ty \rangle = \sup_{y \in V} \langle T^*x, y \rangle.$$

The symmetries of the cone V manifest themselves in the Laplace transform. That is, for any $T \in \text{Aut}(V)$ and $y \in \text{Int}(V^*)$,

$$\Phi_V(T^*y) = \log \int_V e^{\langle y, Tx \rangle} dx = \log \int_V e^{\langle y, x \rangle} dx - \log |\det T| = \Phi_V(y) - \log |\det T|. \quad (42)$$

Consequently, for any $x \in \text{Int}(V)$ and $T \in \text{Aut}(V)$,

$$\Phi_V^*(Tx) = \sup_{y \in \text{Int}(V^*)} [\langle x, T^*y \rangle - \Phi_V(y)] = \Phi_V^*(x) - \log |\det T|. \quad (43)$$

Proof of Theorem 1.4. We may assume that the barycenter of K lies at the origin and define V as in (34). Then V is a convex, homogeneous cone. From (42) and (43) we know that for any $x \in \text{Int}(V)$ and $T \in \text{Aut}(V)$,

$$J(Tx) = \Phi_{V^*}(Tx) - \Phi_V^*(Tx) = (\Phi_{V^*}(x) - \log |\det T|) - (\Phi_V^*(x) - \log |\det T|) = J(x).$$

However, for any $x, y \in \text{Int}(V)$ there is $T \in \text{Aut}(V)$ with $Tx = y$. Consequently $J : \text{Int}(V) \rightarrow \mathbb{R}$ is a constant function. In particular, the gradient and the Hessian matrix of J vanish. We may now apply the computations of Lemma 3.1. First, we deduce that $b(K^\circ) = 0$, and hence the Santaló point of K coincides with its barycenter. Second, we obtain

$$(n+2) \cdot \text{Cov}(K^\circ) = \frac{1}{n+2} \cdot \text{Cov}(K)^{-1}.$$

In particular, $L_K^{2n} \cdot L_{K^\circ}^{2n} \cdot s(K)^2 = \det \text{Cov}(K) \cdot \det \text{Cov}(K^\circ) = (n+2)^{-(2n)}$ by (3). \square

Remark 3.2. A convex body $K \subseteq \mathbb{R}^n$ is affinely equivalent to a hyperplane section of a homogeneous cone if and only if every projective image of K is affinely equivalent to K . This follows from the fact that two hyperplane sections of a proper, convex cone $V \subset \mathbb{R}^{n+1}$ coincide if and only if their barycenters coincide. A corollary is that the dual to a homogeneous cone is homogeneous in itself. These standard facts are not used in this paper.

Theorem 1.1 will be deduced from the next proposition:

Proposition 3.3. *Let $T \subseteq \mathbb{R}^n$ be a convex body which is a local minimizer of the Mahler volume $s(T)$ in the class of the projective images of T . Then,*

$$(n+2)^2 \cdot \text{Cov}(T^\circ) \geq \text{Cov}(T)^{-1}. \quad (44)$$

Moreover, if T is a local maximizer of the Mahler volume in the class of projective images of T with barycenter at the origin, then $(n+2)^2 \cdot \text{Cov}(T^\circ) \leq \text{Cov}(T)^{-1}$.

Proof. A translation of T is a particular case of a projective image of T . Assume that T is a local minimizer. From (10) we learn that the Santaló point of T lies at the origin. Denote $K = T^\circ$, so $b(K) = 0$. Consider the proper, convex cone $V \subset \mathbb{R} \times \mathbb{R}^n$ defined in (34). For $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^n$ we have

$$T_e = \{-1\} \times T.$$

Therefore T_x is a projective image of T for any $x \in \text{Int}(V)$. When x is close to e , the Santaló point of T_x is close to the Santaló point of T_e . By our local minimum assumption, for any x in some neighborhood of e , we have

$$\bar{s}(T_x) \geq \bar{s}(T_e) = \bar{s}(T) = s(T).$$

Recall from Corollary 2.4 that $J(x) = \Phi_{V^*}(x) - \Phi_V^*(x) = \kappa_n + \log \bar{s}(T_x)$. We thus conclude that e is a local minimum of the function $J : \text{Int}(V) \rightarrow \mathbb{R}$. Thus $\nabla J(e) = 0$ and $\nabla^2 J(e) \geq 0$. From Lemma 3.1 we obtain

$$(n+2) \cdot \text{Cov}(T) - \frac{1}{n+2} \cdot \text{Cov}(T^\circ)^{-1} = (n+2) \cdot \text{Cov}(K^\circ) - \frac{1}{n+2} \cdot \text{Cov}(K)^{-1} \geq 0.$$

This proves (44). Similarly, if T is a local maximizer of $\bar{s}(T_x)$, then $\nabla J(e) = 0$ and $\nabla^2 J(e) \leq 0$, and from Lemma 3.1 we obtain $(n+2)^2 \cdot \text{Cov}(T^\circ) \leq \text{Cov}(T)^{-1}$. \square

Proof of Theorem 1.1. Assume that $K \subseteq \mathbb{R}^n$ is a local minimizer of the Mahler volume in the class of convex bodies containing the origin in their interior. In particular, K is a local minimizer in the class of projective images of K , and by Proposition 3.3,

$$\text{Cov}(K^\circ) - (n+2)^{-2} \cdot \text{Cov}(K)^{-1} \geq 0. \quad (45)$$

In order to prove (4), we use the fact that $\det(A) \geq \det(B)$ whenever $A \geq B \geq 0$. Thus, by (3) and (45),

$$L_K^{2n} \cdot L_{K^\circ}^{2n} \cdot s(K)^2 = \det \text{Cov}(K) \cdot \det \text{Cov}(K^\circ) \geq \frac{1}{(n+2)^{2n}},$$

proving (4). Note that if K is a global minimizer, then $s(K) \leq s(\Delta^n)$ and from (4),

$$L_K \cdot L_{K^\circ} \geq L_{\Delta^n}^2$$

where we used the fact that $L_{\Delta^n}^2 \cdot s(\Delta^n)^{1/n} = 1/(n+2)$. Hence $L_K \geq L_{\Delta^n}$ or $L_{K^\circ} \geq L_{\Delta^n}$ in the case of a global maximizer. \square

Remark 3.4. Suppose that $K \subseteq \mathbb{R}^n$ is a convex body which is not an ellipsoid, whose boundary is smooth with Gauss curvature that is always positive (i.e., the boundary is

strongly-convex). Assume that the barycenter of K lies at the origin, set $T = K^\circ$, and let V be defined as in (34). Then by Corollary 2.4 with $x = e = (1, 0) \in \mathbb{R} \times \mathbb{R}^n$,

$$J(e) = \kappa_n + \log \bar{s}(T) < \kappa_n + \log s(B^n),$$

where in the last passage we used the equality case in the Santaló inequality (see Meyer and Pajor [20]). We claim that for $J = \Phi_{V^*} - \Phi_V^*$,

$$\lim_{x \rightarrow \infty} J(x) = \kappa_n + \log s(B^n) > J(e). \quad (46)$$

Indeed, the behavior of the functional $J : \text{Int}(V) \rightarrow \mathbb{R}$ at infinity is simple to understand, as the corresponding hyperplane sections T_x are close to ellipsoids, and hence $\bar{s}(T_x)$ is close to $s(B^n)$. From (46) we see that the infimum of J is necessarily attained at some point $x \in \text{Int}(V)$. From Lemma 3.1 we thus obtain that whenever $T \subseteq \mathbb{R}^n$ is a convex body with a smooth and strongly-convex boundary, it has a projective image \tilde{T} whose barycenter and Santaló point are at the origin and

$$(n+2)^2 \cdot \text{Cov}(\tilde{T}^\circ) \geq \text{Cov}(\tilde{T})^{-1}.$$

4 Examples

Let us begin this section by inspecting the simplest example, the case of the orthant

$$V = \mathbb{R}_+^{n+1} = \{x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}; \forall i, x_i \geq 0\}.$$

The proper, convex cone V is homogeneous, with an automorphism group consisting of all diagonal transformations with positive entries on the diagonal. Moreover, in this case $V^* = -V$, and a homogeneous cone with this property is called a *symmetric cone*. For any $y \in \text{Int}(V^*)$, the set K_y is an n -dimensional simplex. The logarithmic Laplace transform is given by

$$\Phi_V(y) = - \sum_{i=0}^n \log |y_i| \quad \text{for } y \in \text{Int}(V^*).$$

Since $\sup_{s>0} [-st + \log(s)] = -1 - \log(t)$ for any positive t , we have

$$\Phi_V^*(x) = -(n+1) - \sum_{i=0}^n \log |x_i| \quad \text{for } x \in \text{Int}(V).$$

It follows that $\Phi_{V^*}(x) - \Phi_V^*(x) = n+1$ for any $x \in \text{Int}(V)$. From Corollary 2.4 and this example we conclude the following:

Corollary 4.1. *The Mahler conjecture (1) is equivalent to the assertion that for any proper, convex cone $V \subset \mathbb{R}^{n+1}$ and for any point $x \in \text{Int}(V)$ we have $\Phi_{V^*}(x) - \Phi_V^*(x) \geq n+1$.*

The second example we consider is where

$$V = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1}; x_0 \geq \sqrt{\sum_{i=1}^n x_i^2} \right\}$$

is the Lorentz cone. Here again $V^* = -V$. For any $y \in \text{Int}(V^*)$, the set K_y is an ellipsoid. Denoting $Q(x) = x_0^2 - \sum_{i=1}^n x_i^2$ for $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$, we have

$$\Phi_V(y) = -\frac{n+1}{2} \log Q(y) + C_n, \quad \text{for } y \in \text{Int}(V^*),$$

as dictated by the symmetries of the problem, where $C_n = \log(\pi^{n/2} \cdot \Gamma(n+1)/\Gamma(1+n/2))$. Moreover, here

$$\Phi_{V^*} - \Phi_V^* \equiv 2C_n - (n+1) \log \frac{n+1}{e}, \quad (47)$$

which by Santaló's inequality is the maximal possible value of $\Phi_{V^*} - \Phi_V^*$ for any proper, convex cone $V \subset \mathbb{R}^{n+1}$. The right-hand side of (47) is asymptotically $(\log(2\pi) + o(1)) \cdot n$, according to Stirling's approximation. The third example we consider is the cone of positive semi-definite matrices

$$V = \{A \in \mathbb{R}^{n \times n}; A^* = A, A \geq 0\}. \quad (48)$$

This is a proper, convex cone in the linear space $X_n \subseteq \mathbb{R}^{n \times n}$ of all real, symmetric matrices. We endow X_n with the scalar product

$$\langle A, B \rangle = \text{Tr}[AB] = \sum_{i=1}^n A_{ii}B_{ii} + 2 \sum_{i < j} A_{ij}B_{ij}$$

where $A = (A_{ij})_{i,j=1,\dots,n}$ and $B = (B_{ij})_{i,j=1,\dots,n}$. With this scalar product, we have $V^* = -V$. The volume form in X_n which is induced by this scalar product is the volume form $dA := 2^{n(n-1)/4} \prod_{i < j} dA_{ij}$, up to a sign which corresponds to a choice of orientation in X_n . The logarithmic Laplace transform is given by

$$\Phi_V(-A) = \log \int_V e^{-\text{Tr}[AB]} dB = -\frac{n+1}{2} \cdot \log \det A + C_n \quad (49)$$

for some constant C_n . Indeed, for any map $T \in \mathbb{R}^{n \times n}$ with $\det(T) = 1$ we know that $\Phi_V(T^*AT) = \Phi_V(A)$. This shows that $\Phi_V(A)$ depends only on the determinant of A . The homogeneity property $\Phi_V(tA) = -\log(t) \cdot n(n+1)/2 + \Phi_V(A)$ implies formula (49). The computation of C_n by induction on n is well-known and it is explained, e.g., in [24, Section 5.7]. For completeness, and since our notation is a bit different, the following lemma includes the standard computation:

Lemma 4.2. *In the case where $V = V_n$ is given by (48), the constant C_n from (49) satisfies*

$$C_n = \log \int_{V_n} e^{-\text{Tr}[A]} dA = \frac{n(n-1)}{4} \log(2\pi) + \sum_{k=1}^n \log \Gamma\left(\frac{k+1}{2}\right). \quad (50)$$

Proof. For $A \in X_n$ let us write

$$A = \left(\begin{array}{c|c} B & u \\ \hline u^* & s \end{array} \right)$$

where $B \in X_{n-1}$ is a symmetric matrix, $u \in \mathbb{R}^{n-1}$ and $s \in \mathbb{R}$. Then $dA = \pm 2^{(n-1)/2} dB \wedge du \wedge ds$, where we recall that $dB := 2^{(n-1)(n-2)/4} \prod_{i \leq j} dB_{ij}$ while $du = \prod_i du_i$. By setting $v = u/s$, and $D = B - svv^*$ we obtain

$$A = \left(\begin{array}{c|c} B & u \\ \hline u^* & s \end{array} \right) = \left(\begin{array}{c|c} D + svv^* & sv \\ \hline sv^* & s \end{array} \right) = \left(\begin{array}{c|c} 1 & v \\ \hline 0 & 1 \end{array} \right) \cdot \left(\begin{array}{c|c} D & 0 \\ \hline 0 & s \end{array} \right) \cdot \left(\begin{array}{c|c} 1 & 0 \\ \hline v^* & 1 \end{array} \right).$$

Note that the map from $A \in \text{Int}(V_n)$ to $(D, v, s) \in \text{Int}(V_{n-1}) \times \mathbb{R}^{n-1} \times (0, \infty)$ is a diffeomorphism. Moreover, $du = s^{n-1}dv + \alpha \wedge ds$ for some α and hence $dB \wedge du \wedge ds = s^{n-1}dD \wedge dv \wedge ds$. Consequently,

$$\begin{aligned} e^{C_n} &= \int_{V_n} e^{-\text{Tr}[A]} dA = 2^{\frac{n-1}{2}} \int_{V_{n-1} \times \mathbb{R}^{n-1} \times (0, \infty)} e^{-\text{Tr}[D] - s|v|^2 - s} s^{n-1} dD \wedge dv \wedge ds \\ &= e^{C_{n-1}} \cdot 2^{\frac{n-1}{2}} \int_0^\infty s^{n-1} e^{-s} \left(\int_{\mathbb{R}^{n-1}} e^{-s|v|^2} dv \right) ds = e^{C_{n-1}} 2^{\frac{n-1}{2}} \int_0^\infty s^{n-1} e^{-s} \left(\frac{\pi}{s} \right)^{(n-1)/2} ds. \end{aligned}$$

It follows that

$$C_n = C_{n-1} + \frac{n-1}{2} \log(2\pi) + \log \Gamma \left(\frac{n+1}{2} \right).$$

Since $C_1 = 0$, formula (50) follows by a simple induction. \square

It follows that in the case where V is given by (48),

$$\Phi_{V^*} - \Phi_V^* \equiv 2C_n - \frac{n(n+1)}{2} \log \frac{n+1}{2e} = \frac{n(n+1)}{2} \cdot [\log(2\pi) - 1/2 + o(1)]$$

where the asymptotics as $n \rightarrow \infty$ follows from Stirling's formula. Since $\log(2\pi) - 1/2 > 1$, in view of Corollary 4.1 we have verified the Mahler conjecture for hyperplane sections of the cone of positive-definite, symmetric matrices in a sufficiently high dimension. Moreover, thanks to Theorem 1.4, we see that in high dimensions, the isotropic constant of such hyperplane sections is smaller than that of the simplex in the corresponding dimension. We proceed with the case of complex-valued matrices and the cone

$$V = \{A \in \mathbb{C}^{n \times n}; A^* = A, A \geq 0\}. \quad (51)$$

Now we write X_n for the space of Hermitian $n \times n$ matrices, equipped with the scalar product $\langle A, B \rangle = \text{Tr}[A^*B]$. This space has real dimension n^2 , and we have $V^* = -V$. The induced volume form is $dA := 2^{n(n-1)/2} \prod_{i \leq j} dA_{ij}$, where $A_{ii} \in \mathbb{R}$ and $A_{ij} \in \mathbb{C}$ for $i \neq j$. The logarithmic Laplace transform is given by

$$\Phi_V(-A) = \log \int_V e^{-\text{Tr}[AB]} dB = -n \cdot \log \det A + C_n$$

where

$$C_n = \frac{n(n-1)}{2} \log(2\pi) + \sum_{k=1}^{n-1} \log(k!).$$

Here,

$$\Phi_{V^*} - \Phi_V^* \equiv 2C_n - n^2 \log\left(\frac{n}{e}\right) = n^2 \cdot [\log(2\pi) - 1/2 + o(1)].$$

Once again we see the numerical constant $\log(2\pi) - 1/2$ which appeared in the case of real, symmetric matrices. The same numerical constant also appears in the quaternionic case.

A natural operation on convex cones is that of Cartesian products. If $V_1 \subseteq \mathbb{R}^{n_1+1}$ and $V_2 \subseteq \mathbb{R}^{n_2+1}$ are proper, convex cones, then so is the Cartesian product $V_1 \times V_2 \subseteq \mathbb{R}^{n_1+1} \times \mathbb{R}^{n_2+1}$ whose dual is $V_1^* \times V_2^*$. Moreover,

$$\Phi_{V_1 \times V_2}(x, y) = \Phi_{V_1}(x) + \Phi_{V_2}(y) \quad (x \in \text{Int}(V_1^*), y \in \text{Int}(V_2^*))$$

and similarly $\Phi_{V_1^* \times V_2^*}(x, y) = \Phi_{V_1^*}(x) + \Phi_{V_2^*}(y)$. Write $J_V(x) = \Phi_{V^*}(x) - \Phi_V^*(x)$ and let

$$J_{n+1} := \inf_{V \subseteq \mathbb{R}^{n+1}} \inf_{x \in \text{Int}(V)} J_V(x)$$

where the first infimum runs over all proper, convex cones $V \subseteq \mathbb{R}^{n+1}$. Since $J_{V_1 \times V_2}(x, y) = J_{V_1}(x) + J_{V_2}(y)$, we see that $J_{n+m} \leq J_n + J_m$. The subadditivity property of J_n implies that

$$\lim_{n \rightarrow \infty} \frac{J_n}{n} = \inf_{n \rightarrow \infty} \frac{J_n}{n}$$

thanks to the Fekete lemma. Thus, in view of Corollary 4.1, the Mahler conjecture would follow from an asymptotic estimate of the form $J_n \geq (1 + o(1)) \cdot n$. We move on to discuss the case where for $i = 1, 2$, the cone $V_i \subseteq \mathbb{R}^{n_i}$ takes the form

$$V_i = \{(t, tx); t \geq 0, x \in K_i\} \subseteq \mathbb{R} \times \mathbb{R}^{n_i} \quad (52)$$

for some convex body $K_i \subseteq \mathbb{R}^{n_i}$. Consider the hyperplane section of the cone $V_1 \times V_2$ that consists of all 4-tuples (t, x, s, y) with $t + s = 1$. This hyperplane section is

$$\{(t, tx, 1-t, (1-t)y); x \in K_1, y \in K_2, 0 \leq t \leq 1\},$$

which is affinely equivalent to the *geometric join* of K_1 and K_2 , defined via

$$K_1 \diamond K_2 := \sqrt{2} \cdot \{(t - 1/2, tx, 1/2 - t, (1-t)y); x \in K_1, y \in K_2, 0 \leq t \leq 1\}. \quad (53)$$

The geometric join of $K_1 \subseteq \mathbb{R}^{n_1}$ and $K_2 \subseteq \mathbb{R}^{n_2}$ is an $(n_1 + n_2 + 1)$ -dimensional compact, convex set with a non-empty interior relative to the ambient linear subspace

$$H_{n_1, n_2} = \{(t, x, -t, y); t \in \mathbb{R}, x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}\} \subseteq \mathbb{R}^{n_1 + n_2 + 2}.$$

Our definition of a geometric join is slightly different from the perhaps more standard notation in [15], yet the two definitions are affinely equivalent. The geometric join of Δ^{n_1} and Δ^{n_2} is an $(n_1 + n_2 + 1)$ -dimensional simplex. Geometric joins fit well with duality:

Proposition 4.3. *Let $K_1 \subseteq \mathbb{R}^{n_1}$ and $K_2 \subseteq \mathbb{R}^{n_2}$ be convex bodies containing the origin in their interior. Then,*

$$(K_1 \diamond K_2)^\circ = \pi(K_1^\circ \diamond K_2^\circ), \quad (54)$$

where we view $K_1 \diamond K_2$ and $K_1^\circ \diamond K_2^\circ$ as convex bodies in the subspace $H_{n_1, n_2} \subseteq \mathbb{R}^{n_1+n_2+2}$ which is equipped with the induced scalar product from $\mathbb{R}^{n_1+n_2+2}$, and where

$$\pi(t, x, -t, y) = (-t, x, t, y) \quad \text{for } (t, x, -t, y) \in H_{n_1, n_2}.$$

Proof. This follows from the fact that $(V_1 \times V_2)^* = V_1^* \times V_2^*$ where V_i is given by (52). Alternatively, we may argue directly as follows. A point $\sqrt{2} \cdot (1/2 - t, x, t - 1/2, y) \in H_{n_1, n_2}$ belongs to $(K_1 \diamond K_2)^\circ$ if and only if for all $s \in [0, 1]$, $x' \in K_1$ and $y' \in K_2$,

$$-2(t - 1/2)(s - 1/2) + s\langle x, x' \rangle + (1 - s)\langle y, y' \rangle \leq \frac{1}{2}.$$

This happens if and only if for all $s \in [0, 1]$,

$$-2(t - 1/2)(s - 1/2) + s\|x\|_{K_1^\circ} + (1 - s)\|y\|_{K_2^\circ} \leq \frac{1}{2} \quad (55)$$

where $\|x\|_{K_1^\circ} = \sup_{x' \in K_1} \langle x, x' \rangle$. Since the left-hand side of (55) is an affine function of s , it suffices to look at the two values $s = 0, 1$. Hence the point $\sqrt{2} \cdot (1/2 - t, x, t - 1/2, y)$ belongs to $(K_1 \diamond K_2)^\circ$ if and only if

$$(1/2 - t) + \|x\|_{K_1^\circ} \leq 1/2 \quad \text{and} \quad (t - 1/2) + \|y\|_{K_2^\circ} \leq 1/2,$$

i.e., if and only if $x \in tK_1^\circ$ and $y \in (1 - t)K_2^\circ$. This completes the proof. \square

The geometric join may be viewed as a variant of the Cartesian product. For example, it may be verified using Corollary 2.4 and the connection between geometric joins and Cartesian products of cones, that

$$\bar{s}(K_1 \diamond K_2) = C_{n_1, n_2} \cdot \bar{s}(K_1) \cdot \bar{s}(K_2)$$

for any convex bodies $K_1 \subseteq \mathbb{R}^{n_1}$ and $K_2 \subseteq \mathbb{R}^{n_2}$, where

$$C_{n_1, n_2} = \left(\frac{n_1! \cdot n_2!}{(n_1 + n_2 + 1)!} \right)^2 \cdot \frac{(n_1 + n_2 + 2)^{n_1+n_2+2}}{(n_1 + 1)^{n_1+1} (n_2 + 1)^{n_2+1}}.$$

In comparison, for Cartesian products we know that for any convex bodies $K_1 \subseteq \mathbb{R}^{n_1}$ and $K_2 \subseteq \mathbb{R}^{n_2}$,

$$\bar{s}(K_1 \times K_2) = \tilde{C}_{n_1, n_2} \cdot \bar{s}(K_1) \cdot \bar{s}(K_2)$$

where $\tilde{C}_{n_1, n_2} = n_1! n_2! / (n_1 + n_2)!$. For more relations between the geometric join and various inequalities, see Rogers and Shephard [27].

5 Covariance of a body and its polar

In this section we prove Proposition 1.5. Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body. Then $b(K) = b(K^\circ) = 0$. From Corollary 2.4 and Lemma 3.1 we learn the following: The functional

$$T \mapsto \bar{s}(T),$$

restricted to the class of projective images T of the body K , has a stationary point at K . If this stationary point were in fact a *local maximum* then by Proposition 3.3 we would have

$$\phi(K) = \text{Tr}[\text{Cov}(K^\circ) \cdot \text{Cov}(K)] \leq n/(n+2)^2. \quad (56)$$

Inequality (56) is equivalent to Conjecture 5.1 from [14]. This local maximum property indeed holds in the case where K is the unit ball of ℓ_p^n for $1 \leq p \leq \infty$, see Alonso-Gutiérrez [1]. However, Proposition 1.5 above implies that this local maximum property fails in general. The remainder of this section is concerned with the proof of Proposition 1.5. Set

$$K_0 = B_1^{n-1} = \left\{ x \in \mathbb{R}^{n-1}; \sum_{i=1}^{n-1} |x_i| \leq 1 \right\}.$$

Define $K_1 = K_0 \cap (\sqrt{3/n})B_2^{n-1}$ where $B_2^{n-1} = \{x \in \mathbb{R}^{n-1}; \sum_i |x_i|^2 \leq 1\}$, and

$$K = \{(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}; |t| \leq 1, x \in (1 - |t|)K_0 + |t|K_1\}. \quad (57)$$

We claim that

$$\text{Vol}_{n-1}(K_1) \geq \frac{1}{3} \cdot \text{Vol}_{n-1}(K_0). \quad (58)$$

Indeed, a direct computation shows that $\int_{K_0} x_i^2 dx = 2\text{Vol}_{n-1}(K_0)/[n(n+1)]$ for all i . Therefore the average of the function $x \mapsto |x|^2$ on K_0 is at most $2/n$. Now (58) follows by the Markov-Chebychev inequality. From (57) and (58) we conclude that

$$\forall x_1 \in (-1, 1), \quad \frac{1}{3} \leq \frac{\text{Vol}_{n-1}(\{x \in \mathbb{R}^{n-1}; (x_1, x) \in K\})}{\text{Vol}_{n-1}(K_0)} \leq 1,$$

where again we use the coordinates $(x_1, x) \in \mathbb{R} \times \mathbb{R}^{n-1} \cong \mathbb{R}^n$. Consequently $\text{Vol}_n(K) \leq 2\text{Vol}_{n-1}(K_0)$ and

$$\int_K x_1^2 \frac{dx}{\text{Vol}_n(K)} = \int_{-1}^1 x_1^2 \cdot \frac{\text{Vol}_{n-1}(\{x \in \mathbb{R}^{n-1}; (x_1, x) \in K\})}{\text{Vol}_n(K)} dx_1 \geq \int_{-1}^1 \frac{x_1^2}{6} dx_1 = \frac{1}{9}. \quad (59)$$

We now move on to discuss the integral of x_1^2 over K° . We require the following:

Lemma 5.1. *Let X_1, \dots, X_{n-1} be independent random variables, distributed uniformly in the interval $[-1, 1]$. Then with a probability of at least $1/6$, there exists a decomposition of $X = (X_1, \dots, X_{n-1})$ as*

$$X = (Y + Z)/2$$

with $Y \in (8/9)B_\infty^{n-1}$ and $Z \in \sqrt{3n/10} \cdot B_2^{n-1}$. Here, $B_\infty^{n-1} = [-1, 1]^{n-1}$.

Proof. We set

$$Y_i = \begin{cases} 8/9 & \text{if } X_i > 4/9 \\ 2X_i & \text{if } -4/9 \leq X_i \leq 4/9 \\ -8/9 & \text{if } X_i < -4/9 \end{cases}$$

and $Z_i = 2X_i - Y_i$. Then for any i ,

$$\mathbb{E}Z_i^2 = \int_{4/9}^1 (2t - 8/9)^2 dt = \int_0^{5/9} (2s)^2 ds = \frac{4}{3} \cdot \left(\frac{5}{9}\right)^3 < \frac{1}{4}.$$

Hence $\mathbb{E}|Z|^2 < n/4$, and consequently $\mathbb{P}(|Z| \geq \sqrt{3n/10}) \leq 5/6$ by the Markov-Chebyshev inequality. \square

It follows from (57) that

$$K^\circ = \{(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}; |t| \leq 1, x \in K_0^\circ \text{ and } x \in (1 - |t|) \cdot K_1^\circ\}.$$

Note that $K_0^\circ = B_\infty^{n-1}$ while K_1° is the convex hull of B_∞^{n-1} with $\sqrt{n/3} \cdot B_2^{n-1}$. Lemma 5.1 implies that for any $|t| \leq 1/20$,

$$\begin{aligned} \text{Vol}_{n-1}(\{x \in \mathbb{R}^{n-1}; (t, x) \in K^\circ\}) &\geq \text{Vol}_{n-1}\left(\left\{x \in B_\infty^{n-1}; x \in \frac{19}{20}K_1^\circ\right\}\right) \\ &\geq \text{Vol}_{n-1}\left(\left\{x \in B_\infty^{n-1}; x \in \frac{19}{20} \cdot \frac{B_\infty^{n-1} + \sqrt{n/3} \cdot B_2^{n-1}}{2}\right\}\right) \geq \frac{1}{6} \cdot \text{Vol}_{n-1}(B_\infty^{n-1}). \end{aligned} \quad (60)$$

Write $\alpha(t) = \text{Vol}_{n-1}(\{x \in \mathbb{R}^{n-1}; (t, x) \in K^\circ\})$. Then α is supported in $[-1, 1]$, and its maximum is attained at $t = 0$ by the Brunn-Minkowski inequality. From (60) we learn that $\alpha(t) \geq \alpha(0)/6$ for $|t| \leq 1/20$. Therefore,

$$\int_{K^\circ} x_1^2 \frac{dx}{\text{Vol}_n(K^\circ)} = \frac{\int_{-1}^1 s^2 \alpha(s) ds}{\int_{-1}^1 \alpha(s) ds} \geq \frac{\int_{-1/20}^{1/20} s^2 \cdot (\alpha(0)/6) ds}{2\alpha(0)} \geq 10^{-6}. \quad (61)$$

Glancing at (57) we see that the compact set $K \subseteq \mathbb{R}^n$ is convex and unconditional. The conclusion (9) of Proposition 1.5 thus follows from (59) and (61). Since K and K° are unconditional, their covariance matrices are positive-definite and diagonal. Hence, with $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$,

$$\begin{aligned} \phi(K) &= \text{Tr}[\text{Cov}(K^\circ) \cdot \text{Cov}(K)] \geq \langle \text{Cov}(K^\circ)e_1, e_1 \rangle \cdot \langle \text{Cov}(K)e_1, e_1 \rangle \\ &= \int_K x_1^2 \frac{dx}{\text{Vol}_n(K)} \cdot \int_{K^\circ} x_1^2 \frac{dx}{\text{Vol}_n(K^\circ)} \geq c. \end{aligned}$$

This completes the proof of Proposition 1.5.

6 The floating body of a cone and self-convolution

In this section we describe various relations between the floating body of a convex cone, its Laplace transform and its self-convolution. Given a convex set $A \subseteq \mathbb{R}^n$ and a parameter $\delta > 0$, Schütt and Werner [32] define the *floating body* A_δ as the intersection of all closed half-spaces $H \subseteq \mathbb{R}^n$ for which

$$\text{Vol}_n(A \cap H) \geq \delta.$$

The floating body $A_\delta \subseteq A$ is closed and convex. When $V \subset \mathbb{R}^{n+1}$ is a proper, convex cone, by homogeneity we have

$$V_\delta = \delta^{1/(n+1)} \cdot V_1 \quad \text{for all } \delta > 0.$$

Clearly $V_\delta \subseteq \text{Int}(V)$. Recall the logarithmic Laplace transform Φ_V and its Legendre transform Φ_V^* .

Proposition 6.1. *For any proper, convex cone $V \subset \mathbb{R}^{n+1}$ and $\delta > 0$, we have*

$$V_\delta = \{x \in \text{Int}(V) ; \Phi_V^*(x) \leq \kappa_n - \log \delta\}$$

where $\kappa_n = \log \left[\left(\frac{n+1}{e} \right)^{n+1} / (n+1)! \right]$.

Proof. Recall from (12) above that for any $y \in \mathbb{R}^{n+1}$,

$$e^{\Phi_V(y)} = \int_V e^{\langle y, x \rangle} dx = \frac{n!}{|y|} \cdot \text{Vol}_{n-1}(K_y) = (n+1)! \cdot \text{Vol}_n(C_y), \quad (62)$$

where $K_y = \{x \in V ; \langle x, y \rangle = -1\}$ while

$$C_y = \{x \in V ; \langle x, y \rangle \geq -1\}. \quad (63)$$

A point $x \in \mathbb{R}^{n+1}$ belongs to V_δ if and only if the following holds: The point x belongs to $\text{Int}(V)$ and for any $y \in \text{Int}(V^*)$ with $\langle x, y \rangle = -1$,

$$\text{Vol}_{n+1}(C_y) \geq \delta.$$

By homogeneity, we see that for any $x \in \text{Int}(V)$,

$$x \in V_\delta \iff \forall y \in \text{Int}(V^*), \quad (-\langle x, y \rangle)^{n+1} \cdot \text{Vol}_{n+1}(C_y) \geq \delta.$$

From (62), for any $x \in \text{Int}(V)$ we see that $x \in V_\delta$ if and only if

$$(n+1)! \cdot \delta \leq \inf_{y \in \text{Int}(V^*)} e^{\Phi_V(y)} \cdot (-\langle x, y \rangle)^{n+1} = \left(\frac{n+1}{e} \right)^{n+1} \cdot e^{-\Phi_V^*(x)},$$

where the last passage is the content of formula (26) above. □

Remark 6.2. Given a boundary point $x \in \partial V_\delta$ we may look at the normal $N(x)$ to the smooth hypersurface ∂V_δ at the point x , pointing outwards of V_δ , and satisfying

$$|\langle N(x), x \rangle| = 1.$$

Then $N(x) = \nabla \Phi_V^*(x)/(n+1)$ by Proposition 6.1, and by (25),

$$\Phi_V(N(x)) = (n+1) \log \frac{n+1}{e} - \Phi_V^*(x) = \log((n+1)!) + \log \delta.$$

It follows that the *polar hypersurface* to ∂V_δ , which is defined as the left-hand side of the following formula, satisfies

$$\{N(x); x \in \partial V_\delta\} = \{y \in V^*; \Phi_V(y) = \log((n+1)!) + \log \delta\}.$$

In other words, the level sets of the Laplace transform of the cone V are the polar hypersurfaces to the boundaries of the floating bodies V_δ .

In addition to the convex functions Φ_{V^*} and Φ_V^* , we shall introduce yet another convex function that is canonically defined on a proper, convex cone V . It is influenced by Schmuckenschläger's work [30]. For a proper, convex cone $V \subset \mathbb{R}^{n+1}$ and $x \in \text{Int}(V)$ we define

$$\Psi_V(x) = -\log(1_V * 1_V)(x) = -\log \text{Vol}_{n+1}(V \cap (x - V)),$$

the *self-convolution function* of the cone. Here 1_V is the characteristic function of the set V , which attains the value 1 in V and vanishes elsewhere. Since V is convex, the convolution $1_V * 1_V$ is a log-concave function by the Brunn-Minkowski inequality and hence Ψ_V is a convex function which is finite in $\text{Int}(V)$. Moreover,

$$\Psi_V(tx) = -(n+1) \log t - \log \text{Vol}_{n+1}(V/t \cap (x - V/t)) = -(n+1) \log t + \Psi_V(x). \quad (64)$$

Thus the convex function Ψ_V has the same homogeneity as its sisters Φ_{V^*} and Φ_V^* . The convex function $\Psi_V : \text{Int}(V) \rightarrow \mathbb{R}$ does not seem smooth in general. However, it is certainly smooth when the boundary of K_y is smooth and strongly convex for some (and hence for all) $y \in \text{Int}(V^*)$. Recall that for $x \in \text{Int}(V)$ we denote

$$T_x = \{z \in V^*; \langle x, z \rangle = -1\}.$$

We claim that the point $y = \nabla \Phi_V^*(x)/(n+1)$ is the Santaló point of T_x . Indeed, since $\nabla \Phi_V^*$ is (-1) -homogeneous, $x = \nabla \Phi_V(y)/(n+1)$. From Lemma 2.5 we know that x is the barycenter of K_y . Consequently y is the Santaló point of T_x as explained in Remark 2.3.

Proposition 6.3. *Let $V \subset \mathbb{R}^{n+1}$ be a proper, convex cone. Then for any $x \in \text{Int}(V)$,*

$$\Psi_V(x) \geq \Phi_V^*(x) + \kappa_n, \quad (65)$$

where $\kappa_n = \log(2^n(n+1)!) - (n+1) \log \left(\frac{n+1}{e}\right)$. There is equality in (65) if and only if T_x is centrally-symmetric with respect to some point in T_x .

Moreover, consider the case where T_x is centrally-symmetric with respect to some point in T_x , and where the boundary of T_x is smooth and strongly convex. Then the equality in (65) holds to first order in x , and consequently in this case,

$$\nabla^2 \Psi_V(x) \geq \nabla^2 \Phi_V^*(x). \quad (66)$$

Proof. Set $y = \nabla \Phi_V^*(x)/(n+1)$, the Santaló point of T_x . Then $\langle x, y \rangle = -1$ as $y \in T_x$. For any point $z \in V \cap (x - V)$, the point $x - z$ also belongs to V . Therefore the convex body $V \cap (x - V)$ is centrally-symmetric around the point $x/2$. Consequently,

$$\text{Vol}_{n+1} \left(\left\{ z \in V \cap (x - V); \left\langle z - \frac{x}{2}, y \right\rangle \geq 0 \right\} \right) = \frac{1}{2} \cdot \text{Vol}_{n+1}(V \cap (x - V)). \quad (67)$$

From (67) and from the fact that $\langle x, y \rangle = -1$,

$$\text{Vol}_{n+1} \left(\left\{ z \in V; \langle z, y \rangle \geq -\frac{1}{2} \right\} \right) \geq \frac{1}{2} \cdot \text{Vol}_{n+1}(V \cap (x - V)). \quad (68)$$

The left-hand side of (68) equals $\text{Vol}_{n+1}(C_{2y})$ while the right-hand side equals $e^{-\Psi_V(x)}/2$. Thanks to (62) we may rephrase (68) as

$$\Phi_V(2y) - \log(n+1)! \geq -\Psi_V(x) - \log 2. \quad (69)$$

Since $y = \nabla \Phi_V^*(x)/(n+1)$, from properties (22) and (25) we obtain

$$\Phi_V^*(x) + \Phi_V(2y) = (n+1) \log[(n+1)/(2e)]. \quad (70)$$

Now (65) follows from (69) and (70).

Equality in (65) is equivalent to equality in (68). If equality holds in (68) then the closed convex set $V \cap (x - V)$ must contain the entire slice K_{2y} , or equivalently,

$$K_{2y} \cap (x - K_{2y}) \supseteq K_{2y}.$$

This means that K_{2y} is centrally-symmetric around the point $x/2 \in K_{2y}$. This central symmetry condition is not only necessary but it is also sufficient for equality in (68), as it implies that $V \cap (x - V)$ is a double cone with base K_{2y} and apices 0 and x , which leads to equality in (68). We have thus proven that equality holds in (65) if and only if K_y is centrally-symmetric around x , which according to Remark 2.3 happens if and only if T_x is centrally-symmetric around y . Note that if T_x is centrally-symmetric with respect to some point, then this point must be the Santaló point y .

We move on to the ‘‘Moreover’’ part. Assume that T_x has a smooth and strongly convex boundary, and that it is centrally-symmetric with respect to a certain point $y \in T_x$. Then $y \in \text{Int}(V^*)$ with $\langle x, y \rangle = -1$, and K_y is centrally-symmetric around the point x . We thus see that the cone V has a non-trivial symmetry, which is the linear map

$$S : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$$

with $S(x) = x$ and $S(z) = -z$ for any $z \in y^\perp$. Since $S(V) = V$, the functions $\Phi_V^* \circ S - \Phi_V^*$ and $\Psi_V \circ S - \Psi_V$ are constant in $\text{Int}(V)$. It follows that for any $z \in \text{Int}(V)$,

$$S^*(\nabla\Phi_V^*(Sz)) = \nabla\Phi_V^*(z) \quad \text{and} \quad S^*(\nabla\Psi_V(Sz)) = \nabla\Psi_V(z).$$

In particular, at the point x , which is a fixed point of the symmetry S , the gradients $\nabla\Phi_V^*(x)$ and $\nabla\Psi_V(x)$ are both fixed points of S^* . However, the eigenspace of S^* that corresponds to the eigenvalue one is 1-dimensional, as $S^*(y) = y$ and $S^*(z) = -z$ for $z \in x^\perp$. Hence the vectors $\nabla\Phi_V^*(x)$ and $\nabla\Psi_V(x)$ are proportional. The homogeneity relations (22) and (64) yield

$$\langle \nabla\Psi_V(x), x \rangle = -(n+1) = \langle \nabla\Phi_V^*(x), x \rangle.$$

Since the gradients are proportional, then necessarily $\nabla\Psi_V(x) = \nabla\Phi_V^*(x)$, and the equality in (65) is to first order. The Hessian inequality (66) follows. \square

The following is a crude reverse form of Proposition 6.3:

Proposition 6.4. *Let $V \subset \mathbb{R}^{n+1}$ be a proper, convex cone. Then for any $x \in \text{Int}(V)$,*

$$\Psi_V(x) \leq \Phi_V^*(x) + Cn, \tag{71}$$

where $C > 0$ is a universal constant.

Proof. Set $y = \nabla\Phi_V^*(x)/(n+1)$, the Santaló point of T_x . Then $b(K_y) = x$ by Lemma 2.5. By Fubini's theorem,

$$\text{Vol}_{n+1}(V \cap (x - V)) = \frac{1}{|y|} \int_0^1 \text{Vol}_n[tK_y \cap (x - (1-t)K_y)] dt.$$

The barycenter of $K_y - x$ lies at the origin, and by Milman and Pajor [22, page 321],

$$\text{Vol}_n[tK_y \cap (x - (1-t)K_y)] = \text{Vol}_n[(t(K_y - x) \cap (1-t)(x - K_y))] \geq t^n(1-t)^n \text{Vol}_n(K_y).$$

Therefore,

$$e^{-\Psi_V(x)} = \text{Vol}_{n+1}(V \cap (x - V)) \geq \frac{\text{Vol}_n(K_y)}{|y|} \int_0^1 t^n(1-t)^n dt = \frac{n! \cdot e^{\Phi_V(y)}}{(2n+1)!}, \tag{72}$$

where we used (29) in the last passage. As in (70) above, we know that

$$\Phi_V^*(x) + \Phi_V(y) = (n+1) \log[(n+1)/e]. \tag{73}$$

Now (71) follows from (72), (73) and the fact that $(2n+1)! \cdot e^{n+1} \leq C^n \cdot n! \cdot (n+1)^{n+1}$. \square

7 The isomorphic slicing problem

In this section we prove Theorem 1.3. We begin with a formula for the isotropic constant of a hyperplane section of V . Recall that for any $A \in \mathbb{R}^{(n+1) \times (n+1)}$ and $v \in \mathbb{R}^{n+1}$,

$$\det(A + vv^*) = \det(A) + v^* \text{Adj}(A)v$$

where $\text{Adj}(A)$ is the adjoint matrix.

Lemma 7.1. *For any proper, convex cone $V \subset \mathbb{R}^{n+1}$ and $y \in \text{Int}(V^*)$,*

$$\det \nabla^2 \Phi_V(y) = \kappa_n \cdot L_{K_y}^{2n} \cdot e^{2\Phi_V(y)} \quad (74)$$

with $\kappa_n = (n+1)^{n+1} \cdot (n+2)^n / (n!)^2$.

Proof. By Lemma 2.5,

$$\frac{\nabla^2 \Phi_V(y)}{(n+1)(n+2)} = \text{Cov}(K_y) + \frac{\text{b}(K_y)\text{b}^*(K_y)}{n+2}.$$

The symmetric matrix $\text{Cov}(K_y)$ is of rank n , with the vector y spanning its kernel. Therefore the adjoint matrix of $\text{Cov}(K_y)$ is

$$\det_n \text{Cov}(K_y) \cdot \frac{yy^*}{|y|^2},$$

where $\det_n(A)$ stands for the sum of the determinants of all principal $n \times n$ minors of a matrix $A \in \mathbb{R}^{(n+1) \times (n+1)}$. Consequently,

$$\det \nabla^2 \Phi_V(y) = (n+1)^{n+1} \cdot (n+2)^{n+1} \cdot \frac{\det_n \text{Cov}(K_y)}{|y|^2} \cdot \frac{\langle y, \text{b}(K_y) \rangle^2}{n+2}.$$

However, $\langle \text{b}(K_y), y \rangle = -1$ as $\text{b}(K_y) \in K_y = \{x \in V; \langle x, y \rangle = -1\}$. Hence, by (29),

$$L_{K_y}^{2n} = \frac{\det_n \text{Cov}(K_y)}{\text{Vol}_n(K_y)^2} = \frac{(n!)^2}{(n+1)^{n+1} \cdot (n+2)^n} \cdot \frac{\det \nabla^2 \Phi_V(y)}{e^{2\Phi_V(y)}},$$

and the formula follows. \square

The role of the determinant $\nabla^2 \Phi_V$ is twofold: First, it appears in the expression for the isotropic constant in Lemma 7.1. Second, it is the Jacobian determinant of the diffeomorphism $\nabla \Phi_V : \text{Int}(V^*) \rightarrow \text{Int}(V)$. The next lemma describes a certain geometric property of this map. Recall that for $y \in \text{Int}(V^*)$ we denote $C_y = \{x \in V; \langle x, y \rangle \geq -1\}$, a truncated cone.

Lemma 7.2. *Let $V \subset \mathbb{R}^{n+1}$ be a proper, convex cone and let $y \in \text{Int}(V^*)$. Then,*

$$\nabla\Phi_V(y + V^*) \subseteq (n + 1) \cdot C_y.$$

Proof. It suffices to prove that the image of the open set $y + \text{Int}(V^*)$ under the diffeomorphism $\nabla\Phi_V$ is contained in the closed set $(n + 1)C_y$. Let $z \in y + \text{Int}(V^*)$. Then for any $x \in K_y$ we have $\langle x, z - y \rangle < 0$. This means that

$$K_y \cap K_z = \emptyset,$$

as there is no point $x \in K_y$ with $\langle x, z \rangle = \langle x, y \rangle = -1$. The convex set K_y disconnects the cone V into two connected components. The set K_z must be contained in the convex set C_y , and hence its barycenter satisfies $\text{b}(K_z) \in C_y$. By Lemma 2.5, we know that $\nabla\Phi_V(z) = (n + 1) \cdot \text{b}(K_z)$, and the conclusion of the lemma follows. \square

Remark 7.3. From the proof of Lemma 7.2 we obtain a simple geometric interpretation of the set $\nabla\Phi_V(y + \text{Int}(V^*))$. Namely, this set consists of all barycenters of all hyperplane sections of the truncated cone $(n + 1) \cdot C_y$ that are disjoint from the base of this truncated cone. Additionally, a simple modification of the proof of Lemma 7.2 shows that

$$\nabla\Phi_V(y - V^*) \subseteq V \setminus (n + 1)C_y.$$

Lemma 7.4. *Let $V \subset \mathbb{R}^{n+1}$ be a proper, convex cone, $y_0 \in \text{Int}(V^*)$ and let $0 < \varepsilon < 1$. Then there exists a point $y \in \text{Int}(V^*)$ such that $y - y_0 \in V^* \cap (\varepsilon y_0 - V^*)$ and*

$$L_{K_y} \leq C/\sqrt{\varepsilon} \tag{75}$$

where $C > 0$ is a universal constant.

Proof. In this proof $C, \tilde{C}, \bar{C}, \hat{C} > 0$ denote various positive universal constants, whose value may change from one line to the next. We may assume that $\varepsilon > e^{-n}$ since otherwise conclusion (75) follows from the trivial upper bound $L_{K_{y_0}} \leq C\sqrt{n}$ (see, e.g., [9]). Define

$$S = (y_0 + V^*) \cap (y_0 + \varepsilon y_0 - V^*).$$

According to Proposition 6.4,

$$\text{Vol}_{n+1}(S) = \text{Vol}_{n+1}(V^* \cap (\varepsilon y_0 - V^*)) = \exp(-\Psi_{V^*}(\varepsilon y_0)) \geq \exp(-Cn - \Phi_{V^*}^*(\varepsilon y_0)). \tag{76}$$

We would like to get rid of the expression $\Phi_{V^*}^*(\varepsilon y_0)$, and replace it by $\Phi_V(\varepsilon y_0)$ plus an error term. To this end, we may use the ‘‘commutation relation’’ of Corollary 2.4, according to which for any $y \in \text{Int}(V^*)$,

$$\Phi_V(y) - \Phi_{V^*}^*(y) = \log \frac{(n!)^2 \cdot e^{n+1}}{(n + 1)^{n+1}} + \log \bar{s}(K_y) \geq n \log n - Cn + \log \bar{s}(K_y). \tag{77}$$

However, from the Bourgain-Milman inequality (2), we know that $\bar{s}(K_y) \geq (c/n)^n$ for some universal constant $c > 0$. Hence, from (76) and (77),

$$Vol_{n+1}(S) \geq \exp(-\tilde{C}n - \Phi_V(\varepsilon y_0)). \quad (78)$$

For any $y \in S$ we know that $y - (1+\varepsilon)y_0 \in -V^*$, and hence the scalar product of $y - (1+\varepsilon)y_0$ with $\nabla\Phi_V(y_0 + \varepsilon y_0) \in V$ is non-negative. By the convexity of Φ_V , for any $y \in S$,

$$\Phi_V(y) \geq \Phi_V(y_0 + \varepsilon y_0) + \langle \nabla\Phi_V(y_0 + \varepsilon y_0), y - (1 + \varepsilon)y_0 \rangle \geq \Phi_V(y_0 + \varepsilon y_0). \quad (79)$$

From (78) and (79)

$$\int_S e^{2\Phi_V(y)} dy \geq e^{-\tilde{C}n} \cdot e^{2\Phi_V((1+\varepsilon)y_0) - \Phi_V(\varepsilon y_0)} = e^{-\tilde{C}n} \cdot \left(\frac{\varepsilon}{(1+\varepsilon)^2} \right)^{n+1} \cdot e^{\Phi_V(y_0)}, \quad (80)$$

where we used the homogeneity relation (22) in the last passage. According to Lemma 7.2, the set $\nabla\Phi_V(S)$ is contained in the truncated cone $(n+1) \cdot C_{y_0}$. Corollary 2.6 states that the map $\nabla\Phi_V$ is a diffeomorphism. By changing variables,

$$Vol_{n+1}((n+1) \cdot C_{y_0}) \geq Vol_{n+1}(\nabla\Phi_V(S)) = \int_S \det \nabla^2\Phi_V(y) dy = \kappa_n \int_S e^{2\Phi_V(y)} \cdot L_{K_y}^{2n} dy, \quad (81)$$

where $\kappa_n = (n+1)^{n+1} \cdot (n+2)^n / (n!)^2 \geq 1$ is the coefficient from Lemma 7.1. Recall the formula $Vol_{n+1}(C_{y_0}) = \exp(\Phi_V(y_0)) / (n+1)!$ according to (62). We thus deduce from (81) that

$$\frac{(n+1)^{n+1}}{(n+1)!} e^{\Phi_V(y_0)} \geq \inf_{y \in S} L_{K_y}^{2n} \cdot \int_S e^{2\Phi_V(y)} dy. \quad (82)$$

From (80) and (82),

$$\frac{(n+1)^{n+1}}{(n+1)!} \cdot e^{\Phi_V(y_0)} \geq \inf_{y \in S} L_{K_y}^{2n} \cdot e^{-\tilde{C}n} \cdot \left(\frac{\varepsilon}{(1+\varepsilon)^2} \right)^{n+1} \cdot e^{\Phi_V(y_0)}.$$

This implies that there exists $y \in S$ for which

$$L_{K_y}^{2n} \leq \left(\frac{\hat{C}}{\varepsilon} \right)^{n+1} \leq \left(\frac{\bar{C}}{\varepsilon} \right)^n,$$

where we used the assumption that $\varepsilon > e^{-n}$ in the last passage. This completes the proof of the lemma. \square

Given a convex body $K \subseteq \mathbb{R}^n$ with the origin in its interior, we consider the associated (non-symmetric) norm

$$\|x\|_K = \inf\{\lambda \geq 0; x \in \lambda K\} \quad (x \in \mathbb{R}^n).$$

The supporting functional of K is

$$h_K(y) = \|y\|_{K^\circ} = \sup_{z \in K} \langle y, z \rangle = \sup_{0 \neq z \in \mathbb{R}^n} \frac{\langle y, z \rangle}{\|z\|_K} \quad (y \in \mathbb{R}^n).$$

We write $\pi(t, x) = x$ for $(t, x) \in \mathbb{R} \times \mathbb{R}^n \cong \mathbb{R}^{n+1}$.

Lemma 7.5. *Let $K \subseteq \mathbb{R}^n$ be a convex body with the origin in its interior, and set*

$$V = \{(t, tx) \in \mathbb{R} \times \mathbb{R}^n; t \geq 0, x \in K\}. \quad (83)$$

Then for any $y \in \text{Int}(V^)$ and $x \in \mathbb{R}^n$, denoting $y = (y_1, \pi(y)) \in \mathbb{R} \times \mathbb{R}^n$ we have*

$$\|x\|_{\pi(K_y)} = -y_1 \|x\|_K - \langle x, \pi(y) \rangle, \quad (84)$$

and when setting $T = \pi(K_y) \subseteq \mathbb{R}^n$ we obtain

$$T^\circ = -y_1 K^\circ - \pi(y). \quad (85)$$

Proof. By definition,

$$K_y = \{(t, tx); t \geq 0, x \in K, ty_1 + \langle tx, \pi(y) \rangle = -1\}.$$

It follows that

$$\pi(K_y) = \left\{ x \in \mathbb{R}^n; \|x\|_K \leq -\frac{1 + \langle x, \pi(y) \rangle}{y_1} \right\},$$

from which (84) follows. Next, for any $z \in \mathbb{R}^n$, we see that $z \in T^\circ$ if and only if

$$\langle z, x \rangle \leq \|x\|_{\pi(K_y)} = -y_1 \|x\|_K - \langle x, \pi(y) \rangle \quad \text{for all } x \in \mathbb{R}^n. \quad (86)$$

Condition (86) is equivalent to $\langle z + \pi(y), x \rangle \leq -y_1 \|x\|_K$ for all $x \in \mathbb{R}^n$. Thus $z \in T^\circ$ if and only if $h_K(z + \pi(y)) \leq -y_1$, or equivalently if and only if $z + \pi(y) \in -y_1 K^\circ$. The relation (85) follows. \square

Proof of Theorem 1.3: Define V as in (83). Since the barycenter of K lies at the origin, the point $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^n$ is the barycenter of $K_{-e} = \{1\} \times K$. We will apply Lemma 7.4 with $y_0 = -e$. From the conclusion of this lemma, there exists $y \in \text{Int}(V^*)$ such that

$$L_{K_y} \leq C/\sqrt{\varepsilon}.$$

Moreover, $y + e \in V^* \cap (-\varepsilon e - V^*)$. Since $\langle z, e \rangle \leq 0$ for any $z \in V^*$, by setting $y_1 := \langle y, e \rangle$ we have

$$-1 - \varepsilon \leq y_1 \leq -1. \quad (87)$$

Since $y + e \in V^*$ we obtain from (87) that

$$y + e \in \{z \in V^*; -\varepsilon \leq z_1 \leq 0\} = \{(t, tw) \in \mathbb{R} \times \mathbb{R}^n; -\varepsilon \leq t \leq 0, w \in -K^\circ\}.$$

Thus $\pi(y) \in \varepsilon K^\circ$. Since $y + e \in -\varepsilon e - V^*$ we obtain from (87) that

$$y + e \in \{z \in -\varepsilon e - V^*; -\varepsilon \leq z_1 \leq 0\} = \{(-\varepsilon + t, tw) \in \mathbb{R} \times \mathbb{R}^n; 0 \leq t \leq \varepsilon, w \in -K^\circ\}.$$

Thus $\pi(y) \in -\varepsilon K^\circ$. To summarize,

$$\pi(y) \in \varepsilon(K^\circ \cap (-K^\circ)). \quad (88)$$

Denote

$$T = -y_1 \cdot \pi(K_y).$$

Then $T \subseteq \mathbb{R}^n$ is a convex body that is affinely equivalent to K_y , and hence $L_T = L_{K_y} < C/\sqrt{\varepsilon}$. Furthermore, by Lemma 7.5,

$$T^\circ = K^\circ + \frac{\pi(y)}{y_1}, \quad (89)$$

and T° is a translate of K° . From (87) and (88) we know that $\pi(y)/y_1$ belongs to $\varepsilon(-K^\circ) \cap \varepsilon K^\circ$. Hence, from (89),

$$T^\circ \subseteq K^\circ + \varepsilon K^\circ \quad \text{and} \quad K^\circ \subseteq T^\circ + \varepsilon K^\circ.$$

Equivalently,

$$(1 - \varepsilon)K^\circ \subseteq T^\circ \subseteq (1 + \varepsilon)K^\circ.$$

Since $0 < \varepsilon < 1/2$, by dualizing this inclusion we obtain

$$(1 - \varepsilon) \cdot K \subseteq \frac{1}{1 + \varepsilon} \cdot K \subseteq T \subseteq \frac{1}{1 - \varepsilon} \cdot K \subseteq (1 + 2\varepsilon) \cdot K,$$

and the theorem follows by adjusting the universal constant C . □

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