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A coordinate-free proof of the finiteness principle for the Whitney extension problem

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Abstract

We present a coordinate-free version of Fefferman's solution of Whitney's extension problem in the space $C^{m-1,1}(\mathbb{R}^n)$. While the original argument relies on an elaborate induction on collections of partial derivatives, our proof uses the language of ideals and translation-invariant subspaces in the ring of polynomials. We emphasize the role of compactness in the proof, first in the familiar sense of topological compactness, but also in the sense of finiteness theorems arising in logic and semialgebraic geometry. In a follow-up paper, we apply these ideas to study extension problems for a class of sub-Riemannian manifolds where global coordinates may be unavailable.

1 Introduction

Whitney's extension problem asks, given a subset $E \subset \mathbb{R}^n$ and function $f : E \to \mathbb{R}$, how can one determine whether f admits an extension $F : \mathbb{R}^n \to \mathbb{R}$ in a prescribed regularity class (e.g., Hölder, C^m, Sobolev, etc.)? In [23, 24, 25], H. Whitney developed characterizations for the existence of extensions in the class C^m (i.e., functions which are continuously differentiable up to order m). In particular, in dimension $n = 1$, he proved that certain natural conditions on the continuity of the finite difference quotients of a function $f : E \to \mathbb{R}$ (for $E \subset \mathbb{R}$) are necessary and sufficient for the existence of a C^m-extension to the real line. In higher dimensions there is no analogue of finite difference quotients and the problem is far more difficult. Several years ago, a complete characterization of C^m-extendibility in arbitrary dimensions was developed by C. Fefferman [11, 12], building on the work of Y. Brudnyi and P. Shvartsman [4, 5, 6, 7, 8, 17, 19, 20], who solved the extension problem in $C^{1,1}(\mathbb{R}^n)$, work of G. Glaeser on C^1-extendibility [15], and work of E. Bierstone, P. Milman, and W. Pawłucki on C^m-extendibility for functions on subanalytic sets [2, 3].

In this article we will focus on the Hölder class $C^{m-1,1}(\mathbb{R}^n)$, which consists of all C^{m-1} functions $F : \mathbb{R}^n \to \mathbb{R}$ whose $(m-1)$-st order derivatives are Lipschitz continuous. This

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space is equipped with a seminorm

$$\|F\|_{C^{m-1,1}(\mathbb{R}^n)} := \sup_{x,y\in\mathbb{R}^n} \left(\sum_{|\alpha|=m-1} \frac{(\partial^\alpha F(x) - \partial^\alpha F(y))^2}{|x-y|^2} \right)^{\frac{1}{2}}, \quad F \in C^{m-1,1}(\mathbb{R}^n), \tag{1}$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_n$ denotes the order of a multiindex $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$.

In [16, 19], Shvartsman considers Whitney's extension problem in the space $C^{1,1}(\mathbb{R}^n)$. One of his main results is the following *finiteness principle* (see also [4, 17]): Suppose that the restriction of a function $f : E \to \mathbb{R}$ (for $E \subset \mathbb{R}^n$) to every subset $S \subset E$ of cardinality at most $3 \cdot 2^{n-1}$ can be extended to a function $F_S \in C^{1,1}(\mathbb{R}^n)$ with $\|F_S\|_{C^{1,1}(\mathbb{R}^n)} \leq M$. Then the function f itself can be extended to a function $F \in C^{1,1}(\mathbb{R}^n)$ with norm $\|F\|_{C^{1,1}(\mathbb{R}^n)} \leq \gamma(n)M$. Brudnyi and Shvartsman conjectured in [5] and [8] (see also [17, 18, 19]) that a similar result should hold for the entire range of Hölder spaces (i.e., for any order of smoothness $m \geq 2$). In [10], Fefferman verified their conjecture with the following theorem:

Theorem 1.1 (The Brudnyi-Shvartsman-Fefferman finiteness principle)**.** *For any $m, n \geq 1$, there exist constants $C^\# \geq 1$ and $k^\# \in \mathbb{N}$ such that the following holds.*

Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}$ be given. Suppose that there exists $M > 0$ so that for all subsets $S \subset E$ satisfying $\#(S) \leq k^\#$ there exists a function $F^S \in C^{m-1,1}(\mathbb{R}^n)$ with $\|F^S\|_{C^{m-1,1}(\mathbb{R}^n)} \leq M$ and $F^S = f$ on S.

Then there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ with $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C^\# \cdot M$ and $F = f$ on E.

The finiteness principle says that a function $f : E \to \mathbb{R}$ admits a $C^{m-1,1}$ extension if and only if for every $k^\#$-point subset $S \subset E$, the restriction $f|_S$ admits a $C^{m-1,1}$ extension with a uniform bound on the seminorm. The parameters $k^\#$ and $C^\#$ in Theorem 1.1 are often referred to as *finiteness constants* for the function space $C^{m-1,1}(\mathbb{R}^n)$.

In this article we describe a proof of Theorem 1.1 based on a coordinate-free version of Fefferman's stopping time argument. Our approach emphasizes the metric and symmetry structure of $\mathbb{R}^n$ and shortens several components of the analysis through the use of compactness arguments. There are two types of compactness arguments that are relevant here. The first is topological compactness, which is the common compactness used in Analysis. The second is logic-type compactness results from the theory of semialgebraic sets. We will explain how to replace the basis-dependent notion of *monotonic multiindex sets* from Fefferman's argument with the basis-independent notion of *transverse dilation-and-translation-invariant subspaces*. Our use of the latter concept is likely adaptable to the study of extension problems on sub-Riemannian manifolds, where global coordinates are often unavailable. In a follow-up paper [9], we will address this topic for the special class of sub-Riemannian manifolds known as Carnot groups.

Our main result is a finiteness principle for $C^{m-1,1}$-extension on finite subsets $E \subset \mathbb{R}^n$, where the constants depend on a parameter $\mathcal{C}(E) = \mathcal{C}_m(E) \in \{0, 1, 2, \cdots\}$, called the "complexity" of E. We refer the reader to section 4 for the definition of this term.

Theorem 1.2. *Fix $m, n \geq 1$. There exist constants $\lambda_1, \lambda_2 \geq 1$, determined by m and n such that the following holds. Fix a finite set $E \subset \mathbb{R}^n$ and a function $f : E \to \mathbb{R}$. Set $k^\# = 2^{\lambda_1 \mathcal{C}(E)}$ and $C^\# = 2^{\lambda_2 \mathcal{C}(E)}$. Suppose that for all subsets $S \subset E$ with $\#(S) \leq k^\#$ there exists $F^S \in C^{m-1,1}(\mathbb{R}^n)$ with $F^S = f$ on S and $\|F^S\|_{C^{m-1,1}(\mathbb{R}^n)} \leq 1$. Then there exists a function $F \in C^{m-1,1}(\mathbb{R}^n)$ with $F = f$ on E and $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C^\#$.*

In order to deduce Theorem 1.1 from Theorem 1.2, we will prove the following lemma:

Lemma 1.3. *There exists a constant K_0, determined only by m and n, such that $\mathcal{C}(E) \leq K_0$ for any finite set $E \subset \mathbb{R}^n$.*

Together, Theorem 1.2 and Lemma 1.3 imply Theorem 1.1 in the case when E is a finite subset of $\mathbb{R}^n$ and $M = 1$. By a compactness argument involving the Arzela-Ascoli theorem, one can extend this result to infinite sets. Finally, by a trivial rescaling argument we deduce Theorem 1.1 for arbitrary $M > 0$.

Fefferman's proof of Theorem 1.1 yields the constants $k^\# = C^\# = O(\exp(\exp(D))$, where $D = \binom{n+m-1}{n}$ is the dimension of the jet space for $C^{m-1,1}(\mathbb{R}^n)$, or equivalently, the number of multiindices $(\alpha_1, \cdots, \alpha_n)$ of order at most $m-1$. Bierstone and Milman [1] and Shvartsman [21] obtain the improvement $k^\# = 2^D$ at the expense of multiplying $C^\#$ by a multiplicative factor which does not affect the asymptotics $C^\# = O(\exp(\exp(D))$. In [13], Fefferman and Klartag show that the finiteness principle fails to hold for $C^\# = 1 + \epsilon$ for a small absolute constant $\epsilon > 0$, no matter the choice of $k^\#$.

We emphasize the use of compactness arguments and algebraic methods to prove our results. For this reason, some of the constants are either non-explicit or depend poorly on m and n. In particular, the constant K_0 in Lemma 1.3 is not explicit. By the use of more direct methods (which will lengthen the proofs), it is possible to obtain $K_0 = \exp(\exp(\gamma D))$, where γ is a numerical constant independent of m and n. We conjecture that this dependence is far from optimal. In fact, evidence suggests that it is possible to take K_0 to be an explicit polynomial function of the dimension D. By following through the proofs below one may check that the constants λ_1 and λ_2 in Theorem 1.2 are also harmless polynomial functions of D. This leads us to conjecture that the finiteness principle will hold with the constants $k^\# = 2^D$ and $C^\# = \exp(\text{poly}(D))$.

Throughout the proof, the symbols $C, C', c,$, etc., will be used to denote *universal constants* which are determined only by m and n. The same symbol may be used to denote a different constant in separate appearances, even within the same line.

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2 Notation, definitions, and preliminary lemmas

Given a convex domain $G \subset \mathbb{R}^n$ with nonempty interior, we let $C^{m-1,1}(G)$ denote the space of real-valued functions $F : G \to \mathbb{R}$ whose $(m-1)$-st order partial derivatives are Lipschitz continuous. Define a seminorm on $C^{m-1,1}(G)$ by

$$\|F\|_{C^{m-1,1}(G)} := \sup_{x,y\in G} \left(\sum_{|\alpha|=m-1} \frac{(\partial^\alpha F(x) - \partial^\alpha F(y))^2}{|x-y|^2} \right)^{\frac{1}{2}}, \quad F \in C^{m-1,1}(G).$$

The seminorm on $C^{m-1,1}(\mathbb{R}^n)$ is abbreviated by $\|F\| := \|F\|_{C^{m-1,1}(\mathbb{R}^n)}$.

Let $\mathcal{P}$ be the space of polynomials of degree at most $m-1$ in n real variables. Let us review some of the structure and basic properties of $\mathcal{P}$. First, $\mathcal{P}$ is a vector space of dimension $D := \#\{\alpha \in \mathbb{Z}^n_{\geq 0} : |\alpha| \leq m-1\}$. For $x \in \mathbb{R}^n$, define an inner product on $\mathcal{P}$:

$$\langle P, Q\rangle_x := \sum_{|\alpha|\leq m-1} (\frac{1}{\alpha!})\partial^\alpha P(x) \cdot \partial^\alpha Q(x),$$

where $\alpha! = \prod_{i=1}^n \alpha_i!$ and we also set $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$. If $P(z) = \sum_{|\alpha|\leq m-1} a_\alpha \cdot (z-x)^\alpha$ and $Q(z) = \sum_{|\alpha|\leq m-1} b_\alpha \cdot (z-x)^\alpha$, then $\langle P, Q\rangle_x = \sum_{|\alpha|\leq m-1} \alpha! \cdot a_\alpha b_\alpha$. Therefore, the inner product space $(\mathcal{P}, \langle\cdot,\cdot\rangle_x)$ admits an orthonormal basis of monomials $\{\sqrt{\alpha!}\cdot(z-x)^\alpha\}_{|\alpha|\leq m-1}$. We define a norm on $\mathcal{P}$ by $|P|_x := \sqrt{\langle P, P\rangle_x}$.

We define translation operators $T_h : \mathcal{P} \to \mathcal{P}$ (for $h \in \mathbb{R}^n$) by $T_h(P)(z) := P(z-h)$, and dilation operators $\tau_{x,\delta} : \mathcal{P} \to \mathcal{P}$ (for $(x,\delta) \in \mathbb{R}^n \times (0,\infty)$) by $\tau_{x,\delta}(P)(z) := \delta^{-m} P(x + \delta\cdot(z-x))$. The dilation operators lead us to define a scaled inner product on $\mathcal{P}$: For $(x,\delta) \in \mathbb{R}^n \times (0,\infty)$, let

$$\langle P, Q\rangle_{x,\delta} := \langle \tau_{x,\delta}(P), \tau_{x,\delta}(Q)\rangle_x \qquad (P, Q \in \mathcal{P}),$$

and the corresponding scaled norm is denoted by $|P|_{x,\delta} := \sqrt{\langle P, P\rangle_{x,\delta}}$. The unit ball associated to this norm is the subset

$$\mathcal{B}_{x,\delta} := \left\{ P : \ |P|_{x,\delta} = \left(\sum_{|\alpha|\leq m-1} (\frac{1}{\alpha!})(\delta^{|\alpha|-m} \cdot \partial^\alpha P(x))^2 \right)^{\frac{1}{2}} \leq 1 \right\} \subset \mathcal{P}.$$

We write $\langle\cdot,\cdot\rangle$ and $|\cdot|$ to denote the "standard" inner product $\langle\cdot,\cdot\rangle_{0,1}$ and norm $|\cdot|_{0,1}$ on $\mathcal{P}$, and $\mathcal{B} = \mathcal{B}_{0,1}$ for the corresponding unit ball.

Given $\Omega \subset \mathcal{P}$, $P_0 \in \mathcal{P}$, and $r \in \mathbb{R}$, let $r\Omega := \{rP : P \in \Omega\}$ and $P_0 + \Omega := \{P_0 + P : P \in \Omega\}$. For future use, we record below a few identities and inequalities which connect the dilation and translation operators with the scaled inner products, norms, and balls.

(a) (i) $T_{h_1} \circ T_{h_2} = T_{h_1+h_2}$.

(ii) $\tau_{x,\delta_1} \circ \tau_{x,\delta_2} = \tau_{x,\delta_1\cdot\delta_2}$.

(iii) $T_h \circ \tau_{x,\delta} = \tau_{x+h,\delta} \circ T_h$.

(b) (i) $\langle \tau_{x,\rho}(P), \tau_{x,\rho}(Q)\rangle_{x,\delta} = \langle P, Q\rangle_{x,\delta\cdot\rho}$.

(ii) $|\tau_{x,\rho}(P)|_{x,\delta} = |P|_{x,\delta\cdot\rho}$.

(iii) $\tau_{x,\rho}\mathcal{B}_{x,\delta} = \mathcal{B}_{x,\delta/\rho}$.

(c) (i) $\langle T_h(P), T_h(Q)\rangle_{x,\delta} = \langle P, Q\rangle_{x-h,\delta}$. (iii) $T_h\mathcal{B}_{x,\delta} = \mathcal{B}_{x+h,\delta}$.

(ii) $|T_h(P)|_{x,\delta} = |P|_{x-h,\delta}$.

Furthermore, for any $\delta \geq \rho > 0$,

$$\begin{cases} (\rho/\delta)^m \cdot |P|_{x,\rho} \leq |P|_{x,\delta} \leq (\rho/\delta) \cdot |P|_{x,\rho}, \text{ and hence} \\ (\delta/\rho) \cdot \mathcal{B}_{x,\rho} \subset \mathcal{B}_{x,\delta} \subset (\delta/\rho)^m \cdot \mathcal{B}_{x,\rho}. \end{cases} \tag{2}$$

Let $J_xF \in \mathcal{P}$ denote the $(m-1)$-jet of a function $F \in C^{m-1,1}(\mathbb{R}^n)$ at x, namely, the Taylor polynomial

$$(J_xF)(z) := \sum_{|\alpha|\leq m-1} (\frac{1}{\alpha!})\partial^\alpha F(x) \cdot (z-x)^\alpha \quad (z \in \mathbb{R}^n).$$

The importance of the norms $|\cdot|_{x,\delta}$ on $\mathcal{P}$ stems from the Taylor and Whitney theorems. According to Taylor's theorem, if $F \in C^{m-1,1}(G)$, where G is any convex domain in $\mathbb{R}^n$ with nonempty interior, then

$$|\partial^\beta(F - J_yF)(x)| \leq C \cdot \|F\|_{C^{m-1,1}(G)} \cdot |x-y|^{m-|\beta|}, \quad \text{for } x, y \in G,\ |\beta| \leq m-1.$$

This implies

$$\begin{cases} |J_xF - J_yF|_{x,\delta} \leq C_T\|F\|_{C^{m-1,1}(G)}, \text{ or equivalently} \\ J_xF - J_yF \in C_T\|F\|_{C^{m-1,1}(G)} \cdot \mathcal{B}_{x,\delta} \quad \text{for } x, y \in G,\ \delta \geq |x-y|, \end{cases} \tag{3}$$

where $C_T = C_T(m,n)$ is a constant determined by m and n. Therefore the norm $|\cdot|_{x,\delta}$ may be used to describe the compatibility conditions on the $(m-1)$-jets of a $C^{m-1,1}$ function at two points x, y in $\mathbb{R}^n$, whenever $|x-y| \leq \delta$. The conditions in (3) capture the essence of the concept of a $C^{m-1,1}$ function in the following sense: Whitney's theorem [23] states that whenever $E \subset \mathbb{R}^n$ is an arbitrary set, $M > 0$, and $\{P_x\}_{x\in E}$ is a collection of polynomials with

$$|P_x - P_y|_{x,\delta} \leq M \qquad \text{for } x, y \in E,\ \delta = |x-y|, \tag{4}$$

then there exists a $C^{m-1,1}$ function $F : \mathbb{R}^n \to \mathbb{R}$ with $\|F\| \leq CM$ and $J_xF = P_x$ for all $x \in E$. As usual, C is a constant depending solely on m and n.

The vector space of $(m-1)$-jets is a ring, denoted by $\mathcal{P}_x$, equipped with the product $\odot_x$ (indexed by a basepoint $x \in \mathbb{R}^n$) defined by $P \odot_x Q = J_x(P \cdot Q)$. The product and translation/dilation operators are related by

$$\begin{cases} \tau_{x,\delta}(P \odot_x Q) = \delta^m \cdot \tau_{x,\delta}(P) \odot_x \tau_{x,\delta}(Q), \\ T_h(P \odot_x Q) = T_h(P) \odot_{x+h} T_h(Q) \qquad \text{for } x, h \in \mathbb{R}^n,\ \delta > 0. \end{cases} \tag{5}$$

The following lemma, taken verbatim from [14, section 12], summarizes a few basic properties of the product and norms introduced above. See the proof of Lemma 1 in [14, section 12] for a direct argument that leads to explicit constants. Our argument below emphasizes the rôle of rescaling and compactness.

Lemma 2.1. *Let $x, y \in \mathbb{R}^n$ and $\delta, \rho > 0$. Assume that $|x - y| \leq \rho \leq \delta$. Then for any $P, Q \in \mathcal{P}$,*

(i) $|P|_{y,\rho} \leq \widetilde{C}|P|_{x,\rho}$.

(ii) $|P \odot_x Q|_{x,\rho} \leq \widetilde{C}\delta^m |P|_{x,\delta}|Q|_{x,\rho}$.

(iii) $|(P \odot_y Q) - (P \odot_x Q)|_{x,\rho} \leq \widetilde{C}\delta^m |P|_{x,\delta}|Q|_{x,\delta}$.

Here, $\widetilde{C} > 0$ is a constant depending solely on m and n.

Proof. The main step is to use (5) and observe that by translating and rescaling, we may reduce matters to the case $x = 0$ and $\rho = 1$. Next, note that it suffices to prove the lemma for non-zero polynomials P and Q. Normalizing, we assume that $|P|_{0,1} = |Q|_{0,1} = 1$.

In order to prove (i), observe that the space of all relevant parameters is compact, since $|y| \leq 1$ and $|P|_{0,1} = 1$. The left-hand side of (i) is a continuous function on this space of parameters, hence the maximum is attained, and yields the constant $\widetilde{C}$ on the right-hand side. In order to prove (ii), observe that the left-hand side in (ii) is bounded from above by a constant $\widetilde{C}$ by compactness, while

$$\delta^m |P|_{0,\delta} \geq |P|_{0,1} = 1$$

for any $\delta \geq 1$, according to (2). Hence (ii) holds true as well. In order to prove (iii), it is more convenient to rescale so that $\delta = 1$, rather than $\rho = 1$. We may still assume that $|P|_{0,1} = |Q|_{0,1} = 1$. Consider the unit ball $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and the function $F(x) = P(x)Q(x)$. Yet another compactness argument yields that $\|F\|_{C^{m-1,1}(B)} \leq C_0$ for a constant C_0 determined by m and n. From Taylor's theorem, rendered above as (3),

$$|(P \odot_y Q) - (P \odot_0 Q)|_{0,\rho} = |J_y F - J_0 F|_{0,\rho} \leq C_T \cdot C_0,$$

and the lemma is proven. □

If $|x - y| \leq \lambda\delta$ for some $\lambda \geq 1$, then we have the inequality

$$|P|_{y,\delta} \leq \widetilde{C}\lambda^{m-1}|P|_{x,\delta}, \tag{6}$$

or the equivalent inclusion $\mathcal{B}_{x,\delta} \subset \widetilde{C}\lambda^{m-1}\mathcal{B}_{y,\delta}$. Indeed, this follows from (2) and Lemma 2.1:

$$|P|_{y,\delta} \leq \lambda^m |P|_{y,\lambda\delta} \leq \widetilde{C}\lambda^m |P|_{x,\lambda\delta} \leq \widetilde{C}\lambda^{m-1}|P|_{x,\delta}.$$

Furthermore, if $\theta \in C^{m-1,1}(\mathbb{R}^n)$ is supported on a ball $B \subset \mathbb{R}^n$, then

$$|J_x(\theta)|_{x,\mathrm{diam}(B)} \leq C_T\|\theta\| \quad (x \in \mathbb{R}^n). \tag{7}$$

Indeed, this inequality is trivial if $x \in \mathbb{R}^n \setminus B$, as then $J_x(\theta) = 0$. Fix $x_0 \in \partial B$. Then $J_{x_0}(\theta) = 0$. As $|x - x_0| \leq \mathrm{diam}(B)$ for any $x \in B$, we may apply Taylor's theorem (rendered as (3)) and obtain $|J_x(\theta)|_{x,\mathrm{diam}(B)} = |J_x(\theta) - J_{x_0}(\theta)|_{x,\mathrm{diam}(B)} \leq C_T\|\theta\|$, which yields (7).

We next give a more general form of Lemma 2.1(iii) involving products of up to three polynomials which are allowed to vary from point to point.

Lemma 2.2. *Fix polynomials P_x, Q_x, R_x and P_y, Q_y, R_y in $\mathcal{P}$, for $|x - y| \leq \rho \leq \delta$. Suppose that $P_x, P_y \in M_0\mathcal{B}_{x,\delta}$, $Q_x, Q_y \in M_1\mathcal{B}_{x,\delta}$, and $R_x, R_y \in M_2\mathcal{B}_{x,\delta}$. Also suppose that $P_x - P_y \in M_0\mathcal{B}_{x,\rho}$, $Q_x - Q_y \in M_1\mathcal{B}_{x,\rho}$, and $R_x - R_y \in M_2\mathcal{B}_{x,\rho}$. Then*

$$|P_x \odot_x Q_x \odot_x R_x - P_y \odot_y Q_y \odot_y R_y|_{x,\rho} \leq C\delta^{2m} M_0 M_1 M_2,$$

where C is a constant determined by m and n.

Proof. In view of (5), we may assume that $\delta = 1$. By renormalizing, we may assume $M_0 = M_1 = M_2 = 1$. Then all six polynomials belong to $\mathcal{B}_{x,1}$, and the three differences $P_x - P_y$, $Q_x - Q_y$, and $R_x - R_y$ belong to $\mathcal{B}_{x,\rho}$. The letter x appears five times in the expression $P_x \odot_x Q_x \odot_x R_x$, and we will change these five x's to five y's one by one. We first apply Lemma 2.1(ii) three times and replace R_x, Q_x, and P_x by R_y, Q_y, and P_y, in that respective order, as follows:

$$|P_x \odot_x Q_x \odot_x R_x - P_y \odot_x Q_y \odot_x R_y|_{x,\rho} \leq C.$$

This step also requires the bounds $|P_x \odot_x Q_x|_{x,1} \leq C$, $|P_x \odot_x R_y|_{x,1} \leq C$, and $|Q_y \odot_x R_y|_{x,1} \leq C$, which are all consequences of Lemma 2.1(ii). Next we apply Lemma 2.1(iii) twice, and deduce that

$$|P_y \odot_x Q_y \odot_x R_y - P_y \odot_y Q_y \odot_y R_y|_{x,\rho} \leq C.$$

This step requires the bounds $|P_y \odot_x Q_y|_{x,1} \leq C$ and $|Q_y \odot_y R_y|_{x,1} \leq C$, which follow from Lemma 2.1(ii) and, for the second inequality, also Lemma 2.1(iii). This concludes the proof of the lemma. □

Remark 2.3. *We can obtain a version of Lemma 2.2 also for products of two polynomials. Notice that $1 \in \delta^{-m}\mathcal{B}_{x,\delta}$ for any $\delta > 0$. Thus, by taking $P_x = P_y = 1$, under the hypotheses of Lemma 2.2, $|Q_x \odot_x R_x - Q_y \odot_y R_y|_{x,\rho} \leq C\delta^m M_1 M_2$.*

Finally, we state a few elementary facts from convex geometry. A convex set Ω in a finite-dimensional vector space $\mathcal{V}$ is said to be *symmetric* if $P \in \Omega \implies -P \in \Omega$. If A, K, and T are symmetric convex sets then

$$K \subset T \implies (A + K) \cap T \subset (A \cap 2T) + K, \tag{8}$$

and also if K is bounded then

$$K \subset T + K/3 \implies K \subset 2T. \tag{9}$$

To prove (8), pick $x \in (A + K) \cap T$. Then $x = a + k$ with $a \in A$ and $k \in K$. It suffices to show that $a \in 2T$. This holds since $a = x - k \in T - K \subset 2T$. Next observe that the condition $K \subset T + K/3$ implies $\sup_{x\in K} f(x) \leq \sup_{x\in T} f(x) + \frac{1}{3}\sup_{x\in K} f(x)$ for any linear functional $f : \mathcal{V} \to \mathbb{R}$. If K is bounded, this implies $\frac{2}{3}\sup_{x\in K} f(x) \leq \sup_{x\in T} f(x)$. From the Hahn-Banach theorem, K is contained in the closure of $\frac{3}{2}T$, and therefore $K \subset 2T$.

2.1 Taylor polynomials of functions with prescribed values.

Fix a finite subset $E \subset \mathbb{R}^n$ and a function $f : E \to \mathbb{R}$ satisfying the hypothesis of Theorem 1.2. That is, we assume that for some natural number $k^\# \in \mathbb{N}$, the following holds:

$$\mathcal{FH}(k^\#) \begin{cases} \text{For all } S \subset E \text{ with } \#(S) \leq k^\# \text{ there exists } F^S \in C^{m-1,1}(\mathbb{R}^n) \\ \quad \text{with } F^S = f \text{ on } S \text{ and } \|F^S\| \leq 1. \end{cases} \tag{10}$$

We call $\mathcal{FH}(k^\#)$ the *finiteness hypothesis* and $k^\#$ the *finiteness constant*. We aim to construct a function $F \in C^{m-1,1}(\mathbb{R}^n)$ satisfying $F = f$ on E and $\|F\| \leq C^\#$ for a suitable constant $C^\# \geq 1$. We first introduce a family of convex subsets of $\mathcal{P}$ that contain information on the Taylor polynomials of extensions associated to subsets of E:

$$\Gamma_S(x, f, M) := \{J_x F : F \in C^{m-1,1}(\mathbb{R}^n),\ F = f \text{ on } S,\ \|F\| \leq M\},$$
$$\text{for } S \subset E,\ x \in \mathbb{R}^n,\ f : E \to \mathbb{R}, \text{ and } M > 0.$$

We also denote $\Gamma(x, f, M) := \Gamma_E(x, f, M)$. Notice that $\Gamma_S(x, f, M)$ is nonempty if and only if there exists an extension of the restricted function $f|_S$ with $C^{m-1,1}$ seminorm at most M. Therefore the finiteness hypothesis $\mathcal{FH}(k^\#)$ is equivalent to the condition that $\Gamma_S(x, f, 1) \neq \emptyset$ for all $S \subset E$ with $\#(S) \leq k^\#$. Now, for $\ell \in \mathbb{Z}_{\geq 0}$ we define

$$\Gamma_\ell(x, f, M) := \{P \in \mathcal{P} : \forall S \subset E,\ \#(S) \leq (D+1)^\ell,\ \exists F^S \in C^{m-1,1}(\mathbb{R}^n),$$
$$F^S = f \text{ on } S,\ J_x F^S = P,\ \|F^S\| \leq M\};$$

here, recall that $D = \dim \mathcal{P}$. In other words, an element of $\Gamma_\ell(x, f, M)$ is simultaneously the jet of a solution to any extension problem associated to a subset $S \subset E$ of cardinality at most $(D+1)^\ell$. The sets denoted by $\Gamma_\ell(\cdot, \cdot, \cdot)$ were introduced in [10] as a tool to demonstrate that $\Gamma(x, f, M)$ is nonempty – the latter condition is relevant because it implies, in particular, the existence of an extension of f with $C^{m-1,1}$ seminorm at most M. We note the identity

$$\Gamma_\ell(x, f, M) = \bigcap_{S \subset E,\ \#(S) \leq (D+1)^\ell} \Gamma_S(x, f, M). \tag{11}$$

Given $x \in \mathbb{R}^n$ and $S \subset E$, let

$$\sigma(x, S) := \{J_x \varphi : \varphi \in C^{m-1,1}(\mathbb{R}^n),\ \varphi = 0 \text{ on } S,\ \|\varphi\| \leq 1\},$$

and given $\ell \in \mathbb{Z}_{\geq 0}$, let

$$\sigma_\ell(x) = \bigcap_{S \subset E,\ \#(S) \leq (D+1)^\ell} \sigma(x, S). \tag{12}$$

We also denote $\sigma(x) := \sigma(x, E)$.

Note that $\sigma(x)$ and $\sigma_\ell(x)$ are symmetric convex subsets of $\mathcal{P}$, whereas $\Gamma(x, f, M)$ and $\Gamma_\ell(x, f, M)$ are only convex. By a straightforward application of the Arzela-Ascoli theorem one can show that $\sigma(x)$, $\sigma_\ell(x)$, $\Gamma(x, f, M)$, and $\Gamma_\ell(x, f, M)$ are closed. Finally, we observe the relationships $\sigma(x, S) = \Gamma_S(x, 0, 1)$, $\sigma_\ell(x) = \Gamma_\ell(x, 0, 1)$, and $\sigma(x) = \Gamma(x, 0, 1)$.

Lemma 2.4 (Relationship between Γ_ℓ and σ_ℓ)**.** *For any $\ell \in \mathbb{Z}_{\geq 0}$,*

$$\Gamma_\ell(x, f, M/2) + (M/2)\sigma_\ell(x) \subset \Gamma_\ell(x, f, M), \quad \textit{and}$$
$$\Gamma_\ell(x, f, M) - \Gamma_\ell(x, f, M) \subset 2M\sigma_\ell(x).$$

Proof. By definition we have $\Gamma_S(x, f, M/2) + (M/2)\sigma(x, S) \subset \Gamma_S(x, f, M)$ and $\Gamma_S(x, f, M) - \Gamma_S(x, f, M) \subset 2M\sigma(x, S)$. The conclusion of the lemma then follows from the definition of Γ_ℓ and σ_ℓ in (11) and (12). □

Remark 2.5. *Lemma 2.4 implies that $P_x + \frac{M}{2} \cdot \sigma_\ell(x) \subset \Gamma_\ell(x, f, M) \subset P_x + 2M \cdot \sigma_\ell(x)$, for any $P_x \in \Gamma_\ell(x, f, M/2)$. Later on we will be concerned with the geometry of the set $\Gamma_\ell(x, f, M)$ at various points $x \in \mathbb{R}^n$. Lemma 2.4 implies that it is sufficient to understand the geometry of the set $\sigma_\ell(x)$ (which depends on fewer parameters and is therefore more manageable).*

Recall the translation and scaling transformations T_h and $\tau_{x,\delta}$ on $\mathcal{P}$. With a slight abuse of notation, we also denote the transformations T_h and $\tau_{x,\delta}$ on $\mathbb{R}^n$ given by

$$T_h(y) = y + h, \quad \tau_{x,\delta}(y) = x + \delta \cdot (y - x) \qquad (x, y, h \in \mathbb{R}^n, \delta > 0).$$

Then,

$$\sigma(T_h(y), T_h(S)) = T_h \{\sigma(y, S)\}, \text{ and } \sigma(\tau_{x,\delta}(y), \tau_{x,\delta}(S)) = \tau_{x,\delta} \{\sigma(y, S)\}, \tag{13}$$

for any $x, y, h \in \mathbb{R}^n$, $\delta > 0$, and $S \subset \mathbb{R}^n$, as may be verified directly. Here in our notation, if $T : \mathbb{R}^n \to \mathbb{R}^n$ then $T(S) = \{T(y) : y \in S\}$.

In the next lemma we establish two important properties of the sets $\Gamma_\ell(x, f, M)$. We show that the finiteness hypothesis $\mathcal{FH}(k^\#)$ (see (10)) implies that $\Gamma_\ell(x, f, M)$ is non-empty if ℓ and $k^\#$ are suitably related and if $M \geq 1$. We also show that the mappings $x \mapsto \Gamma_\ell(x, f, M)$ are "quasicontinuous" in a sense to be made precise below.

Lemma 2.6. *If $x \in \mathbb{R}^n$, $(D + 1)^{\ell+1} \leq k^\#$, and $M \geq 1$, then*

$$\mathcal{FH}(k^\#) \implies \Gamma_\ell(x, f, M) \neq \emptyset. \tag{14}$$

If $x, y \in \mathbb{R}^n$, $\ell \geq 1$, $\delta \geq |x - y|$, and $M > 0$, then

$$\Gamma_\ell(x, f, M) \subset \Gamma_{\ell-1}(x, f, M) + C_T M \cdot \mathcal{B}_{x,\delta} \tag{15}$$

and

$$\sigma_\ell(x) \subset \sigma_{\ell-1}(x) + C_T \cdot \mathcal{B}_{x,\delta}, \tag{16}$$

where C_T is the constant in (3)*.*

Proof. We first show that the finiteness hypothesis with constant $k^\# \geq (D + 1)^{\ell+1}$ implies the intersection of the sets in (11) is nonempty for $M = 1$. As $\Gamma(x, f, M) \supset \Gamma(x, f, 1)$ for $M \geq 1$, the implication (14) will then follow. By Helly's theorem and the fact that $\dim \mathcal{P} = D$, it suffices to show that the intersection of any $(D + 1)$-element subcollection is nonempty. Fix $S_1, \cdots, S_{D+1} \subset E$ with $\#(S_i) \le (D+1)^\ell$. Let $S := S_1 \cup \cdots \cup S_{D+1}$. Note that $\Gamma_{S_1}(x, f, 1) \cap \cdots \cap \Gamma_{S_{D+1}}(x, f, 1) \supset \Gamma_S(x, f, 1)$. Furthermore, $\#(S) \le (D+1) \cdot (D+1)^\ell \le k^\#$, and so $\Gamma_S(x, f, 1) \neq \emptyset$ by the finiteness hypothesis $\mathcal{FH}(k^\#)$. This finishes the proof of (14).

To prove (15) and (16) we reproduce the proof of [10, Lemma 10.2]. Note (16) is a special case of (15), as $\sigma_\ell(x) = \Gamma_\ell(x, 0, 1)$. So it suffices to prove (15). Given $P \in \Gamma_\ell(x, f, M)$, we will find $Q \in \Gamma_{\ell-1}(y, f, M)$ with

$$|P - Q|_{x,\delta} \le C_T M. \tag{17}$$

For a subset $S \subset E$, consider

$$\mathcal{K}(S) := \left\{ J_y F : F \in C^{m-1,1}(\mathbb{R}^n),\ F = f \text{ on } S,\ \|F\| \le M,\ J_x F = P \right\}.$$

Then $\mathcal{K}(S) \subset \mathcal{P}$ is convex, and according to (3),

$$\mathcal{K}(S) \subset P + C_T M \cdot \mathcal{B}_{x,\delta}. \tag{18}$$

Note that $\mathcal{K}(S) \neq \emptyset$ whenever $\#(S) \le (D+1)^\ell$, due to the fact that $P \in \Gamma_\ell(x, f, M)$. We will show that

$$\emptyset \neq \bigcap_{\substack{S \subset E \\ \#(S) \le (D+1)^{\ell-1}}} \mathcal{K}(S) \subset \Gamma_{\ell-1}(y, f, M). \tag{19}$$

The inclusion on the right-hand side of (19) is immediate from the definition of $\Gamma_{\ell-1}(y, f, M)$. All that remains is to show that the intersection of the collection of sets in (19) is nonempty. By Helly's theorem it suffices to show that the intersection of any $(D+1)$-element subcollection is nonempty. Thus, pick $S_1, \ldots, S_{D+1} \subset E$ with $\#(S_i) \le (D+1)^{\ell-1}$. Then $S = S_1 \cup \ldots \cup S_{D+1}$ is of cardinality at most $(D+1)(D+1)^{\ell-1} = (D+1)^\ell$, and thus $\mathcal{K}(S) \neq \emptyset$. Clearly, $\mathcal{K}(S) \subset \mathcal{K}(S_1) \cap \cdots \cap \mathcal{K}(S_{D+1})$. This finishes the proof of (19). Fix a polynomial Q belonging to the intersection in (19). According to (19), $Q \in \Gamma_{\ell-1}(y, f, M)$. By (18), $Q \in \mathcal{K}(\emptyset) \subset P + C_T M \cdot \mathcal{B}_{x,\delta}$, and so $Q - P \in C_T M \cdot \mathcal{B}_{x,\delta}$, giving (17). ☐

Lemma 2.7. *If $x, y \in \mathbb{R}^n$, and $\delta \ge |x - y|$, then $\sigma(x) \subset \sigma(y) + C_T \cdot \mathcal{B}_{x,\delta}$.*

Proof. Let $P \in \sigma(x)$. Then there exists $\varphi \in C^{m-1,1}(\mathbb{R}^n)$ with $\varphi = 0$ on E, $\|\varphi\| \le 1$, and $J_x \varphi = P$. Let $Q = J_y \varphi$. Then $Q \in \sigma(y)$, and by (3) we have $P - Q \in C_T \cdot \mathcal{B}_{x,\delta}$. ☐

Remark 2.8. *By (2), $\mathcal{B}_{x,\delta} \subset \delta \cdot \mathcal{B}_{x,1}$ for $\delta \le 1$. Therefore, Lemma 2.7 implies the mapping $x \mapsto \sigma(x)$ is continuous, where the space of subsets of $\mathcal{P}$ carries the topology induced by the Hausdorff metric with respect to any of the topologically equivalent scaled norms.*

Lemma 2.9. *There exists a constant $C \ge 1$ determined by m and n so that, for any ball $B \subset \mathbb{R}^n$ and $z \in \frac{1}{2}B$, we have*

$$\sigma(z, E \cap B) \cap \mathcal{B}_{z,\mathrm{diam}(B)} \subset C \cdot \sigma(z, E).$$

Proof. Choose a cutoff function $\theta \in C^{m-1,1}(\mathbb{R}^n)$ which is supported on B, equal to 1 on $(\frac{1}{2})B$, and satisfies $\|\theta\| \le C \cdot \delta^{-m}$. Fix $z \in (\frac{1}{2})B$ and a polynomial $P \in \sigma(z, E \cap B) \cap \mathcal{B}_{z,\delta}$.

Since $P \in \sigma(z, E \cap B)$ there exists $\varphi \in C^{m-1,1}(\mathbb{R}^n)$ with $\varphi = 0$ on $E \cap B$, $\|\varphi\| \le 1$, and $J_z(\varphi) = P$. Define $\widetilde{\varphi} = \varphi\theta$. This function clearly vanishes on all of E. Since z belongs to the ball $(\frac{1}{2})B$ on which θ is identically 1, we have $J_z(\widetilde{\varphi}) = J_z(\varphi) = P$. To prove $P \in C\sigma(z, E)$, all that remains is to establish the seminorm bound $\|\widetilde{\varphi}\| \le C$. As $\widetilde{\varphi}$ vanishes on $\mathbb{R}^n \setminus B$, it suffices to prove $\|\widetilde{\varphi}\|_{C^{m-1,1}(B)} \le C$. To do so, we will prove that

$$|J_x(\widetilde{\varphi}) - J_y(\widetilde{\varphi})|_{x,\rho} = |J_x(\varphi) \odot_x J_x(\theta) - J_y(\varphi) \odot_y J_y(\theta)|_{x,\rho} \le C \quad \text{for } x, y \in B,\ \rho = |x-y|. \tag{20}$$

To prove this estimate we will apply Lemma 2.2. According to (7), $J_x(\theta) \in C\delta^{-m}\mathcal{B}_{x,\delta}$. On the other hand, by (6) and the fact $|x - y| \le \delta$, also $J_y(\theta) \in C\delta^{-m}\mathcal{B}_{y,\delta} \subset C'\delta^{-m}\mathcal{B}_{x,\delta}$. By Taylor's theorem (in the form (3)), $J_x(\theta) - J_y(\theta) \in C\|\theta\|\mathcal{B}_{x,\rho} \subset C\delta^{-m}\mathcal{B}_{x,\rho}$.

Note that $|x - z| \le \delta$, since $x \in B$ and $z \in (\frac{1}{2})B$. Thus, by Taylor's theorem (see (3)) and (6), $J_x(\varphi) = (J_x(\varphi) - J_z(\varphi)) + P \in C_T\mathcal{B}_{x,\delta} + \mathcal{B}_{z,\delta} \subset C_T\mathcal{B}_{x,\delta} + \widetilde{C}\mathcal{B}_{x,\delta} \subset C\mathcal{B}_{x,\delta}$. On the other hand, by Taylor's theorem, $J_x(\varphi) - J_y(\varphi) \in C_T\mathcal{B}_{x,\rho}$. We are therefore in a position to apply Lemma 2.2 (see Remark 2.3), with Q_x, Q_y, R_x, and R_y picked to be the jets at x and y of φ and θ, respectively. This finishes the proof of (20). □

2.2 Whitney convexity

The next definition illustrates an additional important property of the sets $\sigma_\ell(x)$ beyond convexity.

Definition 2.10 (Whitney convexity)**.** *Given a symmetric convex set Ω in $\mathcal{P}$, and $x \in \mathbb{R}^n$, the Whitney coefficient of Ω at x is the infimum over all $R > 0$ such that $(\Omega \cap \mathcal{B}_{x,\delta}) \odot_x \mathcal{B}_{x,\delta} \subset R\delta^m\Omega$ for all $\delta > 0$. Denote the Whitney coefficient of Ω at x by $w_x(\Omega)$. If no finite R exists, then $w_x(\Omega) = +\infty$. If $w_x(\Omega) < +\infty$ then we say that Ω is Whitney convex at x.*

The term "Whitney convexity" was coined by Fefferman [11]. It is a quantitative analogue of the concept of an ideal; roughly, a small Whitney coefficient means that Ω is "close" to an ideal. For example, any ideal I in $\mathcal{P}_x$ is Whitney convex at x with $w_x(I) = 0$.

For $x \in \mathbb{R}^n$, a symmetric convex set $\Omega \subset \mathcal{P}$ and $r \ge 1$, it holds that $w_x(r\Omega) \le w_x(\Omega)$. One can also check that $w_x(\Omega_1 \cap \Omega_2) \le \max\{w_x(\Omega_1), w_x(\Omega_2)\}$. Furthermore, it follows from (5) that $w_x(\Omega) = w_x(\tau_{x,\delta}(\Omega))$ and $w_x(\Omega) = w_{x+h}(T_h\Omega)$ for any $\delta > 0$.

Lemma 2.11. *For any $z \in \mathbb{R}^n$, the sets $\sigma_\ell(z)$ and $\sigma(z)$ are Whitney convex at z with Whitney coefficient at most C_0, for a universal constant $C_0 = C_0(m, n)$.*

Proof. Note that $w_x(\sigma_\ell(z)) \le \max\{w_z(\sigma(x, S)) : S \subset E,\ \#(S) \le (D + 1)^\ell\}$. Hence, it will be sufficient to show that the Whitney coefficient of $\sigma(z, S)$ at z is at most C for any subset $S \subset E$ and $z \in \mathbb{R}^n$, where C is a constant determined by m and n. Fix $\delta > 0$, and let $P \in \sigma(z, S) \cap \mathcal{B}_{z,\delta}$ and $\widetilde{P} \in \mathcal{B}_{z,\delta}$. In order to prove the lemma, we need to show that

$$P \odot_z \widetilde{P} \in C\delta^m\sigma(z, S). \tag{21}$$

Since $P \in \sigma(z, S)$, there exists $\varphi \in C^{m-1,1}(\mathbb{R}^n)$ with $\varphi = 0$ on S, $J_z(\varphi) = P$, and $\|\varphi\| \leq 1$. Fix a C^∞-function $\theta : \mathbb{R}^n \to \mathbb{R}$, which is supported on the ball $B = \{y \in \mathbb{R}^n : |y - z| \leq \frac{\delta}{2}\}$, which equals one in a neighborhood of z, and satisfies $\|\theta\| \leq C\delta^{-m}$ for a constant C determined by m and n. Since $J_z(\theta) = 1$ and $J_z(\varphi) = P$, we conclude that $J_z(\theta \widetilde{P} \varphi) = 1 \odot_z \widetilde{P} \odot_z P = \widetilde{P} \odot_z P$. In order to establish (21) and conclude the proof of the lemma, it therefore suffices to show $J_z(\theta \widetilde{P} \varphi) \in C\delta^m \sigma(z, S)$. Since the function $\theta \widetilde{P} \varphi$ vanishes on S (as does φ), all that remains is to establish the seminorm bound $\|\theta \widetilde{P} \varphi\| \leq C\delta^m$, and as this function vanishes on $\mathbb{R}^n \setminus B$, it suffices to establish $\|\theta \widetilde{P} \varphi\|_{C^{m-1,1}(B)} \leq C\delta^m$. To that end, we need to show that

$$\left| J_x(\theta) \odot_x \widetilde{P} \odot_x J_x(\varphi) - J_y(\theta) \odot_y \widetilde{P} \odot_y J_y(\varphi) \right|_{x,\rho} \leq C\delta^m, \quad \text{for } x, y \in B,\ \rho = |x - y|. \quad (22)$$

We prepare to apply Lemma 2.2 to prove this estimate.

As in the proof of Lemma 2.9 (using that $J_z(\varphi) = P \in \mathcal{B}_{z,\delta}$ and $\operatorname{diam}(\{x, y, z\}) \leq \delta = \operatorname{diam}(B)$), and by (7), the jets $J_x(\varphi)$, $J_y(\varphi)$ belong to $C\mathcal{B}_{x,\delta}$; and $J_x(\theta)$, $J_y(\theta)$ belong to $C\delta^{-m}\mathcal{B}_{x,\delta}$. Furthermore, $\widetilde{P} \in \mathcal{B}_{z,\delta}$, and hence by (6), $\widetilde{P} \in \widetilde{C}\mathcal{B}_{x,\delta}$. Finally, by Taylor's theorem (rendered as (3)), $J_x(\varphi) - J_y(\varphi) \in C\mathcal{B}_{x,\rho}$ and $J_x(\theta) - J_y(\theta) \in C\delta^{-m}\mathcal{B}_{x,\rho}$.

We are in a position to apply Lemma 2.2, with P_x, P_y, R_x, and R_y picked to be the jets at x and y of φ and θ, respectively, and with $Q_x = Q_y = \widetilde{P}$. This finishes the proof of the estimate (22), and with it the proof of (21). □

Lemma 2.12. *If Ω is Whitney convex at x, then* $\operatorname{span}(\Omega)$ *is an $\odot_x$-ideal in $\mathcal{P}_x$.*

Proof. Choose any $R \in (w_x(\Omega), \infty)$. Then $(\Omega \cap \mathcal{B}_{x,\delta}) \odot_x \mathcal{B}_{x,\delta} \subset R\delta^m\Omega$ for all $\delta > 0$, and so

$$\Omega \odot_x \mathcal{P}_x = \bigcup_{\delta > 0} (\Omega \cap \mathcal{B}_{x,\delta}) \odot_x \mathcal{B}_{x,\delta} \subset \bigcup_{\delta > 0} R\delta^m \Omega = \operatorname{span}(\Omega).$$

Thus, $\operatorname{span}(\Omega) \odot_x \mathcal{P}_x = \bigcup_{r>0} r \cdot \Omega \odot_x \mathcal{P}_x \subset \operatorname{span}(\Omega)$, and hence $\operatorname{span}(\Omega)$ is an $\odot_x$-ideal. □

2.3 Covering lemmas

This section contains the covering lemmas that will be used later in the paper. Given a ball $B \subset \mathbb{R}^n$ and $\lambda > 0$, we write λB to denote the ball with the same center as B and with radius equal to λ times the radius of B.

2.3.1 Whitney covers

Definition 2.13. *A finite collection of closed balls $\mathcal{W}$ is a* Whitney cover *of a ball $\widehat{B} \subset \mathbb{R}^n$ if (a) $\mathcal{W}$ is a cover of $\widehat{B}$, (b) the collection of third-dilates $\{\frac{1}{3}B : B \in \mathcal{W}\}$ is pairwise disjoint, and (c)* $\operatorname{diam}(B_1)/\operatorname{diam}(B_2) \in [1/8, 8]$ *for all balls $B_1, B_2 \in \mathcal{W}$ with $\frac{6}{5}B_1 \cap \frac{6}{5}B_2 \neq \emptyset$.*

Lemma 2.14 (Bounded overlap)**.** *If $\mathcal{W}$ is Whitney cover of $\widehat{B}$ then $\#\{B \in \mathcal{W} : x \in \frac{6}{5}B\} \leq 100^n$ for all $x \in \mathbb{R}^n$.*

Proof. We may assume $\mathcal{W}_x := \{B \in \mathcal{W} : x \in \frac{6}{5}B\}$ is nonempty, and fix $B_0 \in \mathcal{W}_x$ of maximal radius. By rescaling, we may assume $\operatorname{diam}(B_0) = 1$. If $B \in \mathcal{W}_x$ then $\frac{6}{5}B \cap \frac{6}{5}B_0 \neq \emptyset$, and so condition (c) of Definition 2.13 implies that $\operatorname{diam}(B) \in [\frac{1}{8}, 1]$; thus, by the triangle inequality, $\frac{1}{3}B \subset (\frac{12}{5} + \frac{1}{3})B_0 = \frac{41}{15}B_0$ for all $B \in \mathcal{W}_x$. Since the collection $\{\frac{1}{3}B\}_{B\in\mathcal{W}}$ is pairwise disjoint, a volume comparison shows that $\#\mathcal{W}_x \leq (24 \cdot \frac{41}{15})^n \leq 100^n$. □

2.3.2 Partitions of unity

Lemma 2.15 (Existence of partitions of unity)**.** *If $\mathcal{W}$ is a Whitney cover of $\widehat{B} \subset \mathbb{R}^n$, then there exist non-negative C^∞ functions $\theta_B : \widehat{B} \to [0,\infty)$ $(B \in \mathcal{W})$ such that*

1. *$\theta_B = 0$ on $\widehat{B} \setminus \frac{6}{5}B$.*
2. *$|\partial^\alpha \theta_B(x)| \leq C \operatorname{diam}(B)^{-|\alpha|}$ for all $|\alpha| \leq m$ and $x \in \widehat{B}$.*
3. *$\sum_{B\in\mathcal{W}} \theta_B = 1$ on $\widehat{B}$.*

Here, C is a constant determined by m and n.

Proof. For each $B \in \mathcal{W}$, fix a C^∞ cutoff function $\psi_B : \mathbb{R}^n \to \mathbb{R}$ which is supported on $\frac{6}{5}B$, equals 1 on B, and satisfies the natural derivative bounds $|\partial^\alpha \psi_B(x)| \leq C \operatorname{diam}(B)^{-|\alpha|}$ for $x \in \mathbb{R}^n$, $|\alpha| \leq m$. Set $\Psi = \sum_{B\in\mathcal{W}} \psi_B$ and define

$$\theta_B(x) := \psi_B(x)/\Psi(x), \quad x \in \widehat{B}.$$

Each point in $\widehat{B}$ belongs to some $B \in \mathcal{W}$, thus $\Psi \geq 1$ on $\widehat{B}$. Thus $\theta_B \in C^\infty(\widehat{B})$ is well-defined. Property 1 follows because ψ_B is supported on $\frac{6}{5}B$. Furthermore, $\sum_B \theta_B = \sum_B \psi_B/\Psi = 1$ on $\widehat{B}$, yielding property 3.

Property 2 is trivial for $x \in \widehat{B} \setminus \frac{6}{5}B$, as then $J_x(\theta_B) = 0$. Now fix $x \in \frac{6}{5}B \cap \widehat{B}$. If $\psi_{B'}(x) \neq 0$ then $x \in \frac{6}{5}B'$. In particular, $\frac{6}{5}B \cap \frac{6}{5}B' \neq \emptyset$, and hence $\operatorname{diam}(B')/\operatorname{diam}(B) \in [\frac{1}{8}, 8]$. Furthermore, by Lemma 2.14, the cardinality of $\mathcal{W}_x := \{B' : x \in \frac{6}{5}B'\}$ is at most 100^n. Hence,

$$|\partial^\alpha \Psi(x)| \leq \sum_{B'\in\mathcal{W}_x} |\partial^\alpha \psi_{B'}(x)| \leq \sum_{B'\in\mathcal{W}_x} C \operatorname{diam}(B')^{-|\alpha|} \leq C' \operatorname{diam}(B)^{-|\alpha|}. \tag{23}$$

By a repeated application of the quotient rule for differentiation, and substituting the bounds (23) and $|\partial^\alpha \psi_B(x)| \leq C \operatorname{diam}(B)^{-|\alpha|}$, we conclude that $|\partial^\alpha \theta_B(x)| = |\partial^\alpha(\psi_B/\Psi)(x)| \leq C'' \operatorname{diam}(B)^{-|\alpha|}$. □

We mention a few additional properties of the partition of unity $\{\theta_B\}$ in Lemma 2.15. First, by property 2 of Lemma 2.15 and the definition of the scaled norm $|\cdot|_{x,\delta}$,

$$|J_x(\theta_B)|_{x,\operatorname{diam}(B)} \leq C \operatorname{diam}(B)^{-m} \quad (x \in \widehat{B}). \tag{24}$$

By the equivalence of $C^{m-1,1}(\widehat{B})$ and the homogeneous Sobolev space $\dot{W}^{m,\infty}(\widehat{B})$ and by property 2 of Lemma 2.15,

$$\|\theta_B\|_{C^{m-1,1}(\widehat{B})} \leq C \max_{|\alpha|=m} \|\partial^\alpha \theta_B\|_{L^\infty(\widehat{B})} \leq C \operatorname{diam}(B)^{-m}. \tag{25}$$

Lemma 2.16 (Gluing lemma)**.** *Fix a Whitney cover $\mathcal{W}$ of $\widehat{B}$, a partition of unity $\{\theta_B\}_{B\in\mathcal{W}}$ as in Lemma 2.15, and points $x_B \in \frac{6}{5}B$ for each $B \in \mathcal{W}$. Suppose $\{F_B\}_{B\in\mathcal{W}}$ is a collection of functions in $C^{m-1,1}(\mathbb{R}^n)$ with the following properties:*

- $\|F_B\| \leq M_0$.
- $F_B = f$ *on* $E \cap \frac{6}{5}B$.
- $|J_{x_B}F_B - J_{x_{B'}}F_{B'}|_{x_B,\operatorname{diam}(B)} \leq M_0$ *whenever* $\frac{6}{5}B \cap \frac{6}{5}B' \neq \emptyset$.

Let $F = \sum_{B\in\mathcal{W}} \theta_B F_B$. Then $F \in C^{m-1,1}(\widehat{B})$ with $F = f$ on $E \cap \widehat{B}$ and $\|F\|_{C^{m-1,1}(\widehat{B})} \leq CM_0$, where C is a constant determined by m and n.

Proof. The nonzero terms in the sum $F(x) = \sum_B \theta_B(x)F_B(x)$, $x \in E \cap \widehat{B}$, occur when $x \in \frac{6}{5}B$. By assumption, $F_B(x) = f(x)$ for such B. Thus $F(x) = \sum_B \theta_B(x)f(x) = f(x)$. Therefore, $F = f$ on $E \cap \widehat{B}$.

We will now bound the seminorm of F. Recall the following well-known characterization: $F \in C^{m-1,1}(\widehat{B})$ if and only if there exists $\epsilon > 0$ and $M \geq 0$ such that for any $x, y \in \widehat{B}$ with $|x-y| \leq \epsilon$ and any multiindex α with $|\alpha| = m-1$, we have $|\partial^\alpha F(x) - \partial^\alpha F(y)| \leq M \cdot |x-y|$. Furthermore, the seminorm $\|F\|_{C^{m-1,1}(\widehat{B})}$ is comparable to the least possible M up to constant factors depending on m and n. This characterization is an easy consequence of the triangle inequality on $\mathbb{R}^n$, and the proof is left as an exercise for the reader. Thus, it suffices to prove that if $|x-y| \leq \frac{1}{100}\delta_{\min}$ for $\delta_{\min} := \min_{B\in\mathcal{W}} \operatorname{diam}(B)$, then

$$|J_x(F) - J_y(F)|_{x,\rho} \leq CM_0, \text{ for } \rho := |x-y|. \tag{26}$$

Fix an arbitrary ball $B_0 \in \mathcal{W}$ with $x \in B_0$. Since $|x-y| \leq \frac{1}{100}\operatorname{diam}(B_0)$, we have that both x and y belong to $\frac{6}{5}B_0$. Note that $\sum_B J_x(\theta_B) = \sum_B J_y(\theta_B) = 1$. This lets us write

$$J_x(F) - J_y(F) = \sum_{B\in\mathcal{W}} \Big[(J_x(F_B) - J_x(F_{B_0})) \odot_x J_x(\theta_B) - (J_y(F_B) - J_y(F_{B_0})) \odot_y J_y(\theta_B) \Big] + (J_x(F_{B_0}) - J_y(F_{B_0})).$$

The summands in the main sum on the right-hand side are nonzero only if $x \in \frac{6}{5}B$ or $y \in \frac{6}{5}B$. By Lemma 2.14, there can be at most $2 \cdot 100^n$ many elements $B \in \mathcal{W}$ with this property. Therefore, to prove inequality (26) it suffices to show that the $|\cdot|_{x,\rho}$ norm of each summand on the right-hand side is at most CM_0. To start, consider the last term and apply Taylor's theorem (in the form (3)):

$$|J_x(F_{B_0}) - J_y(F_{B_0})|_{x,\rho} \leq C_T \|F_{B_0}\| \leq CM_0.$$

Next we select a summand in the main sum by fixing an element $B \in \mathcal{W}$ with either $x \in \frac{6}{5}B$ or $y \in \frac{6}{5}B$. In either case, $\frac{6}{5}B \cap \frac{6}{5}B_0 \neq \emptyset$. Let $\delta := \operatorname{diam}(B)$. By condition (c) in the definition of a Whitney cover (see Definition 2.13), we have $\delta / \operatorname{diam}(B_0) \in [\frac{1}{8}, 8]$. Define four polynomials $P_x = J_x(F_B) - J_x(F_{B_0})$ and $R_x = J_x(\theta_B)$, and similarly $P_y = J_y(F_B) - J_y(F_{B_0})$ and $R_y = J_y(\theta_B)$. We will be finished once we show that

$$|P_x \odot_x R_x - P_y \odot_y R_y|_{x,\rho} \leq C M_0. \tag{27}$$

We will prove (27) using Lemma 2.2 (specifically, the form in Remark 2.3). Let us verify that the hypotheses of this lemma are satisfied. Using $|x-y| = \rho$ and Taylor's theorem (see (3)),

$$|P_x - P_y|_{x,\rho} \leq |J_x(F_B) - J_y(F_B)|_{x,\rho} + |J_x(F_{B_0}) - J_y(F_{B_0})|_{x,\rho} \leq C_T \cdot (\|F_B\| + \|F_{B_0}\|) \leq C M_0. \tag{28}$$

Next write $|P_x|_{x,\delta} \leq |P_{x_{B_0}} - P_x|_{x,\delta} + |P_{x_{B_0}}|_{x,\delta}$. As $x \in B_0$ and $x_{B_0} \in \frac{6}{5}B_0$, we have $|x - x_{B_0}| \leq \frac{6}{5}\operatorname{diam}(B_0) \leq 3\delta$. Thus, by (2) and following the proof of (28), $|P_{x_{B_0}} - P_x|_{x,\delta} \leq 3^m |P_{x_{B_0}} - P_x|_{x,3\delta} \leq C' M_0$. Then by (2) and (6), the hypothesis in the third bullet point of this lemma, and another application of Taylor's theorem,

$$\begin{aligned} |P_{x_{B_0}}|_{x,\delta} &\leq |J_{x_B}(F_B) - J_{x_{B_0}}(F_{B_0})|_{x,\delta} + |J_{x_B}(F_B) - J_{x_{B_0}}(F_B)|_{x,\delta} \\ &\leq C|J_{x_B}(F_B) - J_{x_{B_0}}(F_{B_0})|_{x_{B_0},\delta} + C|J_{x_B}(F_B) - J_{x_{B_0}}(F_B)|_{x_{B_0},\delta} \\ &\leq C'|J_{x_B}(F_B) - J_{x_{B_0}}(F_{B_0})|_{x_{B_0},\operatorname{diam}(B_0)} + C'|J_{x_B}(F_B) - J_{x_{B_0}}(F_B)|_{x_{B_0},4\delta} \leq C'' M_0. \end{aligned}$$

Here, note we are using that $|x_B - x_{B_0}| \leq \frac{6}{5}\operatorname{diam}(B) + \frac{6}{5}\operatorname{diam}(B_0) \leq 4\delta$ in the final application of Taylor's theorem. In conclusion, $|P_x|_{x,\delta} \leq C M_0$. By the identical argument, $|P_y|_{y,\delta} \leq C M_0$ – then by (6), $|P_y|_{x,\delta} \leq C' M_0$.

Next, note the estimate $|R_x - R_y|_{x,\rho} \leq C\delta^{-m}$ is a direct consequence of Taylor's theorem and (25). Also, $|R_x|_{x,\delta} \leq C\delta^{-m}$ is a direct consequence of (24). Similarly, $|R_y|_{y,\delta} \leq C\delta^{-m}$, and thus by (6), $|R_y|_{x,\delta} \leq C'\delta^{-m}$.

We obtain (27) by an application of Lemma 2.2 (see Remark 2.3), which finishes the proof of the lemma. □

3 Transversality

Let $(X, \langle\cdot,\cdot\rangle)$ be a real Hilbert space of finite dimension $d := \dim X < \infty$. We denote the norm of X by $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$, and let $\mathcal{B}$ be the unit ball of X. Let $\mathcal{S}$ denote the set of symmetric, closed, convex subsets of X, and let $d_H : \mathcal{S} \times \mathcal{S} \to [0,\infty]$ be the Hausdorff metric, namely,

$$d_H(\Omega_1, \Omega_2) := \inf\{\epsilon > 0 : \Omega_1 \subset \Omega_2 + \epsilon\mathcal{B},\ \Omega_2 \subset \Omega_1 + \epsilon\mathcal{B}\}.$$

Given a set $A \subset X$ and subspace $V \subset X$, let A/V (the *quotient* of A by V) be the image of A under the quotient mapping $\pi : X \to X/V$, i.e., $A/V := \{a + V : a \in A\}$.

Definition 3.1. *Let V be a linear subspace of X, let $\Omega \in \mathcal{S}$, and let $R \geq 1$. We say that Ω is R-transverse to V if (1) $\mathcal{B}/V \subset R \cdot (\Omega \cap \mathcal{B})/V$, and (2) $\Omega \cap V \subset R \cdot \mathcal{B}$.*

Remark 3.2. *Transversality captures the idea that there is a uniform lower bound on the angle between the subspace V and the "large" vectors of Ω. If Ω is an ellipsoid in X, it is equivalent (modulo multiplicative factors in the constants) to say that the principle axes of Ω of length at least R make an angle of at least $\frac{1}{R}$ with V; furthermore, Ω will be 1-transverse to the subspace V spanned by the principle axes of Ω of length at most 1. By approximation with John ellipsoids, this shows that every symmetric, closed, convex set $\Omega \subset X$ is $\sqrt{d}$-transverse to some subspace V.*

Lemma 3.3 (Stability I)**.** *If Ω is R-transverse to V, then $\Omega + \lambda\mathcal{B}$ is $(R + 3R^2\lambda)$-transverse to V for any $\lambda > 0$.*

Proof. Note that $\mathcal{B}/V \subset R\cdot(\Omega\cap\mathcal{B})/V \subset R\cdot((\Omega + \lambda\mathcal{B}) \cap \mathcal{B})/V$. All that remains is to show

$$(\Omega + \lambda\mathcal{B}) \cap V \subset (R + 3R^2\lambda)\mathcal{B}.$$

Fix $P \in (\Omega + \lambda\mathcal{B}) \cap V$. Write $P = P_0 + P_1$ with $P_0 \in \Omega$ and $P_1 \in \lambda\mathcal{B}$. By the transversality of Ω and V, we have $\lambda\mathcal{B}/V \subset R\lambda(\Omega \cap \mathcal{B})/V$. Since $P_1 \in \lambda\mathcal{B}$, there exists a polynomial $P_2 \in R\lambda(\Omega\cap\mathcal{B})$ with $P_1/V = P_2/V$ – or rather, $P_1 - P_2 \in V$. Define $\tilde{P} := P - (P_1 - P_2) \in V$. We write $\tilde{P} = P_0 + P_2$, where $P_0 \in \Omega$ and $P_2 \in R\lambda \cdot \Omega$, and thus $\tilde{P} \in (R\lambda + 1) \cdot (\Omega \cap V) \subset (R\lambda + 1) \cdot R\mathcal{B}$, where the second containment is by transversality of Ω and V. Therefore,

$$P = \tilde{P} + P_1 - P_2 \in (R^2\lambda + R)\mathcal{B} + \lambda\mathcal{B} + R\lambda\mathcal{B} \subset (R^2\lambda + R + \lambda + R\lambda)\mathcal{B}.$$

We conclude that $P \in (R + 3R^2\lambda)\mathcal{B}$, which completes the proof of the lemma.

□

Lemma 3.4 (Stability II)**.** *Let $\Omega_1, \Omega_2 \in \mathcal{S}$, and let $R \geq 1$. If Ω_1 is R-transverse to V, then the following holds:*

- *If $d_H(\Omega_1, \Omega_2) \leq \frac{1}{4R}$ then Ω_2 is $4R$-transverse to V.*
- *If $d_H(\Omega_1 \cap \tilde{R}\mathcal{B}, \Omega_2 \cap \tilde{R}\mathcal{B}) \leq \frac{1}{4R}$ for any $\tilde{R} \geq 4R$, then Ω_2 is $4R$-transverse to V.*

Proof. For the proof of the first bullet point, we may suppose $\Omega_1 \subset \Omega_2 + \lambda\mathcal{B}$ and $\Omega_2 \subset \Omega_1 + \lambda\mathcal{B}$ for $\lambda = \frac{1}{3R}$. According to Lemma 3.3, $\Omega_1 + \lambda\mathcal{B}$ is $2R$-transverse to V. Thus,

$$\Omega_2 \cap V \subset (\Omega_1 + \lambda\mathcal{B}) \cap V \subset 2R \cdot \mathcal{B}. \tag{29}$$

Also,

$$\mathcal{B}/V \subset R \cdot (\Omega_1 \cap \mathcal{B})/V \subset R \cdot ((\Omega_2 + \lambda\mathcal{B}) \cap \mathcal{B})/V.$$

By (8), $(\Omega_2 + \lambda\mathcal{B}) \cap \mathcal{B} \subset (\Omega_2 \cap 2\mathcal{B}) + \lambda\mathcal{B}$, hence,

$$\mathcal{B}/V \subset R \cdot (\Omega_2 \cap 2\mathcal{B} + \lambda\mathcal{B})/V = R \cdot (\Omega_2 \cap 2\mathcal{B})/V + R\lambda \cdot \mathcal{B}/V.$$

Recall $R\lambda = \frac{1}{3}$, hence $K \subset T + K/3$ for $K = \mathcal{B}/V$ and $T = R \cdot (\Omega_2 \cap 2\mathcal{B})/V$. From (9) we conclude that $K \subset 2T$, i.e.,

$$\mathcal{B}/V \subset 2R \cdot (\Omega_2 \cap 2\mathcal{B})/V \subset 4R \cdot (\Omega_2 \cap \mathcal{B})/V. \tag{30}$$

From (29) and (30) we conclude that Ω_2 is $4R$-transverse to V.

Note Ω_1 is R-transverse to V iff $\Omega_1 \cap \tilde{R}\mathcal{B}$ is R-transverse to V (since $\tilde{R} \geq R$), and similarly, Ω_2 is $4R$-transverse to V iff $\Omega_2 \cap \tilde{R}\mathcal{B}$ is $4R$-transverse to V (since $\tilde{R} \geq 4R$). Thus, by applying the first bullet point to the sets $\Omega_1 \cap \tilde{R}\mathcal{B}$ and $\Omega_2 \cap \tilde{R}\mathcal{B}$, we obtain the conclusion in the second bullet point. □

Lemma 3.5 (Stability III)**.** *Suppose Ω is R-transverse to V, and let If $U : X \to X$ be a unitary transformation. Then $U(\Omega)$ is R-transverse to $U(V)$. If additionally $\|U - id\|_{op} \leq \frac{1}{16R^2}$, then $U(\Omega)$ is $4R$-transverse to V and Ω is $4R$-transverse to $U(V)$.*

Proof. Unitary transformations preserve the metric structure of X, and in particular, they preserve transversality. If $\|U - id\|_{op} \leq \frac{1}{16R^2}$ then

$$d_H(\Omega \cap 4R\mathcal{B}, U(\Omega) \cap 4R\mathcal{B}) = d_H(\Omega \cap 4R\mathcal{B}, U(\Omega \cap 4R\mathcal{B})) \leq \|U - id\|_{op} \cdot 4R \leq \frac{1}{4R}.$$

Therefore, by Lemma 3.4, $U(\Omega)$ is $4R$-transverse to V. Similarly, $U^{-1}(\Omega)$ is $4R$-transverse to V, and thus by the first claim we have that Ω is $4R$-transverse to $U(V)$. □

3.1 Transversality in the space of polynomials

Definition 3.6. *Given a closed, symmetric, convex set $\Omega \subset \mathcal{P}$, a subspace $V \subset \mathcal{P}$, $R \geq 1$, $x \in \mathbb{R}^n$, and $\delta > 0$, we say that Ω is (x, δ, R)-transverse to V if Ω is R-transverse to V with respect to the Hilbert space structure $(\mathcal{P}, \langle\cdot,\cdot\rangle_{x,\delta})$, i.e., (1) $\mathcal{B}_{x,\delta}/V \subset R \cdot (\Omega \cap \mathcal{B}_{x,\delta})/V$, and (2) $\Omega \cap V \subset R \cdot \mathcal{B}_{x,\delta}$.*

Our next result establishes a few basic properties of transversality in this setting.

Lemma 3.7. *If Ω is (x, δ, R)-transverse to V, then the following holds:*

- *$T_h\Omega$ is $(x + h, \delta, R)$-transverse to T_hV.*
- *$\tau_{x,r}\Omega$ is $(x, \delta/r, R)$-transverse to $\tau_{x,r}V$.*
- *If $\delta' \in [\kappa^{-1}\delta, \kappa\delta]$ for some $\kappa \geq 1$, then Ω is $(x, \delta', \kappa^m R)$-transverse to V.*

Proof. The proof of the first and second bullet points is easy: Apply T_h and $\tau_{x,r}$ to both sides of (1) and (2) in Definition 3.6, and use the identities $T_h\mathcal{B}_{x,\delta} = \mathcal{B}_{x+h,\delta}$ and $\tau_{x,r}\mathcal{B}_{x,\delta} = \mathcal{B}_{x,\delta/r}$. The third bullet point follows from the equivalence of the unit balls $\mathcal{B}_{x,\delta} \subset \max\{1, (\delta/\delta')^m\} \cdot \mathcal{B}_{x,\delta'}$ and $\mathcal{B}_{x,\delta'} \subset \max\{1, (\delta'/\delta)^m\} \cdot \mathcal{B}_{x,\delta}$, as well as the property that $A \cap (r \cdot B) \subset r \cdot (A \cap B)$ if A and B are symmetric convex sets, and $r \geq 1$.

□

The continuity of the mapping $x \mapsto \sigma(x)$ can be used to show that the transversality of the set $\sigma(x)$ with respect to a fixed subspace is stable with respect to small perturbations of the basepoint.

Lemma 3.8. *There exists $c_1 = c_1(m,n) > 0$ so that the following holds. Let $V \subset \mathcal{P}$ be a subspace, $x, y \in \mathbb{R}^n$, $\delta > 0$, $R \geq 1$. If $\sigma(x)$ is (x,δ,R)-transverse to V and $|x-y| \leq c_1 \frac{\delta}{R}$ then $\sigma(y)$ is $(y,\delta,8R)$-transverse to V.*

Proof. If $c_1 < \frac{1}{4C_T}$, where C_T is the constant in (3), then by Lemma 2.7,

$$\sigma(y) \subset \sigma(x) + C_T \cdot \mathcal{B}_{x,c_1\cdot(\frac{\delta}{R})} \subset \sigma(x) + C_T \cdot (\frac{c_1}{R}) \cdot \mathcal{B}_{x,\delta} = \sigma(x) + (\frac{1}{4R}) \cdot \mathcal{B}_{x,\delta}.$$

Similarly, $\sigma(x) \subset \sigma(y) + (\frac{1}{4R}) \cdot \mathcal{B}_{x,\delta}$. Thus, $d_H^{x,\delta}(\sigma(x),\sigma(y)) \leq \frac{1}{4R}$, where $d_H^{x,\delta}$ is the Hausdorff distance with respect to the norm $|\cdot|_{x,\delta}$ on $\mathcal{P}$. By Lemma 3.4, since $\sigma(x)$ is (x,δ,R)-transverse to V, we conclude that $\sigma(y)$ is $(x,\delta,4R)$-transverse to V. Since $|x-y| \leq c_1\delta/R \leq c_1\delta$, if $c_1 = c_1(m,n)$ is chosen small enough then $(\frac{9}{10}) \cdot \mathcal{B}_{y,\delta} \subset \mathcal{B}_{x,\delta} \subset (\frac{10}{9}) \cdot \mathcal{B}_{y,\delta}$. Substituting these inclusions in conditions (1) and (2) in the definition of transversality, we learn that $\sigma(y)$ is $(y,\delta,8R)$-transverse to V. ◻

3.2 Ideals in the ring of polynomials and DTI subspaces

Definition 3.9. *A subspace $V \subset \mathcal{P}$ is translation-invariant if $T_h V = V$ for all $h \in \mathbb{R}^n$, and V is dilation-invariant at $x \in \mathbb{R}^n$ if $\tau_{x,\delta} V = V$ for all $\delta > 0$. Say that V is dilation-and-translation-invariant (DTI) if $T_h \tau_{x,\delta} V = V$ for all $x, h \in \mathbb{R}^n$, $\delta > 0$. We write* DTI *to denote the collection of all DTI subspaces of $\mathcal{P}$.*

Remark 3.10. *Equivalently, $V \subset \mathcal{P}$ is translation-invariant if $P \in V, Q \in \mathcal{P} \implies Q(\partial)P \in V$. Since $T_h = \tau_{(1-\delta)^{-1}h,\delta^{-1}} \circ \tau_{0,\delta}$ (for any $\delta > 1$), any translation operator is a composition of dilation operators. Thus, V is DTI if and only if $\tau_{x,\delta} V = V$ for all $(x,\delta) \in \mathbb{R}^n \times (0,\infty)$.*

We now illustrate a connection between translation-invariant subspaces and ideals in $\mathcal{P}_x$.

Lemma 3.11. *Let $(x,\delta) \in \mathbb{R}^n \times (0,\infty)$. Let $V^\perp$ be the orthogonal complement of a subspace $V \subset \mathcal{P}$ with respect to the inner product $\langle \cdot,\cdot \rangle_{x,\delta}$. Then V is translation-invariant if and only if $V^\perp$ is an $\odot_x$-ideal in $\mathcal{P}_x$.*

Proof. Translating, we may assume that $x = 0$. Rescaling preserves the property of V being translation-invariant, and also of $V^\perp$ being an $\odot_x$-ideal, according to (5). Hence we may assume that $\delta = 1$. Note the identity $\langle Q, P \rangle = Q(\partial)(P)(0)$ for any $P, Q \in \mathcal{P}$. Note ∂^α annihilates $\mathcal{P}$ for $|\alpha| \geq m$, and hence $R(\partial)[Q(\partial)P] = (R \odot_0 Q)(\partial)P$ for any $P, Q, R \in \mathcal{P}$. Suppose that V is a translation-invariant subspace, and let $Q \in V^\perp$. Then, for any $h \in \mathbb{R}^n$ and $P \in V$, also $T_h P \in V$ and hence,

$$0 = \langle Q, T_h(P) \rangle = Q(\partial)\left[T_h(P)\right](0) = T_h(Q(\partial)P)(0) = Q(\partial)P(-h).$$

Consequently, $Q(\partial)P = 0$. Thus, for any $R \in \mathcal{P}$, we have $(R \odot_0 Q)(\partial)P = R(\partial)\,[Q(\partial)P] = 0$. In particular, $\langle R \odot_0 Q, P\rangle = 0$ for any $P \in V$ and hence $R \odot_0 Q \in V^\perp$. This shows that $V^\perp$ is an $\odot_0$-ideal.

For the other direction, suppose that $V^\perp$ is an $\odot_0$-ideal. Let $P \in V$ and $R \in \mathcal{P}$. Then for any $Q \in V^\perp$,

$$0 = \langle R \odot_0 Q, P\rangle = Q(\partial)\,[R(\partial)P]\,(0) = \langle Q, R(\partial)P\rangle.$$

This means that $R(\partial)P \in (V^\perp)^\perp = V$. Hence $R(\partial)P \in V$ whenever $P \in V$ and $R \in \mathcal{P}$, and consequently the subspace V is translation-invariant. $\square$

We say that two subspaces $V_1, V_2 \subset \mathcal{P}$ are *complementary* if $V_1 + V_2 = \mathcal{P}$ and $V_1 \cap V_2 = \{0\}$.

Lemma 3.12. *For any $\odot_0$-ideal I in $\mathcal{P}_0$, there exists $V \in \mathrm{DTI}$ that is complementary to I.*

Proof. Set $I_* = \lim_{\delta \to 0} \tau_{0,\delta}(I)$ (where the Grassmanian is endowed with the usual topology). Let us first show that this limit exists: Consider the canonical projection $\pi_k : \mathcal{P}_0 \to \mathcal{P}_0^k$ onto the subspace of k-homogeneous polynomials $\mathcal{P}_0^k := \mathrm{span}\{z^\alpha : |\alpha| = k\}$, and denote the subspace of $(\geq k)$-homogeneous polynomials $\mathcal{P}_0^{\geq k} := \mathrm{span}\{z^\alpha : |\alpha| \geq k\}$. By Gaussian elimination we can pick a basis $\mathcal{B}_1 := \{P_j^k\}_{1 \leq j \leq N_k}^{0 \leq k \leq m-1}$ for I in the *block form*: $P_j^k \in \mathcal{P}_0^{\geq k}$, and $\mathcal{B}_0 := \{\pi_k P_j^k\}_{1 \leq j \leq N_k}^{0 \leq k \leq m-1}$ is linearly independent in $\mathcal{P}_0$. The family $\mathcal{B}_\delta := \{\delta^{m-k}\tau_{0,\delta}(P_j^k)\}_{k,j}$ converges elementwise as $\delta \to 0$ to $\mathcal{B}_0$. Since $\mathcal{B}_\delta$ is a basis for $\tau_{0,\delta}(I)$, and $\mathcal{B}_0$ is a basis for $I_* := \mathrm{span}(\mathcal{B}_0)$, we learn that $\tau_{0,\delta}(I)$ converges to I_*, as desired.

The ideals form a closed subset of the Grassmanian, thus I_* is an ideal in the ring $\mathcal{P}_0$. Let V be the orthogonal complement of I_* with respect to the standard inner product on $\mathcal{P}_0$. Observe that I_* is dilation-invariant at $x = 0$, i.e., $\tau_{0,\delta} I_* = I_*$ for all $\delta > 0$. Equivalently, I_* is a direct sum of homogeneous subspaces of $\mathcal{P}_0$, i.e., $I_* = I^0 + \cdots + I^{m-1}$, with $I^k \subset \mathcal{P}_0^k$. But then V is also a direct sum of homogeneous subspaces of $\mathcal{P}_0$, and so V is dilation-invariant at $x = 0$. From Lemma 3.11, we also know that V is translation-invariant. Thus, $V \in \mathrm{DTI}$. The subspaces I_* and V are complementary and this property is open in $\mathcal{G} \times \mathcal{G}$. By definition of I_* as a limit, $\tau_{0,\delta}(I)$ and V are complementary for some $\delta > 0$. By an application of the isomorphism of vector spaces $\tau_{0,\delta^{-1}}$, we learn that I and $\tau_{0,\delta^{-1}} V$ are complementary. To finish the proof, recall that $V \in \mathrm{DTI}$, and hence $\tau_{0,\delta^{-1}} V = V$. $\square$

Our next result says that every Whitney convex set is transverse to a DTI subspace.

Lemma 3.13. *Given $A \in [1, \infty)$, there exists a constant $R_0 = R_0(A, m, n)$ so that the following holds. Let Ω be a closed, symmetric, convex subset of $\mathcal{P}$. If Ω is Whitney convex at $x \in \mathbb{R}^n$ with $w_x(\Omega) \leq A$, and $\delta > 0$, then there exists $V \in \mathrm{DTI}$ such that Ω is (x, δ, R_0)-transverse to V.*

Proof. By the second bullet point in Lemma 3.7, Ω is (x, δ, R)-transverse to V if and only if $\tau_{x,\delta}\Omega$ is $(x, 1, R)$-transverse to $\tau_{x,\delta} V$. Thus, by the remark following Definition 2.10, we may rescale and assume that $\delta = 1$. Similarly, by translating we may assume that $x = 0$.

Let $\mathcal{S}$ be the set of closed, symmetric, convex subsets of $\mathcal{P}$. We endow $\mathcal{S}$ with the topology of local Hausdorff convergence, i.e., $\Omega_j \to \Omega$ iff $\lim_{j\to\infty} d_H(\Omega_j \cap R\mathcal{B}, \Omega \cap R\mathcal{B}) = 0$ for all $R > 0$ – here, $\mathcal{B} \subset \mathcal{P}$ is the unit ball with respect to the norm $|\cdot| = |\cdot|_{0,1}$ on $\mathcal{P}$, and d_H is the Hausdorff metric with respect to this norm. As a consequence of the Blaschke selection theorem, thus endowed, $\mathcal{S}$ is a compact space. Write $\mathcal{G}$ to denote the Grassmanian of all subspaces of $\mathcal{P}$, and $\mathcal{G}_k \subset \mathcal{G}$ the Grassmanian of all k-dimensional subspaces. We may identify $\mathcal{G}$ as a compact subspace of $\mathcal{S}$.

For any $(x,\delta) \in \mathbb{R}^n \times (0,\infty)$, the isomorphism $\tau_{x,\delta} : \mathcal{P} \to \mathcal{P}$ induces a continuous mapping on the Grassmanian $\tau_{x,\delta} : \mathcal{G} \to \mathcal{G}$. Thus, $\mathrm{DTI} = \{V \in \mathcal{G} : \tau_{x,\delta} V = V \;\; \forall (x,\delta) \in \mathbb{R}^n \times (0,\infty)\}$ is a closed subset of $\mathcal{G}$, and hence DTI is compact.

The conclusion of the lemma is equivalent to the existence of a constant $R_0 = R_0(A,m,n)$ so that $\phi(\Omega) \le R_0$ for all $\Omega \in wc_A$, where

$$wc_A := \{\Omega \in \mathcal{S} : \Omega \text{ is Whitney convex at } 0 \text{ with } w_0(\Omega) \le A\}$$
$$\phi : wc_A \to [0,\infty], \;\; \phi(\Omega) := \inf\{\psi(\Omega, V) : V \in \mathrm{DTI}\}$$
$$\psi : wc_A \times \mathrm{DTI} \to [0,\infty], \;\; \psi(\Omega,V) := \inf\{R : \Omega \cap V \subset R\cdot\mathcal{B}, \;\; \mathcal{B}/V \subset R\cdot(\Omega\cap\mathcal{B})/V\}.$$

If $\Omega_n \to \Omega$, $\Omega_n \in wc_A$, $\delta > 0$, and $A^* > A$, then

$$(\Omega \cap \mathcal{B}_{0,\delta}) \odot_0 \mathcal{B}_{0,\delta} = \lim_{n\to\infty} (\Omega_n \cap \mathcal{B}_{0,\delta}) \odot_0 \mathcal{B}_{0,\delta} \subset \lim_{n\to\infty} A^*\delta^m \Omega_n = A^*\delta^m\Omega,$$

where we used the continuity of $\odot_0$ on $\mathcal{S}\times\mathcal{S}$. So wc_A is closed, and hence compact. We claim that ψ is upper semicontinuous (usc). Indeed, $\psi = \inf_{R>0} \psi_R$, with $\psi_R = R1_{E_R} + \infty 1_{E_R^c}$ and

$$E_R = \{(\Omega,V) \in \mathcal{S}\times\mathrm{DTI} : \text{ for some } R' < R, \; \Omega\cap V \subset R'\cdot\mathcal{B} \text{ and } \mathcal{B}/V \subset R'\cdot(\Omega\cap\mathcal{B})/V\}.$$

As E_R is open, ψ_R is usc. Hence the same is true of ψ, and also of ϕ.

Since ϕ is usc and wc_A is compact, it suffices to show that $\phi(\Omega) < \infty$ for all $\Omega \in wc_A$. Since Ω is Whitney convex at 0, $I = \mathrm{span}(\Omega)$ is an ideal in $\mathcal{P}_0$ (see Lemma 2.12). By Lemma 3.12 there exists a subspace $V \in \mathrm{DTI}$ which is complementary to I, i.e., $V \cap I = \{0\}$ and $V + I = \mathcal{P}$. Note that $\mathrm{span}(\Omega + V) = I + V = \mathcal{P}$, and so by convexity, $\Omega + V$ contains a ball $\epsilon\mathcal{B}$ for some $\epsilon > 0$. If $\epsilon\mathcal{B} \subset \Omega + V$, it follows that $\epsilon\mathcal{B}/V \subset \Omega/V$. Thus,

$$\epsilon\mathcal{B}/V \subset \bigcup_{R>0} (\Omega \cap R\mathcal{B})/V.$$

By compactness, there exists an $R > 0$ with $\frac{\epsilon}{2}\mathcal{B}/V \subset (\Omega\cap R\mathcal{B})/V \subset R(\Omega\cap\mathcal{B})/V$. Thus, $\mathcal{B}/V \subset \frac{2R}{\epsilon}(\Omega\cap\mathcal{B})/V$. Combined with $V\cap\Omega \subset V\cap I = \{0\}$, this implies that $\phi(\Omega) \le \frac{2R}{\epsilon}$. $\square$

For any $x \in \mathbb{R}^n$, the set $\sigma(x) = \sigma(x,E)$ is Whitney convex at x with $w_x(\sigma(x)) \le C_0$ (see Lemma 2.11). Let R_0 be the constant from Lemma 3.13 with $A = C_0$. Then

$$\begin{cases} \text{for any finite set } E \subset \mathbb{R}^n, \text{ for any } (x,\delta) \in \mathbb{R}^n \times (0,\infty), \\ \text{there exists } V \in \mathrm{DTI} \text{ such that } \sigma(x) \text{ is } (x,\delta,R_0)\text{-transverse to } V. \end{cases} \tag{31}$$

Constants: Recall the constant c_1 is defined in Lemma 3.8. We specify constants $R_{\text{label}} \ll R_{\text{med}} \ll R_{\text{big}} \ll R_{\text{huge}}$, C_*, and C_{**}, defined as follows:

$$\begin{cases} R_{\text{label}} := 8R_0 & R_{\text{med}} := 256DR_{\text{label}} & R_{\text{big}} := 10^m R_{\text{med}} \\ R_{\text{huge}} := 2^{m+3}R_{\text{big}} & C_* := 20c_1^{-1}R_{\text{big}} & C_{**} = 1 + 2^m C_T \cdot (1 + R_{\text{label}} \cdot (5C_*)^m). \end{cases} \tag{32}$$

Lemma 3.14. *Let B be a closed ball in $\mathbb{R}^n$. There exists $V \in \mathrm{DTI}$ such that $\sigma(z)$ is $(z, C_* \operatorname{diam}(B), R_{\text{label}})$-transverse to V for all $z \in 100B$.*

Proof. Let $x_0 \in \mathbb{R}^n$ be the center of B. We apply (31) with $x = x_0$ and $\delta = C_* \operatorname{diam}(B)$. Thus, $\sigma(x_0)$ is $(x_0, C_* \operatorname{diam}(B), R_0)$-transverse to some $V \in \mathrm{DTI}$. Let $z \in 100B$ be arbitrary. Then $|z - x_0| \leq 100 \operatorname{diam}(B) \leq c_1 \frac{C_* \operatorname{diam}(B)}{R_0}$ (see (32)). By Lemma 3.8, we conclude that $\sigma(z)$ is $(z, C_* \operatorname{diam}(B), 8R_0)$-transverse to V.

□

4 Complexity

Let $l(I)$ and $r(I)$ denote the left and right endpoints of an interval $I \subset \mathbb{R}$. An interval J is *to the left* of an interval I, written $J < I$, if either $r(J) < l(I)$ or $r(J) = l(I)$ and $l(J) < l(I)$. Let X be a finite-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle_X$, set $d := \dim X < \infty$, and denote the norm and unit ball of X by $|\cdot|_X = \sqrt{\langle \cdot, \cdot \rangle_X}$ and $\mathcal{B} = \{x \in X : |x|_X \leq 1\}$. Let $\Psi : \mathbb{R}^D \to X$ be a coordinate transformation of the form $\Psi(v) = \sum_j v_j e_j$ for an orthonormal basis $\{e_j\}_{1 \leq j \leq d}$ of X. Fix $\vec{m} = (m_1, \cdots, m_d) \in \mathbb{Z}^d_{\geq 0}$ and a 1-parameter family of maps $T_\delta : X \to X$ ($\delta > 0$) of the form $T_\delta = \Psi \widetilde{T}_\delta \Psi^{-1}$, where the transformation $\widetilde{T}_\delta : \mathbb{R}^d \to \mathbb{R}^d$ is represented in standard Euclidean coordinates by a diagonal matrix $D_\delta = \operatorname{diag}(\delta^{-m_1}, \cdots, \delta^{-m_d})$.

Definition 4.1. *Given a closed, symmetric, convex set $\Omega \subset X$, the complexity of Ω relative to the dynamical system $\mathcal{X} = (X, T_\delta)_{\delta > 0}$ at scale $\delta_0 > 0$ with parameter $R \geq 1$– written $\mathcal{C}_{\mathcal{X}, \delta_0, R}(\Omega)$ – is the largest integer $K \geq 1$ such that there exist intervals $I_1 > I_2 > \cdots > I_K$ in $(0, \delta_0]$ and subspaces $V_1, V_2 \cdots, V_K \subset X$, such that $T_{r(I_k)}(\Omega)$ is R-transverse to V_k, but $T_{l(I_k)}(\Omega)$ is not $256dR$-transverse to V_k for all $k = 1, \cdots, K$. If no such K exists, let $\mathcal{C}_{\mathcal{X}, \delta_0, R}(\Omega) := 0$.*

Proposition 4.2. *Given $R \geq 1$ and $\vec{m} \in \mathbb{Z}^d_{\geq 0}$, there is a constant $K_0 = K_0(d, \vec{m}, R)$ such that $\mathcal{C}_{\mathcal{X}, \delta_0, R}(\Omega) \leq K_0$ for all closed, symmetric, convex sets $\Omega \subset X$ and all $\delta_0 > 0$.*

4.1 Background on semialgebraic geometry

We review some standard terminology from semialgebraic geometry: A *basic set* is a subset of $\mathbb{R}^d$ defined by a finite number of polynomial inequalities, i.e., $B = \{x \in \mathbb{R}^d : p_i(x) \leq 0,\ \ q_j(x) < 0\ \forall i \forall j\}$, for polynomials $p_1, \cdots, p_k, q_1, \cdots, q_l$ on $\mathbb{R}^d$. A *semialgebraic set* is a finite union of basic sets. Clearly the class of semialgebraic sets is closed under finite unions/intersections and complements. The celebrated Tarski-Seidenberg theorem on quantifier elimination implies that the class of semialgebraic sets is closed under projections $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$; see [22]. Semialgebraic sets are closely related to *first order formulas* over the reals, which are defined by the following elementary rules: (1) If p is a polynomial on $\mathbb{R}^d$, then "$p \leq 0$" and "$p < 0$" are formulas, (2) If Φ and Ψ are formulas, then "Φ and Ψ", "Φ or Ψ", and "not Φ" are formulas, and (3) If Φ is a formula and x is a variable of Φ (ranging in $\mathbb{R}$), then "$\exists x\ \Phi$" and "$\forall x\ \Phi$" are formulas. A formula is *quantifier-free* if it arises only via (1) and (2). The Tarski-Seidenberg theorem states that every formula is equivalent (i.e., has an identical solution set) to a quantifier-free formula. Accordingly, every semialgebraic set coincides with the solution set of a first-order formula, and visa versa. In the next section, we will consider the set $\mathcal{M}^+ \subset \mathbb{R}^{d \times d}$ of all positive-definite $d \times d$ matrices. Notice that $\mathcal{M}^+$ is semialgebraic because it is the solution set of a formula: $\mathcal{M}^+ = \{(a_{ij})_{1 \leq i,j \leq d} : a_{ij} = a_{ji} \text{ for } i,j = 1, \cdots, d \text{ and } \sum_{i,j=1}^d a_{ij}x_i x_j > 0\ \forall x_1, \cdots, \forall x_d\}$. We will need the following theorem which gives an upper bound on the number of connected components of a semialgebraic set.

Theorem 4.3 (Corollary 3.6, Chapter 3 of [22]). *If $S \subset \mathbb{R}^{k_1+k_2}$ is semialgebraic then there is a natural number M such that for each point $a \in \mathbb{R}^{k_1}$ the fiber $S_a := \{b \in \mathbb{R}^{k_2} : (a,b) \in S\}$ has at most M connected components.*

4.2 Proof of Proposition 4.2

Note that $\Psi^{-1} : X \to \mathbb{R}^d$ is a Hilbert space isomorphism, where $\mathbb{R}^d$ is equipped with the standard Euclidean inner product $\langle \cdot, \cdot \rangle$. Thus $\mathcal{C}_{(X,T_\delta),\delta_0,R}(\Omega) = \mathcal{C}_{(\mathbb{R}^d,\widetilde{T}_\delta),\delta_0,R}(\Psi^{-1}(\Omega))$, where $\widetilde{T}_\delta := \Psi^{-1} T_\delta \Psi$. Therefore, we may reduce to the case where $(X, \langle \cdot, \cdot \rangle_X) = (\mathbb{R}^d, \langle \cdot, \cdot \rangle)$ and the transformation T_δ on $\mathbb{R}^d$ is represented in Euclidean coordinates by the diagonal matrix $D_\delta = \mathrm{diag}(\delta^{-m_1}, \cdots, \delta^{-m_d})$ (i.e., $T_\delta(x) = D_\delta \cdot x$).

We give a proof by contradiction, for a value of K_0 to be determined later. Thus we fix a one-parameter family of linear transformations $T_\delta : \mathbb{R}^d \to \mathbb{R}^d$ of the above form, a closed, symmetric, convex set $\Omega \subset \mathbb{R}^d$, $\delta_0 > 0$, and $R \geq 1$, so that $\mathcal{C}_{(\mathbb{R}^d,T_\delta)_{\delta>0},\delta_0,R}(\Omega) \geq K_0 + 1$. Note that $(T_\delta)_{\delta>0}$ satisfies the semigroup properties $T_1 = id$ and $T_{\delta_1\delta_2} = T_{\delta_1} \circ T_{\delta_2}$. Hence, by exchanging Ω and $T_{\delta_0}(\Omega)$, we may reduce to the case $\delta_0 = 1$. Thus there exist intervals $I_1 > \cdots > I_{K_0+1}$ in $(0,1]$ and subspaces $V_1, \cdots, V_{K_0+1} \subset \mathbb{R}^d$ such that (a) $T_{r(I_k)}(\Omega)$ is R-transverse to V_k, whereas (b) $T_{l(I_k)}(\Omega)$ is not $256dR$-transverse to V_k, for all $1 \leq k \leq K_0 + 1$.

Let $\mathcal{G}$ be the Grassmanian of subspaces of $\mathbb{R}^d$, endowed with the metric

$$d_{\mathcal{G}}(V_1, V_2) := \inf\{\|U - id\| : U : \mathbb{R}^d \to \mathbb{R}^d \text{ unitary},\ U(V_1) = V_2\}.$$

In particular, $d_{\mathcal{G}}(V_1, V_2) < \infty \iff \dim(V_1) = \dim(V_2)$. Let $\epsilon := \frac{1}{2^{12} dR^2}$, and let $\mathcal{N}$ be an ϵ-net in $\mathcal{G}$.

By perturbation, we can approximate Ω by an ellipsoid $\mathcal{E}$ with similar properties. Let

$R_0 := 256dR$. Fix a compact, symmetric, convex set $\tilde{\Omega} \subset \mathbb{R}^d$ with nonempty interior, and

$$\begin{cases} d_H(T_{r(I_k)}(\Omega) \cap R_0\mathcal{B}, T_{r(I_k)}(\tilde{\Omega}) \cap R_0\mathcal{B}) < R_0^{-1}, \\ d_H(T_{l(I_k)}(\Omega) \cap R_0\mathcal{B}, T_{l(I_k)}(\tilde{\Omega}) \cap R_0\mathcal{B}) < R_0^{-1} \text{ for all } 1 \le k \le K_0+1, \end{cases}$$

where d_H is the Hausdorff metric with respect to the Euclidean norm on $\mathbb{R}^d$. By Lemma 3.4 and properties (a),(b), $T_{r(I_k)}(\tilde{\Omega})$ is $4R$-transverse to V_k, but $T_{l(I_k)}(\tilde{\Omega})$ is not $64dR$-transverse to V_k. If $\mathcal{E}$ is the John ellipsoid of $\tilde{\Omega}$, satisfying $\mathcal{E} \subset \tilde{\Omega} \subset \sqrt{d}\mathcal{E}$, then $T_{r(I_k)}(\mathcal{E})$ is $4\sqrt{d}R$-transverse to V_k, but $T_{l(I_k)}(\mathcal{E})$ is not $64\sqrt{d}R$-transverse to V_k. Hence, setting $\widehat{R} = 16\sqrt{d}R$,

$$\begin{cases} T_{r(I_k)}(\mathcal{E}) \text{ is } (\frac{1}{4})\widehat{R}\text{-transverse to } V_k, \text{ but} \\ T_{l(I_k)}(\mathcal{E}) \text{ is not } 4\widehat{R}\text{-transverse to } V_k, \text{ for all } 1 \le k \le K_0+1. \end{cases} \tag{33}$$

We identify an ellipsoid in $\mathbb{R}^d$ with a positive-definite $d \times d$ matrix in the usual way: every ellipsoid has the form $\mathcal{E}_A := \{x \in \mathbb{R}^d : \langle Ax, x \rangle \le 1\}$ for some $A \in \mathcal{M}^+$. Furthermore, every subspace of X has the form $V_C := \mathrm{rowsp}(C)$ for $C \in \mathbb{R}^{d \times d}$, where $\mathrm{rowsp}(C)$ denotes the row space of the matrix C. Consider the set

$$S = \{(C, A, \overline{R}, \delta) \in \mathbb{R}^{d^2} \times \mathcal{M}^+ \times [1,\infty) \times (0,\infty) : T_\delta(\mathcal{E}_A) \text{ is } \overline{R}\text{-transverse to } V_C\}.$$

Here, it is useful to note that $T_\delta(\mathcal{E}_A) = \mathcal{E}_{A_\delta}$, with $A_\delta := D_{\delta^{-1}} A D_{\delta^{-1}}$. Then S is a semialgebraic subset of $\mathbb{R}^{2d^2+2}$ because $\mathcal{M}^+$ is semialgebraic and the statement "$T_\delta(\mathcal{E}_A)$ is $\overline{R}$-transverse to V_C" is expressable by a first order formula in the variables $(A, C, \delta, \overline{R}) \in \mathbb{R}^{2d^2+2}$.

Consider the ellipsoid $\mathcal{E}$ fixed before and fix an arbitrary subspace $V \subset \mathbb{R}^d$. Write $V = V_C$ and $\mathcal{E} = \mathcal{E}_A$ for $C \in \mathbb{R}^{d^2}$, $A \in \mathcal{M}^+$. By Theorem 4.3 there exists $M = M(d, \vec{m}) \ge 1$ so that for any $\overline{R} \ge 1$ there exists a set $\Lambda = \Lambda(V_C, \mathcal{E}_A, \overline{R}) \subset (0,\infty)$ with $\#(\Lambda) \le M$ so that, for any interval $I \subset (0,\infty) \setminus \Lambda$, either $T_\delta(\mathcal{E}_A)$ is $\overline{R}$-transverse to V_C for all $\delta \in I$, or $T_\delta(\mathcal{E}_A)$ is not $\overline{R}$-transverse to V_C for all $\delta \in I$. Set

$$\Lambda_{\text{bad}} := \bigcup_{V \in \mathcal{N}} \Lambda(V, \mathcal{E}, \widehat{R}).$$

Note $\#(\Lambda_{\text{bad}}) \le \#(\mathcal{N}) \cdot M$, and for any interval $I \subset (0,\infty) \setminus \Lambda_{\text{bad}}$, and for all $V \in \mathcal{N}$,
$(*)$ [$T_\delta(\mathcal{E})$ is $\widehat{R}$-transverse to V for all $\delta \in I$] or [$T_\delta(\mathcal{E})$ is not $\widehat{R}$-transverse to V for all $\delta \in I$].

Set $K_0 := 2 \cdot \#(\mathcal{N}) \cdot M$. Then $K_0 + 1 > 2 \cdot \#(\Lambda_{\text{bad}})$. By definition of the order relation on intervals, at most two of the intervals $I_1 > \cdots > I_{K_0+1}$ can contain a given number $\delta \in \mathbb{R}$. Thus, we can find k_* so that $I_* := I_{k_*}$ is disjoint from Λ_{bad}.

Consider the subspace $V_* := V_{k_*}$. By definition of the metric on $\mathcal{G}$ and the fact that $\mathcal{N} \subset \mathcal{G}$ is an ϵ-net, there is a unitary transformation $U : X \to X$ and an element $V \in \mathcal{N}$ with $\|U^{-1} - id\| = \|U - id\| < \epsilon = \frac{1}{2^{12} dR^2} = \frac{1}{16\widehat{R}^2}$ and $U(V_*) = V$. From $(*)$, either $T_\delta(\mathcal{E})$ is $\widehat{R}$-transverse to V for all $\delta \in I_*$, or $T_\delta(\mathcal{E})$ is not $\widehat{R}$-transverse to V for all $\delta \in I_*$. By Lemma 3.5, either $T_\delta(\mathcal{E})$ is $(\frac{1}{4})\widehat{R}$-transverse to V_* for all $\delta \in I_*$, or $T_\delta(\mathcal{E})$ is not $4\widehat{R}$-transverse to V_* for all $\delta \in I_*$. This contradicts (33) for $k = k_*$ and finishes the proof of the proposition.

5 The Local Main Lemma

Definition 5.1. *For $x \in \mathbb{R}^n$, let $\mathcal{P}_x = \mathcal{P}$ be the Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle_x := \langle\cdot,\cdot\rangle_{x,1}$. Write $\mathcal{X}_x$ for the system $(\mathcal{P}_x, \tau_{x,\delta})_{\delta>0}$, where the rescaling transformations $\tau_{x,\delta} : \mathcal{P}_x \to \mathcal{P}_x$ ($\delta > 0$) are given by $\tau_{x,\delta}(P)(z) = \delta^{-m}P(x + \delta(z - x))$; note that $\tau_{x,\delta}$ is represented by a diagonal matrix with negative integer powers of δ along the diagonal with respect to the monomial basis $\{(z - x)^\alpha\}_{|\alpha|\leq m-1}$. Given a ball $B \subset \mathbb{R}^n$ and a finite set $E \subset \mathbb{R}^n$, the local complexity of E on B is the integer-valued quantity*

$$\mathcal{C}(E|B) := \sup_{x\in B} \mathcal{C}_{\mathcal{X}_x, C_* \operatorname{diam}(B), R_{label}}(\sigma(x)).$$

Remark 5.2. *We obtain an equivalent formulation of local complexity by inspection of Definition 4.1: We have $\mathcal{C}(E|B) \geq K$ if and only if there exists $x \in B$ and there exist subspaces $V_1, \cdots, V_K \subset \mathcal{P}$ and intervals $I_1 > I_2 > \cdots > I_K$ in $(0, \operatorname{diam}(B)]$, such that $\tau_{x,r(I_k)}(\sigma(x))$ is (x, C_*, R_{label})-transverse to V_k, but $\tau_{x,l(I_k)}(\sigma(x))$ is not (x, C_*, R_{med})-transverse to V_k for all $k = 1, \cdots, K$. Here, $R_{med} := 256DR_{label}$ (see (32)).*

We have a basic monotonicity property of complexity: $B_1 \subset B_2 \implies \mathcal{C}(E|B_1) \leq \mathcal{C}(E|B_2)$. As a consequence of Proposition 4.2, we also have the following result:

Corollary 5.3. *There exists $K_0 = K_0(m, n)$ such that $\mathcal{C}(E|B) \leq K_0$ for any closed ball $B \subset \mathbb{R}^n$ and finite subset $E \subset \mathbb{R}^n$.*

Next we will define the (global) complexity $\mathcal{C}(E)$ of a finite subset $E \subset \mathbb{R}^n$.

Definition 5.4. *If $E \subset \mathbb{R}^n$ is finite, let $\mathcal{C}(E) := \mathcal{C}(E|5B_0)$, where $B_0 \subset \mathbb{R}^n$ is an (arbitrary) fixed ball satisfying $E \subset B_0$.*

Clearly, Lemma 1.3 from the introduction follows now from Corollary 5.3. The main apparatus that will be used to prove Theorem 1.2 from the introduction is the following:

Lemma 5.5 (Local Main Lemma for K)**.** *Let $K \geq -1$. There exist constants $C^\# = C^\#(K) \geq 1$ and $\ell^\# = \ell^\#(K) \in \mathbb{Z}_{\geq 0}$, depending only on K, m, n, with the following properties.*

Let $E \subset \mathbb{R}^n$ be finite and let $B_0 \subset \mathbb{R}^n$ be a ball. If $\mathcal{C}(E|5B_0) \leq K$ then the following holds:

Local Finiteness Principle on B_0: Suppose $f : E \to \mathbb{R}$, $M > 0$, $x_0 \in B_0$, and $P_0 \in \mathcal{P}$ satisfy the following finiteness hypothesis*: For all $S \subset E$ with $\#(S) \leq (D+1)^{\ell^\#}$ there exists $F^S \in C^{m-1,1}(\mathbb{R}^n)$ with $F^S = f$ on S, $J_{x_0}F^S = P_0$, and $\|F^S\| \leq M$. Then there exists a function $F \in C^{m-1,1}(\mathbb{R}^n)$ with $F = f$ on $E \cap B_0$, $J_{x_0}F = P_0$, and $\|F\| \leq C^\# M$.*

Remark 5.6. *Equivalently, the Local Finiteness Principle on B_0 states that*

$$\Gamma_{\ell^\#}(x_0, f, M) \subset \Gamma_{E\cap B_0}(x_0, f, C^\# M).$$

In particular, by taking $f = 0$ and $M = 1$, we have

$$\sigma_{\ell^\#}(x_0) \subset C^\# \cdot \sigma(x_0, E \cap B_0).$$

5.1 Proof of Theorem 1.2

We now explain why it is that the Local Main Lemma implies Theorem 1.2. Fix a ball B_0 with $E \subset B_0$ as in Definition 5.4. We apply the Local Main Lemma for $K = \mathcal{C}(E) = \mathcal{C}(E|5B_0)$ and deduce that the Local Finiteness Principle for B_0 is true. Therefore, $\Gamma_{\ell^\#}(x_0, f, M) \subset \Gamma_E(x_0, f, C^\# M)$ for any $M > 0$. Our main result, Theorem 1.2, now follows easily: By Lemma 2.6, the Finiteness Hypothesis $\mathcal{FH}(k^\#)$ (see (14)) with constant $k^\# = (D+1)^{\ell^\#+1}$ implies $\Gamma_{\ell^\#}(x_0, f, 1) \neq \emptyset$, and so $\Gamma_E(x_0, f, C^\#) \neq \emptyset$. In particular, there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ with $F = f$ on E and $\|F\| \leq C^\#$.

Remark 5.7. *In section 9.1 we verify that the constant $C^\# = C^\#(K)$ in the Local Main Lemma depends exponentially on K, and the constant $\ell^\# = \ell^\#(K)$ depends linearly on K; thus, $k^\# = (D+1)^{\ell^\#+1}$ will depend exponentially on K. This finishes the proof of Theorem 1.2.*

5.2 Organization

The rest of the paper is organized as follows. In section 6 we formulate the Main Induction Argument that will be used to prove the Local Main Lemma for all values of K. In section 7 we prove the Main Decomposition Lemma which will allow us to pass from a local extension problem on a ball B_0 to a collection of easier subproblems on a family of smaller balls $B \subset 5B_0$; this lemma is the main component in the analysis of the induction step. In section 8, we state a technical lemma that allows us to control the shape of the set $\sigma_\ell(x)$ at lengthscales which are much coarser than the lengthscales of the balls in the decomposition; we next apply this lemma to enforce mutual consistency for a family of jets that are associated to the local extension problems on the smaller balls. In section 9 we will construct a solution to the local extension problem on B_0 by gluing together the solutions to the local problems on the smaller balls by means of a partition of unity; the consistency conditions arranged in the previous step will ensure that the individual local extensions are sufficiently compatible, which will imply the necessary control on the $C^{m-1,1}$ seminorm of the glued-together function.

6 The Main Induction Argument I: Setup

We will prove the Local Main Lemma by induction on the complexity parameter $K \in \{-1, 0, \cdots, K_0\}$ – recall, K_0 is a finite upper bound on the local complexity of any set. When $K = -1$, the Local Main Lemma is vacuously true (say, for $C^\#(-1) = 1$, $\ell^\#(-1) = 0$) since complexity is non-negative. This establishes the base case of the induction.

For the induction step, fix $K \in \{0, 1, \cdots, K_0\}$. The induction hypothesis states that the Local Main Lemma for $K - 1$ is valid. Denote the finiteness constants in the Local Main Lemma for $K-1$ by $\ell_{\text{old}} := \ell^\#(K-1)$ and $C_{\text{old}} := C^\#(K-1)$. Applying the Local Main

Lemma to a closed ball of the form $\frac{6}{5}B$, we obtain

$$\begin{aligned}&\text{If } \ x \in (6/5)\cdot B \text{ and } \mathcal{C}(E|6B) \leq K-1, \text{ then,}\\ &\Gamma_{\ell_{\text{old}}}(x,f,M) \subset \Gamma_{E\cap \frac{6}{5}B}(x,f,C_{\text{old}}M) \text{ for any } f: E \to \mathbb{R},\ M>0.\end{aligned} \tag{34}$$

(Here we use the formulation of the Local Finiteness Principle in Remark 5.6.)

Fix a ball $B_0 \subset \mathbb{R}^n$ with $\mathcal{C}(E|5B_0) \leq K$. To prove the Local Main Lemma for K, we are required to prove the Local Finiteness Principle (LFP) on B_0 for a suitable choice of the constants $\ell^\# \in \mathbb{Z}_{\geq 0}$ and $C^\# \geq 1$, determined by m, n, and K. Thus, our goal is to prove that $\Gamma_{\ell^\#}(x_0, f, M) \subset \Gamma_{E\cap B_0}(x_0, f, C^\# M)$ for any $f : E \to \mathbb{R}$, $x_0 \in B_0$, $M > 0$. A rescaling of the form $f \mapsto f/M$ allows us to reduce to the case $M = 1$. If $\#(B_0 \cap E) \leq 1$ then the LFP is true as long as $C^\# \geq 1$ and $\ell^\# \geq 0$ – indeed,

$$\Gamma_{\ell^\#}(x_0,f,1) \subset \Gamma_0(x_0,f,1) = \bigcap_{S\subset E,\ \#(S)\leq 1} \Gamma_S(x_0,f,1) \subset \Gamma_{E\cap B_0}(x_0,f,1) \subset \Gamma_{E\cap B_0}(x_0,f,C^\#). \tag{35}$$

Accordingly, it suffices to assume that

$$\#(B_0 \cap E) \geq 2. \tag{36}$$

Under these assumptions, we will prove that for any $x_0 \in B_0$ and $f : E \to \mathbb{R}$,

$$\Gamma_{\ell^\#}(x_0,f,1) \subset \Gamma_{E\cap B_0}(x_0,f,C^\#). \tag{37}$$

7 The Main Decomposition Lemma

In this section we fix the following data:

- A closed ball $B_0 \subset \mathbb{R}^n$ and a point $x_0 \in B_0$.
- A finite set $E \subset \mathbb{R}^n$ satisfying $\#(E \cap B_0) \geq 2$ and $\mathcal{C}(E|5B_0) \leq K$.
- A function $f : E \to \mathbb{R}$.
- An integer $\ell^\# \in \mathbb{Z}_{\geq 0}$.
- A polynomial $P_0 \in \Gamma_{\ell^\#}(x_0, f, 1)$.

Our plan is to introduce a cover of the ball $2B_0$ which will later be used to decompose the local extension problem on B_0 into a family of easier subproblems associated to the elements of the cover.

Lemma 7.1 (Main Decomposition Lemma)**.** *Recall that the constants $R_{label} \ll R_{med} \ll R_{big} \ll R_{huge}$, C_*, and C_{**} are defined in* (32)*. Given data $(B_0, E, K, f, \ell^\#, P_0)$ as above, there exists*

a subspace $V \in \mathrm{DTI}$ *such that*
(a) $\sigma(x)$ *is* $(x, C_* \operatorname{diam}(B_0), R_{\scriptscriptstyle label})$*-transverse to* V *for all* $x \in 100B_0$.

There exists a Whitney cover $\mathcal{W}$ *of* $2B_0$ *such that, for all* $B \in \mathcal{W}$,
(b) $B \subset 100B_0$ *and* $\operatorname{diam}(B) \leq \frac{1}{2}\operatorname{diam}(B_0)$.
(c) The subspace $\sigma(x)$ *is* $(x, C_*\delta, R_{\scriptscriptstyle huge})$*-transverse to* V *for all* $x \in 8B$, $\delta \in [\operatorname{diam}(B), \operatorname{diam}(B_0)]$.
(d) Either $\#(6B \cap E) \leq 1$ *or* $\mathcal{C}(E|6B) < K$.

For every $B \in \mathcal{W}$ *there exists a point* $z_B \in \mathbb{R}^n$ *and a jet* $P_B \in \mathcal{P}$ *satisfying*
(e) $z_B \in \frac{6}{5}B \cap 2B_0$*; also, if* $x_0 \in \frac{6}{5}B$ *then* $z_B = x_0$.
(f) $P_B \in \Gamma_{\ell\#-1}(z_B, f, C_{**})$ *and* $P_0 - P_B \in C_{**}\mathcal{B}_{z_B,\operatorname{diam}(B_0)}$*; also, if* $x_0 \in \frac{6}{5}B$ *then* $P_B = P_0$.
(g) $P_0 - P_B \in V$.

We obtain a local finiteness principle on the elements of the cover $\mathcal{W}$ in the next lemma.

Lemma 7.2. *For any* $B \in \mathcal{W}$*, the Local Finiteness Principle on* $\frac{6}{5}B$ *is true for the constants* $\ell_{\scriptscriptstyle old} = \ell^\#(K-1) \in \mathbb{Z}_{\geq 0}$ *and* $C_{\scriptscriptstyle old} = C^\#(K-1) \geq 1$*. That is,* $\Gamma_{\ell_{old}}(x, f, M) \subset \Gamma_{E \cap \frac{6}{5}B}(x, f, C_{\scriptscriptstyle old}M)$*, for all* $x \in \frac{6}{5}B$, $M > 0$.

Proof. If $\mathcal{C}(E|6B) < K$, the result follows from (34). On the other hand, if $\#(E \cap 6B) \leq 1$, the result follows from (35). Condition (d) implies that these cases are exhaustive. ☐

7.1 Proof of the Main Decomposition Lemma

We apply Lemma 3.14 to select a subspace $V \in \mathrm{DTI}$ such that $\sigma(x)$ is $(x, C_* \operatorname{diam}(B_0), R_{\text{label}})$-transverse to V for all $x \in 100B_0$. This establishes property (a). The construction of $\mathcal{W}$ is based on the following definition:

Definition 7.3. *A ball* $B \subset 100B_0$ *is OK if* $\#(B \cap E) \geq 2$ *and if there exists* $z \in B$ *such that* $\sigma(z)$ *is* $(z, C_*\delta, R_{\scriptscriptstyle big})$*-transverse to* V *for all* $\delta \in [\operatorname{diam}(B), \operatorname{diam}(B_0)]$.

The OK property is *inclusion monotone* in the sense that if $B \subset B' \subset 100B_0$ and B is OK then B' is OK.

For each $x \in 2B_0$, let $r(x) := \inf\{r > 0 : B(x,r) \subset 100B_0,\ B(x,r) \text{ is OK}\}$. Every subball of $100B_0$ that contains $2B_0$ is OK, so the infimum is well-defined – this also implies $r(x) \leq 2\operatorname{diam}(B_0)$ for all $x \in 2B_0$. If $B \subset 100B_0$ is sufficiently small then $\#(B \cap E) \leq 1$, and so B is not OK – in particular, this shows that $r(x) \geq \Delta := \min\{|x-y| : x, y \in E, x \neq y\} > 0$ for all $x \in 2B_0$. Let $B_x := B(x, \frac{1}{7}r(x))$ for $x \in 2B_0$. Then

$$70B_x \subset 100B_0, \quad \text{for } x \in 2B_0. \tag{38}$$

Obviously the family $\mathcal{W}^* = \{B_x\}_{x \in 2B_0}$ is a cover of $2B_0$.

Lemma 7.4. *If* $B \in \mathcal{W}^*$ *then* $8B$ *is OK, and* $6B$ *is not OK.*

Proof. We write $B = B(x, \frac{1}{7}r(x))$ for some $x \in 2B_0$. According to (38), $6B \subset 8B \subset 100B_0$. By definition of $r(x)$ as an infimum and the inclusion monotonicity of the OK property, the result follows. $\square$

We apply the Vitali covering lemma to extract a finite subcover $\mathcal{W} \subset \mathcal{W}^*$ of $2B_0$ with the property that the family of third-dilates $\{\frac{1}{3}B\}_{B\in\mathcal{W}}$ is pairwise disjoint.

Lemma 7.5. *$\mathcal{W}$ is a Whitney cover of $2B_0$.*

Proof. We only have to verify condition (c) in the definition of a Whitney cover (see Definition 2.13). Suppose for sake of contradiction that there exist balls $B_j = B(x_j, r_j) \in \mathcal{W}$ for $j = 1, 2$, with $\frac{6}{5}B_1 \cap \frac{6}{5}B_2 \neq \emptyset$ and $r_1 < \frac{1}{8}r_2$. Since $\frac{6}{5}B_1 \cap \frac{6}{5}B_2 \neq \emptyset$, we have $|x_1 - x_2| \leq \frac{6}{5}r_1 + \frac{6}{5}r_2$. If $z \in 8B_1$ then $|z - x_1| \leq 8r_1$, and therefore

$$|z - x_2| \leq |z - x_1| + |x_1 - x_2| \leq 8r_1 + \frac{6}{5}r_1 + \frac{6}{5}r_2 \leq r_2 + \frac{3}{20}r_2 + \frac{6}{5}r_2 \leq 6r_2.$$

Hence, $8B_1 \subset 6B_2$. By Lemma 7.4, $8B_1$ is OK. Thus, by inclusion monotonicity, $6B_2$ is OK. But this contradicts Lemma 7.4. This finishes the proof by contradiction. $\square$

We now establish conditions (b)-(d) in the Main Decomposition Lemma. Fix a ball $B \in \mathcal{W}$.

We will use the following preparatory claim: (PC) If $\#(6B \cap E) \geq 2$ then for all $x \in 6B$ there exists $\delta_x \in [6\,\mathrm{diam}(B), \mathrm{diam}(B_0)]$ so that $\sigma(x)$ is not $(x, C_*\delta_x, R_{\mathrm{big}})$-transverse to V. This follows because $6B$ is not OK.

Proof of (b): The inclusion $B \subset 100B_0$ follows from (38). For sake of contradiction, suppose that $\mathrm{diam}(B) > \frac{1}{2}\mathrm{diam}(B_0)$. Since $B \cap B_0 \neq \emptyset$, we have $B_0 \subset 5B$. Therefore, $\#(5B \cap E) \geq \#(B_0 \cap E) \geq 2$. Fix a point $x \in B$. Then (PC) implies that the interval $[6\,\mathrm{diam}(B), \mathrm{diam}(B_0)]$ is nonempty, thus $\mathrm{diam}(B) \leq \frac{1}{6}\mathrm{diam}(B_0)$, which gives the contradiction.

Proof of (c): Since $8B$ is OK, $\sigma(z)$ is $(z, C_*\delta, R_{\mathrm{big}})$-transverse to V for some $z \in 8B$ and all $\delta \in [8\,\mathrm{diam}(B), \mathrm{diam}(B_0)]$. If $x \in 8B$ then $|x - z| \leq 8\,\mathrm{diam}(B) \leq \frac{c_1}{R_{\mathrm{big}}} \cdot (C_*\delta)$ (see (32)), and so, by Lemma 3.8,

$$\sigma(x) \text{ is } (x, C_*\delta, 8R_{\mathrm{big}})\text{-transverse to } V \text{ for all } \delta \in [8\,\mathrm{diam}(B), \mathrm{diam}(B_0)].$$

Any number in $[\mathrm{diam}(B), \mathrm{diam}(B_0)]$ differs from a number in $[8\,\mathrm{diam}(B), \mathrm{diam}(B_0)]$ by a factor of at most 8. Hence, by Lemma 3.7, $\sigma(x)$ is $(x, C_*\delta, 8^{m+1}R_{\mathrm{big}})$-transverse to V for all $\delta \in [\mathrm{diam}(B), \mathrm{diam}(B_0)]$. Since $R_{\mathrm{huge}} \geq 8^{m+1}R_{\mathrm{big}}$ (see (32)), this implies (c).

Proof of (d): Suppose that $\#(6B \cap E) \geq 2$ and set $J := \mathcal{C}(E|6B)$. According to the formulation of complexity in Remark 5.2, there exist intervals $I_1 > I_2 > \cdots > I_J$ in $(0, 6\,\mathrm{diam}(B)]$, subspaces $V_1, \cdots, V_J \subset \mathcal{P}$, and a point $z \in 6B$, such that
(A) $\tau_{z,r(I_j)}(\sigma(z))$ is $(z, C_*, R_{\mathrm{label}})$-transverse to V_j, and
(B) $\tau_{z,l(I_j)}(\sigma(z))$ is not $(z, C_*, R_{\mathrm{med}})$-transverse to V_j, for $1 \leq j \leq J$, where $R_{\mathrm{med}} = 256DR_{\mathrm{label}}$.

From $B \cap B_0 \neq \emptyset$ and $\operatorname{diam}(B) \leq \frac{1}{2}\operatorname{diam}(B_0)$ (see (b)) it follows that $6B \subset 5B_0$. Hence, $z \in 5B_0$.

Since $\#(6B \cap E) \geq 2$, the condition (PC) implies that there exists $\delta_z \in [6\operatorname{diam}(B), \operatorname{diam}(B_0)]$ such that

$$\sigma(z) \text{ is not } (z, C_*\delta_z, R_{\text{big}})\text{-transverse to } V. \tag{39}$$

We will now establish that (A) and (B) hold for $j = 0$ with $I_0 := [\delta_z, \operatorname{diam}(B_0)]$ and $V_0 := V$. Since V is a DTI subspace, $\tau_{z,l(I_0)}V = V$, and therefore, by rescaling (39),

$$\tau_{z,l(I_0)}(\sigma(z)) \text{ is not } (z, C_*, R_{\text{big}})\text{-transverse to } V. \tag{40}$$

On the other hand, from property (a) we learn that $\sigma(z)$ is $(z, C_*\operatorname{diam}(B_0), R_{\text{label}})$-transverse to V. Therefore, by rescaling,

$$\tau_{z,r(I_0)}(\sigma(z)) \text{ is } (z, C_*, R_{\text{label}})\text{-transverse to } V. \tag{41}$$

The conditions (40) and (41) together imply (A) and (B) for $j = 0$ (recall $R_{\text{big}} \geq R_{\text{med}}$).

Notice that $r(I_1) \leq 6\operatorname{diam}(B) \leq \delta_z = l(I_0)$, thus $I_1 < I_0$. In conclusion, $I_0 > I_1 > \cdots > I_J$ are subintervals of $(0, \operatorname{diam}(B_0)]$.

We produced intervals $I_0 > I_1 > \cdots > I_J$ in $(0, 5\operatorname{diam}(B_0)]$ and subspaces $V_0, \cdots, V_J \subset \mathcal{P}$, so that (A) and (B) hold for $j = 0, 1, \cdots, J$. Since $z \in 5B_0$, by the formulation of complexity in Remark 5.2, we have $\mathcal{C}(E|5B_0) \geq J + 1$. Since $\mathcal{C}(E|5B_0) \leq K$, this completes the proof of (d).

Finally we will define a collection of points $\{z_B\}_{B \in \mathcal{W}}$ and polynomials $\{P_B\}_{B \in \mathcal{W}}$ and prove properties (e)-(g).

Proof of (e): We define the collection $\{z_B\}_{B \in \mathcal{W}}$ to satisfy property (e). For all $B \in \mathcal{W}$ such that $x_0 \in \frac{6}{5}B$ we set $P_B = P_0$. We define P_B for the remaining balls $B \in \mathcal{W}$ in the proof of (f) and (g) below.

Proofs of (f) and (g): If $x_0 \in \frac{6}{5}B$ then $z_B = x_0$ and $P_B = P_0$, in which case (f) and (g) are trivially true (note that $P_0 \in \Gamma_{\ell\#}(x_0, f, 1) \subset \Gamma_{\ell\#-1}(x_0, f, 1)$). Suppose instead $x_0 \notin \frac{6}{5}B$. Then $z_B \in \frac{6}{5}B \cap 2B_0$ and so $|x_0 - z_B| \leq 2\operatorname{diam}(B_0)$. By Lemma 2.6, given that $P_0 \in \Gamma_{\ell\#}(x_0, f, 1)$, we can find $P_B \in \Gamma_{\ell\#-1}(z_B, f, 1)$ with $P_0 - P_B \in C_T\mathcal{B}_{z_B, 2\operatorname{diam}(B_0)} \subset 2^m C_T \mathcal{B}_{z_B, \operatorname{diam}(B_0)}$. We still have to arrange $P_0 - P_B \in V$ as in (g). Unfortunately, there is no reason for this to be true, and we will have to perturb P_B to arrange this property. This is where we use condition (a), which implies that $\sigma(z_B)$ is $(z_B, 5C_*\operatorname{diam}(B_0), R_{\text{label}})$-transverse to V. Therefore,

$$\begin{aligned} \mathcal{B}_{z_B, \operatorname{diam}(B_0)}/V \subset \mathcal{B}_{z_B, 5C_*\operatorname{diam}(B_0)}/V &\subset R_{\text{label}} \cdot (\sigma(z_B) \cap \mathcal{B}_{z_B, 5C_*\operatorname{diam}(B_0)})/V \\ &\subset R_{\text{label}} \cdot (\sigma_{\ell\#-1}(z_B) \cap \mathcal{B}_{z_B, 5C_*\operatorname{diam}(B_0)})/V. \end{aligned}$$

Since $P_0 - P_B \in 2^m C_T \mathcal{B}_{z_B, \operatorname{diam}(B_0)}$, the last containment implies we can find a bounded correction

$$R_B \in 2^m C_T R_{\text{label}} \cdot (\sigma_{\ell\#-1}(z_B) \cap \mathcal{B}_{z_B, 5C_*\operatorname{diam}(B_0)}),$$

so that $R_B/V = (P_0 - P_B)/V$, i.e., $P_0 - P_B - R_B \in V$. Set $\widetilde{P}_B = P_B + R_B$. Then $P_0 - \widetilde{P}_B \in V$ and

$$\widetilde{P}_B \in \Gamma_{\ell^\# - 1}(z_B, f, 1) + 2^m C_T R_{\text{label}} \sigma_{\ell^\# - 1}(z_B) \subset \Gamma_{\ell^\# - 1}(z_B, f, 1 + 2^m C_T R_{\text{label}}).$$

Furthermore,

$$\begin{aligned} P_0 - \widetilde{P}_B = (P_0 - P_B) - R_B &\in 2^m C_T \mathcal{B}_{z_B, \text{diam}(B_0)} + 2^m C_T R_{\text{label}} \mathcal{B}_{z_B, 5C_* \text{diam}(B_0)} \\ &\subset 2^m C_T \cdot (1 + R_{\text{label}} \cdot (5C_*)^m) \cdot \mathcal{B}_{z_B, \text{diam}(B_0)}. \end{aligned}$$

Thus we have proven (f) and (g) for all $B \in \mathcal{W}$ such that $x_0 \notin \frac{6}{5}B$, with $\widetilde{P}_B$ in place of P_B, where $C_{**} = 1 + 2^m C_T \cdot (1 + R_{\text{label}} \cdot (5C_*)^m)$. This finishes the proof of Lemma 7.1.

8 The Main Induction Argument II

We return to the setting of the Main Induction Argument in section 6. Let $\ell_{\text{old}} = \ell^\#(K-1)$ and $C_{\text{old}} = C^\#(K-1)$ be as in (34). We fix data (B_0, x_0, E, f) as in section 6. Recall our goal is to establish the containment (37) for a suitable choice of $\ell^\# = \ell^\#(K)$ and $C^\# = C^\#(K)$ which will be determined by the end of the proof. We fix a polynomial $P_0 \in \Gamma_{\ell^\#}(x_0, f, 1)$, and apply Lemma 7.1 to the data $(B_0, x_0, E, f, \ell^\#, P_0)$. Through this we obtain a Whitney cover $\mathcal{W}$ of $2B_0$, a DTI subspace $V \subset \mathcal{P}$, and two families $\{P_B\}_{B \in \mathcal{W}} \subset \mathcal{P}$ and $\{z_B\}_{B \in \mathcal{W}} \subset \mathbb{R}^n$.

Let $\mathcal{W}_0$ be the collection of all balls $B \in \mathcal{W}$ with $B \cap B_0 \neq \emptyset$. Then $\mathcal{W}_0$ is a Whitney cover of B_0.

The main goal of this section is to prove that the family of polynomials $\{P_B\}_{B \in \mathcal{W}}$ are mutually compatible. Specifically, we will prove:

Lemma 8.1. *There exist constants $\overline{\ell} > \ell_{old}$ and $\overline{C} \geq 1$, determined by m and n, such that the following holds. If $\ell^\# \geq \overline{\ell}$, and $\{P_B\}_{B \in \mathcal{W}}$ is a family of polynomials satisfying the conditions in Lemma 7.1, then $P_B - P_{B'} \in \overline{C} \cdot \mathcal{B}_{z_B, \text{diam}(B)}$ for any $B, B' \in \mathcal{W}_0$ with $(\frac{6}{5})B \cap (\frac{6}{5})B' \neq \emptyset$.*

We will see that Lemma 8.1 follows easily from the next result.

Lemma 8.2. *There exist $\epsilon^* \in (0,1)$, $\ell^* > \ell_{old}$, and $R^* \geq 1$, depending only on m and n, such that the following holds. If $\widehat{B} \in \mathcal{W}_0$ satisfies $\text{diam}(\widehat{B}) \leq \epsilon^* \text{diam}(B_0)$, and if the subspace V is as in Lemma 7.1, then $\sigma_{\ell^*}(x)$ is $(x, \text{diam}(B), R^*)$-transverse to V for any $B \in \mathcal{W}_0$ and $x \in 6B$.*

Lemma 8.2 is difficult for subtle reasons: We know from condition (c) of the Main Decomposition Lemma that $\sigma(x)$ is $(x, \text{diam}(B), R)$-transverse to V for any $B \in \mathcal{W}$ and $x \in 8B$, where $R = R_{\text{huge}} \cdot (6C_*)^m$. But it is not apparent why V would also be transverse to $\sigma_\ell(x)$, which generally can be significantly larger than $\sigma(x)$. The key point in the proof of this proposition is that we are able to use the validity of the Local Finiteness Principle on the balls B in $\mathcal{W}$ to establish a two-sided relationship between the sets $\sigma(x)$ and $\sigma_{\ell^*}(x)$ (for sufficiently large ℓ^*) as long as we are willing to "blur" these sets at a lengthscale larger than diam(B). Since transversality is stable under "blurrings" (e.g., see Lemma 3.3), the result will follow.

The proof of Lemma 8.2 is the most technical part of the paper. We next explain how Lemma 8.1 follows from Lemma 8.2. After this we will establish a preparatory lemma, Lemma 8.3, and finally give the proof of Lemma 8.2 in section 8.2.

Proof of Lemma 8.1. We fix $\epsilon^* \in (0,1)$ and ℓ^* as in Lemma 8.2, and define $\overline{\ell} = \ell^* + 2$. We consider the following two situations:

- *Case 1:* $\operatorname{diam}(B) > \epsilon^* \operatorname{diam}(B_0)$ for all $B \in \mathcal{W}_0$.
- *Case 2:* There exists $\widehat{B} \in \mathcal{W}_0$ with $\operatorname{diam}(\widehat{B}) \leq \epsilon^* \operatorname{diam}(B_0)$.

Fix $B, B' \in \mathcal{W}_0$ with $\frac{6}{5}B \cap \frac{6}{5}B' \neq \emptyset$. In Case 1, by condition (f) in Lemma 7.1, we have

$$P_B - P_{B'} = (P_B - P_0) + (P_0 - P_{B'}) \in C_{**}\mathcal{B}_{z_B,\operatorname{diam}(B_0)} + C_{**}\mathcal{B}_{z_{B'},\operatorname{diam}(B_0)}.$$

Note that $|z_B - z_{B'}| \leq 2\operatorname{diam}(B_0)$ (recall $z_B, z_{B'} \in 2B_0$), and so by (6), $\mathcal{B}_{z_{B'},\operatorname{diam}(B_0)} \subset \widetilde{C}2^{m-1}\mathcal{B}_{z_B,\operatorname{diam}(B_0)}$. As $\operatorname{diam}(B) > \epsilon^* \operatorname{diam}(B_0)$, we have $\mathcal{B}_{z_B,\operatorname{diam}(B_0)} \subset (\epsilon^*)^{-m}\mathcal{B}_{z_B,\operatorname{diam}(B)}$. When put together, we learn that $P_B - P_{B'} \in C_{**} \cdot (\epsilon^*)^{-m}(1+\widetilde{C}2^{m-1})\mathcal{B}_{z_B,\operatorname{diam}(B)}$, which gives the desired result in this case.

Now suppose that Case 2 holds. By property (g) in Lemma 7.1, we have

$$P_B - P_{B'} = (P_B - P_0) + (P_0 - P_{B'}) \in V.$$

By property (1) we have $P_{B'} \in \Gamma_{\ell\#-1}(z_{B'}, f, C)$. By Lemma 2.6, there exists $\tilde{P}_B \in \Gamma_{\ell\#-2}(z_B, f, C)$ with $\tilde{P}_B - P_{B'} \in C' \cdot \mathcal{B}_{z_B,\operatorname{diam}(B)}$. Furthermore, since $\tilde{P}_B \in \Gamma_{\ell\#-2}(z_B, f, C)$ and $P_B \in \Gamma_{\ell\#-1}(z_B, f, C) \subset \Gamma_{\ell\#-2}(z_B, f, C)$, we have

$$\tilde{P}_B - P_B \in 2C \cdot \sigma_{\ell\#-2}(z_B) = 2C \cdot \sigma_{\ell^*}(z_B),$$

where we have used the fact that $\ell^\# - 2 \geq \overline{\ell} - 2 = \ell^*$. Thus,

$$P_B - P_{B'} = (P_B - \tilde{P}_B) + (\tilde{P}_B - P_{B'}) \in 2C \cdot \sigma_{\ell^*}(z_B) + C' \cdot \mathcal{B}_{z_B,\operatorname{diam}(B)}, \text{ and hence}$$
$$P_B - P_{B'} \in (2C \cdot \sigma_{\ell^*}(z_B) + C' \cdot \mathcal{B}_{z_B,\delta_B}) \cap V \subset C'' \cdot (\sigma_{\ell^*}(z_B) + \mathcal{B}_{z_B,\operatorname{diam}(B)}) \cap V.$$

Since $\sigma_{\ell^*}(z_B)$ is $(z_B, \operatorname{diam}(B), R^*)$-transverse to V (see Lemma 8.2), also $\sigma_{\ell^*}(z_B) + \mathcal{B}_{z_B,\operatorname{diam}(B)}$ is $(z_B, \operatorname{diam}(B), R^{**})$-transverse to V, with $R^{**} = R^* + 3 \cdot (R^*)^2$ (see Lemma 3.3). In particular,

$$(\sigma_{\ell^*}(z_B) + \mathcal{B}_{z_B,\operatorname{diam}(B)}) \cap V \subset R^{**} \cdot \mathcal{B}_{z_B,\operatorname{diam}(B)}.$$

Therefore, $P_B - P_{B'} \in C''R^{**} \cdot \mathcal{B}_{z_B,\operatorname{diam}(B)}$, which concludes the proof of the lemma.

□

8.1 Finiteness principles for set unions with weakly controlled constants

Through the use of Lemma 2.16 and Helly's theorem we will obtain the following result: If a ball $\widehat{B} \subset \mathbb{R}^n$ is covered by a collection of balls each of which satisfies a Local Finiteness Principle, then $\widehat{B}$ satisfies a Local Finiteness Principle with constants that may depend on the cardinality of the cover. We should remark that we lack any control on the cardinality of the cover $\mathcal{W}_0$ of B_0, and so this type of result cannot be used to obtain a Local Finiteness Principle on B_0 with any control on the constants. This lemma will be used in the next subsection, however, to obtain a local finiteness principle on a family of intermediate balls that are much larger than the balls of the cover, yet small when compared to B_0.

Lemma 8.3. *Fix $C_0 \geq 1$ and $\ell_0 \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{W}$ be a Whitney cover of a ball $\widehat{B} \subset \mathbb{R}^n$ with cardinality $N = \#\mathcal{W}$. If the Local Finiteness Principle holds on $\frac{6}{5}B$ with constants C_0 and ℓ_0, for all $B \in \mathcal{W}$, then the Local Finiteness Principle holds on $\widehat{B}$ with constants C_1 and $\ell_1 := \ell_0 + \lceil \frac{\log(D\cdot N+1)}{\log(D+1)} \rceil$, where C_1 depends only on C_0, m, and n – in particular, C_1 is independent of the cardinality N of the cover.*

Proof. Let $f : E \to \mathbb{R}$ and $M > 0$. By assumption, $\Gamma_{\ell_0}(x, f, M) \subset \Gamma_{E\cap\frac{6}{5}B}(x, f, C_0M)$ for all $x \in \frac{6}{5}B$, $B \in \mathcal{W}$. Fix a point $x_0 \in \widehat{B}$. Our goal is to prove that

$$\Gamma_{\ell_1}(x_0, f, M) \subset \Gamma_{E\cap\widehat{B}}(x_0, f, C_1M), \tag{42}$$

for a constant $C_1 \geq 1$, to be determined later.

For each $B \in \mathcal{W}$, we fix $x_B \in \frac{6}{5}B$. We demand that

$$x_B = x_0 \iff x_0 \in (6/5)B; \tag{43}$$

otherwise, x_B is an arbitrary element of $\frac{6}{5}B$.

Fix an arbitrary element $P \in \Gamma_{\ell_1}(x_0, f, M)$. We will define a family of auxiliary convex sets to which we will apply Helly's theorem and obtain the desired conclusion. The convex sets will belong to the vector space $\mathcal{P}^N$ consisting of N-tuples of $(m-1)$-st order Taylor polynomials indexed by the elements of the cover $\mathcal{W}$. For each $S \subset E$, define the convex set

$$\mathcal{K}(S, M) := \{(J_{x_B}F)_{B\in\mathcal{W}} : F \in C^{m-1,1}(\mathbb{R}^n),\ \|F\| \leq M,\ F = f \text{ on } S,\ J_{x_0}F = P\} \subset \mathcal{P}^N.$$

If $\#(S) \leq (D+1)^{\ell_1}$ then $P \in \Gamma_{\ell_1}(x_0, f, M) \subset \Gamma_S(x_0, f, M)$. Thus, there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ with $\|F\| \leq M$, $F = f$ on S, and $J_{x_0}F = P$. Therefore, $(J_{x_B}F)_{B\in\mathcal{W}} \in \mathcal{K}(S, M)$. In particular, $\mathcal{K}(S, M) \neq \emptyset$ if $\#(S) \leq (D+1)^{\ell_1}$.

If $S_1, \cdots, S_J \subset E$, with $J := \dim(\mathcal{P}^N) + 1 = D \cdot N + 1$, then

$$\bigcap_{j=1}^{J} \mathcal{K}(S_j, M) \supset \mathcal{K}(S, M), \text{ for } S = S_1 \cup \cdots \cup S_J.$$

If furthermore $\#(S_j) \le (D+1)^{\ell_0}$ for every j, then $\#(S) \le J \cdot (D+1)^{\ell_0} \le (D+1)^{\ell_1}$, and consequently by the previous remark $\mathcal{K}(S,M) \neq \emptyset$. Thus we have shown

$$\bigcap_{j=1}^{J} \mathcal{K}(S_j, M) \neq \emptyset, \text{ if } S_1, \cdots, S_J \subset E, \ J = \dim(\mathcal{P}^N) + 1, \text{ and } \#(S_j) \le (D+1)^{\ell_0} \text{ for all } j.$$

Therefore, by Helly's theorem,

$$\mathcal{K} := \bigcap_{\substack{S \subset E \\ \#(S) \le (D+1)^{\ell_0}}} \mathcal{K}(S,M) \neq \emptyset.$$

Fix an arbitrary element $(P_B)_{B \in \mathcal{W}} \in \mathcal{K}$. By definition of $\mathcal{K}$,
$(*)$ for any $S \subset E$ with $\#(S) \le (D+1)^{\ell_0}$ there exists a function $F^S \in C^{m-1,1}(\mathbb{R}^n)$ with $\|F^S\| \le M$, $F^S = f$ on S, $J_{x_0} F^S = P$, and $J_{x_B} F^S = P_B$ for all $B \in \mathcal{W}$. From this condition we will establish the following properties:
(a) $P_B = P$ if $x_0 \in (6/5)B$,
(b) $|P_B - P_{B'}|_{x_B, \mathrm{diam}(B)} \le CM$ whenever $\frac{6}{5}B \cap \frac{6}{5}B' \neq \emptyset$,
(c) for each $B \in \mathcal{W}$ there exists $F_B \in C^{m-1,1}(\mathbb{R}^n)$ such that $\|F_B\| \le C_0 M$, $F_B = f$ on $E \cap \frac{6}{5}B$, and $J_{x_B} F_B = P_B$.

For the proof of (a) and (b) take $S = \emptyset$ in $(*)$. Then $P_B = J_{x_B} F^{\emptyset} = J_{x_0} F^{\emptyset} = P$ whenever $x_0 \in \frac{6}{5}B$ (see (43)), which yields (a). For (b), note that $x_B \in \frac{6}{5}B$, $x_{B'} \in \frac{6}{5}B'$, and $\frac{6}{5}B \cap \frac{6}{5}B' \neq \emptyset$, and hence by the definition of Whitney covers, $\mathrm{diam}(B)$ and $\mathrm{diam}(B')$ differ by a factor of at most 8. Thus, $|x_B - x_{B'}| \le \frac{6}{5}\mathrm{diam}(B) + \frac{6}{5}\mathrm{diam}(B') \le 11\,\mathrm{diam}(B)$. Thus, by (2) and Taylor's theorem (see (3)),

$$\begin{aligned} |P_B - P_{B'}|_{x_B, \mathrm{diam}(B)} \le 11^m |P_B - P_{B'}|_{x_B, 11\,\mathrm{diam}(B)} &= 11^m |J_{x_B} F^{\emptyset} - J_{x_{B'}} F^{\emptyset}|_{x_B, 11\,\mathrm{diam}(B)} \\ &\le 11^m C_T \|F^{\emptyset}\| \le CM. \end{aligned}$$

For the proof of (c), note that $(*)$ implies $P_B \in \Gamma_{\ell_0}(x_B, f, M)$ for each $B \in \mathcal{W}$. By assumption, the Local Finiteness Principle holds on $\frac{6}{5}B$ with constants C_0 and ℓ_0, and therefore $P_B \in \Gamma_{E \cap \frac{6}{5}B}(x_B, f, C_0 M)$ for each $B \in \mathcal{W}$. This completes the proof of (c).

Fix a partition of unity $\{\theta_B\}$ adapted to the Whitney cover $\mathcal{W}$ as in Lemma 2.15, and set $F = \sum_{B \in \mathcal{W}} \theta_B F_B$. By use of properties (b) and (c), we conclude via Lemma 2.16 that (A) $\|F\|_{C^{m-1,1}(\widehat{B})} \le CM$ and (B) $F = f$ on $E \cap \widehat{B}$. Since $\mathrm{supp}\,\theta_B \subset \frac{6}{5}B$, we learn that $J_{x_0}\theta_B = 0$ if $x_0 \notin \frac{6}{5}B$; on the other hand, $J_{x_0} F_B = J_{x_B} F_B = P_B = P$ if $x_0 \in \frac{6}{5}B$ (see (43)). Thus, if we compare the following sums term-by-term, we obtain the identity

$$J_{x_0} F = \sum_{B \in \mathcal{W}} J_{x_0}\theta_B \odot_{x_0} J_{x_0} F_B = \sum_{B \in \mathcal{W}} J_{x_0}\theta_B \odot_{x_0} P.$$

Recall that $\sum_{B \in \mathcal{W}} \theta_B = 1$ on $\widehat{B}$ and $x_0 \in \widehat{B}$. Thus, $\sum_{B \in \mathcal{W}} J_{x_0}\theta_B = J_{x_0}(1) = 1$. Therefore, (C) $J_{x_0} F = P$. By a standard technique we extend the function $F \in C^{m-1,1}(\widehat{B})$ to a function in $C^{m-1,1}(\mathbb{R}^n)$ with norm bounded by $C\|F\|_{C^{m-1,1}(\widehat{B})} \le C'M$ – by abuse of notation, we denote this extension by the same symbol F. Then (D) $\|F\| \leq C'M$. Furthermore, (B) and (C) continue to hold for this extension. From (B),(C), and (D) we conclude that $P \in \Gamma_{E\cap\widehat{B}}(x_0, f, C'M)$. This finishes the proof of (42).

$\square$

8.2 Proof of Lemma 8.2

We need to generate an upper containment on $\sigma_\ell(x) \cap V$ for a suitable integer constant ℓ. Recall from property (c) in Lemma 7.1, $\sigma(x) \cap V \subset R_{\text{huge}} \cdot \mathcal{B}_{x,\text{diam}(B)}$ for $x \in 8B$ and $B \in \mathcal{W}$. To generate a similar containment for $\sigma_\ell(x) \supset \sigma(x)$ we introduce the idea of "keystone balls" which are elements of the cover for which we may obtain a local finiteness principle on a dilate of the balls by a large constant factor (much larger than the constants C, C_*, R_{huge}, etc.). By an appropriate choice of this factor, we can deduce information about the shape of $\sigma_\ell(x)$ (through the existence of a transverse subspace) on a neighborhood of a keystone ball. This information can then be passed along to the remaining elements of the cover due to the "quasicontinuity" of the sets $\sigma_\ell(x)$ (Lemma 2.6) and the fact that every ball is close to a keystone ball (as established in Lemma 8.6).

8.2.1 Keystone balls

Let $\epsilon^* \in (0, \frac{1}{300}]$ be a free parameter, which will later be fixed to be a small enough constant determined by m and n. In what follows all constants may depend on m and n. If a constant depends additionally on ϵ^* we will be explicit and write it as $C(\epsilon^*)$, $C_0(\epsilon^*)$, etc. Set $A = (3\epsilon^*)^{-\frac{1}{2}} \geq 10$.

By hypothesis of Lemma 8.2, $\text{diam}(\widehat{B}) \leq \epsilon^* \,\text{diam}(B_0)$ for some $\widehat{B} \in \mathcal{W}_0$.

Definition 8.4. *A ball $B^\# \in \mathcal{W}$ is keystone if $\text{diam}(B) \geq \frac{1}{2}\text{diam}(B^\#)$ for every $B \in \mathcal{W}$ with $B \cap A \cdot B^\# \neq \emptyset$. Write $\mathcal{W}^\# \subset \mathcal{W}$ to denote the set of all keystone balls.*

Lemma 8.5. *For each ball $B \in \mathcal{W}$ there exists a keystone ball $B^\# \in \mathcal{W}^\#$ with $B^\# \subset 3AB$, $\text{dist}(B, B^\#) \leq 2A\,\text{diam}(B)$, and $\text{diam}(B^\#) \leq \text{diam}(B)$.*

Proof. If B is itself keystone, take $B^\# = B$ to establish the result. Otherwise, let $B_1 = B$. Since B_1 is not keystone there exists $B_2 \in \mathcal{W}$ with $B_2 \cap AB_1 \neq \emptyset$ and $\text{diam}(B_2) < \frac{1}{2}\text{diam}(B_1)$. Similarly, if B_2 is not keystone there exists $B_3 \in \mathcal{W}$ with $B_3 \cap AB_2 \neq \emptyset$ and $\text{diam}(B_3) < \frac{1}{2}\text{diam}(B_2)$. We continue to iterate this process. As $\mathcal{W}$ is finite, the process must terminate after finitely many steps. By iteration, there exists a sequence of balls $B_1, B_2, \cdots, B_J \in \mathcal{W}$ with $B_j \cap AB_{j-1} \neq \emptyset$ and $\text{diam}(B_j) < \frac{1}{2}\text{diam}(B_{j-1})$ for all j, and with B_J keystone. As $B_j \cap AB_{j-1} \neq \emptyset$ we have $\text{dist}(B_{j-1}, B_j) \leq \frac{A}{2}\text{diam}(B_{j-1})$. Now estimate

$$\begin{aligned}\text{dist}(B_1, B_J) &\leq \sum_{j=2}^{J} \text{dist}(B_{j-1}, B_j) + \sum_{j=2}^{J-1} \text{diam}(B_j) \leq (A/2+1)\sum_{j=1}^{J}\text{diam}(B_j)\\ &\leq (A+2)\,\text{diam}(B_1) \leq 2A\,\text{diam}(B_1).\end{aligned}$$

Since $\operatorname{diam}(B_J) \le \operatorname{diam}(B_1)$, we have $B_J \subset (2A+6)B_1 \subset 3AB_1$. We set $B^\# = B_J$ and this finishes the proof. $\square$

We define a mapping $\kappa : \mathcal{W}_0 \to \mathcal{W}^\#$. By applying Lemma 8.5, we obtain a keystone ball $\widehat{B}^\#$ with $\widehat{B}^\# \subset 3A\widehat{B}$ and $\operatorname{diam}(\widehat{B}^\#) \le \operatorname{diam}(\widehat{B})$. For each $B \in \mathcal{W}_0$, we proceed as follows:

- If $\operatorname{diam}(B) > \epsilon^* \operatorname{diam}(B_0)$ (B is *medium-sized*), set $\kappa(B) := \widehat{B}^\#$.
- If $\operatorname{diam}(B) \le \epsilon^* \operatorname{diam}(B_0)$ (B is *small-sized*), Lemma 8.5 yields a keystone ball $B^\#$ with $B^\# \subset 3AB$; set $\kappa(B) := B^\#$.

Lemma 8.6. *The mapping $\kappa : \mathcal{W}_0 \to \mathcal{W}^\#$ satisfies the following properties: For any $B \in \mathcal{W}_0$, (a) $\operatorname{dist}(B, \kappa(B)) \le C_4 \operatorname{diam}(B)$, for $C_4 = C_4(\epsilon^*)$, (b) $\operatorname{diam}(\kappa(B)) \le \operatorname{diam}(B)$, and (c) $A \cdot \kappa(B) \subset 2B_0$.*

Proof. Suppose B is medium-sized. Then $\kappa(B) = \widehat{B}^\#$. Since $\operatorname{diam}(B) > \epsilon^* \operatorname{diam}(B_0)$ and $B \subset B_0$ we have $9(\epsilon^*)^{-1}B \supset B_0 \supset \widehat{B}$; furthermore, $\widehat{B}^\# \subset 3A\widehat{B}$. Thus, $\widehat{B}^\# \subset 27(\epsilon^*)^{-1}AB$, which gives (a) for $C_4 = 27(\epsilon^*)^{-1}A$. Also, $\operatorname{diam}(\widehat{B}^\#) \le \operatorname{diam}(\widehat{B}) \le \epsilon^* \operatorname{diam}(B_0) < \operatorname{diam}(B)$, which establishes (b). Finally, since $\widehat{B} \subset B_0$ and $\operatorname{diam}(\widehat{B}) \le \epsilon^* \operatorname{diam}(B_0)$, we have $A\widehat{B}^\# \subset 3A^2\widehat{B} \subset (1 + 3\epsilon^* A^2)B_0 = 2B_0$, which gives (c).

Now suppose B is small-sized. Then we defined $\kappa(B) = B^\#$, where $B^\#$ is related to B as in Lemma 8.5. In particular, $\operatorname{dist}(B, B^\#) \le 2A \operatorname{diam}(B)$ and $\operatorname{diam}(B^\#) \le \operatorname{diam}(B)$, yielding (a) and (b). Furthermore, $B^\# \subset 3AB$, and from $B \subset B_0$ and $\operatorname{diam}(B) \le \epsilon^* \operatorname{diam}(B_0)$ we deduce that $AB^\# \subset 3A^2B \subset (1 + 3\epsilon^* A^2)B_0 = 2B_0$, yielding (c). $\square$

This completes the description of the geometric relationship between the balls of $\mathcal{W}_0$ and keystone balls in $\mathcal{W}$. We will next need a lemma about the shape of $\sigma_\ell(z_{B^\#})$ for a keystone ball $B^\#$.

Lemma 8.7. *Let $B^\# \in \mathcal{W}$ be a keystone ball with $AB^\# \subset 2B_0$. Then there exists an integer constant $\ell(\epsilon^*) > \ell_{old}$, determined by ϵ^*, m, and n, and a constant $C \ge 1$ determined by m and n, so that the Local Finiteness Principle holds on $AB^\#$ with constants C and $\ell(\epsilon^*)$, namely, $\Gamma_{\ell(\epsilon^*)}(x, f, M) \subset \Gamma_{E \cap AB^\#}(x, f, CM)$ for all $x \in AB^\#$ and $M > 0$. In particular, by taking $f = 0$ and $M = 1$, we have $\sigma_{\ell(\epsilon^*)}(x) \subset C \cdot \sigma(x, E \cap AB^\#)$ for any $x \in AB^\#$.*

Proof. Let $\mathcal{W}(B^\#)$ be the collection of all elements of $\mathcal{W}$ that intersect $AB^\#$. Since $\mathcal{W}$ is a Whitney cover of $2B_0$ and $AB^\# \subset 2B_0$, we have that $\mathcal{W}(B^\#)$ is a Whitney cover of $AB^\#$. The Local Finiteness Principle holds on $\frac{6}{5}B$ for all $B \in \mathcal{W}(B^\#)$, with constants C_{old} and ℓ_{old} (see Lemma 7.2). Therefore, the Local Finiteness Principle holds on $AB^\#$ with the constant C_1 determined by m and n, and the constant $\ell_1 = \ell_{\text{old}} + \lceil \frac{\log(D \cdot N+1)}{\log(D+1)} \rceil$, where $N = \#\mathcal{W}(B^\#)$; see Lemma 8.3.

We will estimate $N = \#\mathcal{W}(B^\#)$ using a volume comparison bound. By the definition of keystone balls, $\operatorname{diam}(B) \ge \frac{1}{2} \operatorname{diam}(B^\#)$ for all $B \in \mathcal{W}(B^\#)$ – furthermore, we claim that $\operatorname{diam}(B) \le 10A \operatorname{diam}(B^\#)$. We proceed by contradiction. If $\operatorname{diam}(B) > 10A \operatorname{diam}(B^\#)$ for some $B \in \mathcal{W}(B^{\#})$ then $B \cap AB^{\#} \neq \emptyset$, which implies that $(6/5)B \cap B^{\#} \neq \emptyset$. Then $\operatorname{diam}(B) \leq 8 \operatorname{diam}(B^{\#})$ thanks to the definition of a Whitney cover, which gives a contradiction.

For any $B \in \mathcal{W}(B^{\#})$ we have $B \cap AB^{\#} \neq \emptyset$ and $\operatorname{diam}(B) \leq 10A \operatorname{diam}(B^{\#})$, and therefore $B \subset 30AB^{\#}$.

We can estimate the volume of $\Omega := \bigcup_{B \in \mathcal{W}(B^{\#})} \frac{1}{3}B$ in two ways. First, note that $\operatorname{Vol}(\Omega) \leq \operatorname{Vol}(30AB^{\#}) = (30A)^n \operatorname{Vol}(B^{\#})$. Next, using that the collection $\{\frac{1}{3}B\}_{B \in \mathcal{W}}$ is pairwise disjoint, $N = \#\mathcal{W}(B^{\#})$, and $\operatorname{diam}(B) \geq \frac{1}{2} \operatorname{diam}(B^{\#}) \forall B \in \mathcal{W}(B^{\#})$, we have

$$\operatorname{Vol}(\Omega) = \sum_{B \in \mathcal{W}(B^{\#})} 3^{-n} \operatorname{Vol}(B) \geq N 6^{-n} \operatorname{Vol}(B^{\#}).$$

Hence, $N \leq (180A)^n \leq 180^n (\epsilon^*)^{-\frac{n}{2}}$. Thus, $\ell_1 \leq \ell(\epsilon^*) := \ell_{\text{old}} + \lceil \frac{\log(D \cdot 180^n (\epsilon^*)^{-\frac{n}{2}} + 1)}{\log(D+1)} \rceil$. □

Lemma 8.8. *If the parameter ϵ^* is picked sufficiently small depending on m and n, and if $A = (3\epsilon^*)^{-\frac{1}{2}}$ in the definition of keystone balls, then for any keystone ball $B^{\#} \in \mathcal{W}^{\#}$ such that $AB^{\#} \subset 2B_0$, we have*

$$\sigma_\ell(z_{B^{\#}}) \cap V \subset \mathcal{B}_{z_{B^{\#}}, A \operatorname{diam}(B^{\#})}, \qquad \textit{for } \ell = \ell(\epsilon^*) > \ell_{old}.$$

Proof. By Lemma 8.7, and Lemma 2.9 (applied to the ball $AB^{\#}$ and point $z = z_{B^{\#}} \in \frac{1}{2}AB^{\#}$),

$$\sigma_{\ell(\epsilon^*)}(z_{B^{\#}}, E) \cap \mathcal{B}_{z_{B^{\#}}, A \operatorname{diam}(B^{\#})} \subset (C\sigma(z_{B^{\#}}, E \cap AB^{\#})) \cap \mathcal{B}_{z_{B^{\#}}, A \operatorname{diam}(B^{\#})} \subset C_3 \sigma(z_{B^{\#}}, E),$$

for a constant C_3 determined by m and n. Dropping the dependence on E, we have shown that

$$\sigma_{\ell(\epsilon^*)}(z_{B^{\#}}) \cap \mathcal{B}_{z_{B^{\#}}, A \operatorname{diam}(B^{\#})} \subset C_3 \sigma(z_{B^{\#}}). \tag{44}$$

By property (c) in Lemma 7.1, $\sigma(z_{B^{\#}})$ is $(z_{B^{\#}}, C_* \operatorname{diam}(B^{\#}), R_{\text{huge}})$-transverse to V. Hence, $\sigma(z_{B^{\#}}) \cap V \subset R_{\text{huge}} \mathcal{B}_{z_{B^{\#}}, C_* \operatorname{diam}(B^{\#})} \subset \widehat{R} \mathcal{B}_{z_{B^{\#}}, \operatorname{diam}(B^{\#})}$, for $\widehat{R} = R_{\text{huge}}(C_*)^m$. Combined with (44), this implies

$$\sigma_\ell(z_{B^{\#}}) \cap V \cap \mathcal{B}_{z_{B^{\#}}, A \operatorname{diam}(B^{\#})} \subset C_3 \sigma(z_{B^{\#}}) \cap V \subset C_3 \widehat{R} \mathcal{B}_{z_{B^{\#}}, \operatorname{diam}(B^{\#})} \subset \mathcal{B}_{z_{B^{\#}}, C_3 \widehat{R} \operatorname{diam}(B^{\#})}.$$

As long as $A = (3\epsilon^*)^{-\frac{1}{2}} \geq 2C_3 \widehat{R}$, this implies $\sigma_\ell(z_{B^{\#}}) \cap V \subset \mathcal{B}_{z_{B^{\#}}, C_3 \widehat{R} \operatorname{diam}(B^{\#})} \subset \mathcal{B}_{z_{B^{\#}}, A \operatorname{diam}(B^{\#})}$. □

8.2.2 Finishing the proof of Lemma 8.2

We fix $A = (3\epsilon^*)^{-\frac{1}{2}}$, $\epsilon^* = \epsilon^*(m, n)$, and $\ell = \ell(\epsilon^*) > \ell_{\text{old}}$ via Lemma 8.8. The constants ϵ^*, A and ℓ are determined only by m and n.

Fix $B \in \mathcal{W}_0$ and $x \in 6B$. Consider the keystone ball $B^{\#} \in \mathcal{W}$ given by $B^{\#} = \kappa(B)$ which satisfies the conditions in Lemma 8.6, namely, $\operatorname{diam}(B^{\#}) \leq \operatorname{diam}(B)$, $\operatorname{dist}(B^{\#}, B) \leq C_4 \operatorname{diam}(B)$, and $AB^{\#} \subset 2B_0$. By Lemma 8.8,

$$\sigma_\ell(z_{B^{\#}}) \cap V \subset \mathcal{B}_{z_{B^{\#}}, A \operatorname{diam}(B^{\#})} \subset A^m \mathcal{B}_{z_{B^{\#}}, \operatorname{diam}(B^{\#})} \subset A^m \mathcal{B}_{z_{B^{\#}}, \operatorname{diam}(B)}. \tag{45}$$

Now, note that $|x - z_{B\#}| \le 6\,\mathrm{diam}(B) + \mathrm{dist}(B, B^{\#}) + \frac{6}{5}\mathrm{diam}(B^{\#}) \le C_5\,\mathrm{diam}(B)$ for $C_5 = C_4 + 8$. Thus, Lemma 2.6 gives

$$\sigma_{\ell+1}(x) \subset \sigma_\ell(z_{B\#}) + C_T \mathcal{B}_{z_{B\#}, C_5\,\mathrm{diam}(B)} \subset \sigma_\ell(z_{B\#}) + C_T C_5^m \mathcal{B}_{z_{B\#},\mathrm{diam}(B)}.$$

Property (c) in the Main Decomposition Lemma states that $\sigma(z_{B\#})$ is $(z_{B\#}, C_*\delta, R_{\text{huge}})$-transverse to V for all $\delta \in [\mathrm{diam}(B^{\#}), \mathrm{diam}(B_0)]$. We take $\delta = \mathrm{diam}(B)$ in this statement, and apply Lemma 3.7 to deduce that $\sigma(z_{B\#})$ is $(z_{B\#}, \mathrm{diam}(B), R_1)$-transverse to V, for $R_1 = R_{\text{huge}} \cdot (C_*)^m$. Thus, in particular,

$$\mathcal{B}_{z_{B\#},\mathrm{diam}(B)}/V \subset R_1 \cdot (\sigma(z_{B\#}) \cap \mathcal{B}_{z_{B\#},\mathrm{diam}(B)})/V \subset R_1 \cdot (\sigma_\ell(z_{B\#}) \cap \mathcal{B}_{z_{B\#},\mathrm{diam}(B)})/V.$$

Combined with (45), this shows that $\sigma_\ell(z_{B\#})$ is $(z_{B\#}, \mathrm{diam}(B), R_2)$-transverse to V, for $R_2 = \max\{R_1, A^m\}$. Furthermore, by Lemma 3.3, $\sigma_\ell(z_{B\#}) + C_T C_5^m \mathcal{B}_{z_{B\#},\mathrm{diam}(B)}$ is $(z_{B\#}, \mathrm{diam}(B), R_3)$-transverse to V, for $R_3 = R_2 + 3R_2^2 C_T C_5^m$. We conclude that

$$\sigma_{\ell+1}(x) \cap V \subset (\sigma_\ell(z_{B\#}) + C_T C_5^m \mathcal{B}_{z_{B\#},\mathrm{diam}(B)}) \cap V \subset R_3 \mathcal{B}_{z_{B\#},\mathrm{diam}(B)} \subset R_4 \mathcal{B}_{x,\mathrm{diam}(B)}, \quad (46)$$

for $R_4 = R_3 \widetilde{C} C_5^{m-1}$. Here, (6) and $|x - z_{B\#}| \le C_5\,\mathrm{diam}(B)$ are used to obtain the last containment.

On the other hand, property (c) of the Main Decomposition Lemma shows that $\sigma(x)$ is $(x, 6C_*\,\mathrm{diam}(B), R_{\text{huge}})$-transverse to V, and hence $\sigma(x)$ is $(x, \mathrm{diam}(B), R_1)$-transverse to V, for $R_1 = R_{\text{huge}} \cdot (6C_*)^m$ (by Lemma 3.7). In particular,

$$\mathcal{B}_{x,\mathrm{diam}(B)}/V \subset R_1 \cdot (\sigma(x) \cap \mathcal{B}_{x,\mathrm{diam}(B)})/V \subset R_1 \cdot (\sigma_{\ell+1}(x) \cap \mathcal{B}_{x,\mathrm{diam}(B)})/V. \quad (47)$$

Combining (46) and (47), we see that $\sigma_{\ell+1}(z_B)$ is $(z_B, \mathrm{diam}(B), \max\{R_1, R_4\})$-transverse to V. This finishes the proof of Lemma 8.2, with $\epsilon^* = \epsilon^*(m,n)$, $\ell^* = \ell(\epsilon^*) + 1$, and $R^* = \max\{R_1, R_4\}$.

9 The Main Induction Argument III: Putting it all together

Here we finish the proof of the containment (37). Namely, for suitable constants $\ell^{\#} \in \mathbb{Z}_{\ge 0}$ and $C^{\#} \ge 1$, we will prove

$$\Gamma_{\ell^{\#}}(x_0, f, 1) \subset \Gamma_{E \cap B_0}(x_0, f, C^{\#}), \quad \text{for all } x_0 \in B_0,\ f : E \to \mathbb{R}.$$

This will conclude the proof of the Local Finiteness Principle on B_0, and complete the Main Induction Argument.

Continuing with the argument outlined in the beginning of section 8, we fix $P_0 \in \Gamma_{\ell^{\#}}(x_0, f, 1)$. We apply the Main Decomposition Lemma to the data $(x_0, B_0, E, f, \ell^{\#}, P_0)$ to obtain a Whitney cover $\mathcal{W}$ of $2B_0$, a DTI subspace $V \subset \mathcal{P}$, and families $\{P_B\}_{B \in \mathcal{W}}$ and $\{z_B\}_{B \in \mathcal{W}}$. Recall that $\mathcal{W}_0 \subset \mathcal{W}$ is a finite cover of B_0.

We define $\ell^\# = \overline{\ell}$, where $\overline{\ell} > \ell_{\text{old}}$ is defined via Lemma 8.1.

Condition (f) in the Main Decomposition Lemma states that $P_B \in \Gamma_{\ell^\# - 1}(z_B, f, C)$ for all $B \in \mathcal{W}_0$. By Lemma 7.2 and the fact that $\ell^\# - 1 \geq \ell_{\text{old}}$ it follows that $P_B \in \Gamma_{\ell^\# - 1}(z_B, f, C) \subset \Gamma_{\ell_{\text{old}}}(z_B, f, C) \subset \Gamma_{E \cap \frac{6}{5}B}(z_B, f, C \cdot C_{\text{old}})$. So,

$$P_B \in \Gamma_{E \cap \frac{6}{5}B}(z_B, f, C \cdot C_{\text{old}}) \text{ for all } B \in \mathcal{W}.$$

Recall that $z_B \in \frac{6}{5}B$ for all $B \in \mathcal{W}$. By definition of $\Gamma_{E \cap \frac{6}{5}B}(\cdots)$, there exists $F_B \in C^{m-1,1}(\mathbb{R}^n)$ with

$$\begin{cases} F_B = f \text{ on } E \cap (6/5) \cdot B, \ J_{z_B} F_B = P_B, \text{ and} \\ \|F_B\| \leq C \cdot C_{\text{old}}. \end{cases} \tag{48}$$

Since $\ell^\# = \overline{\ell}$, by Lemma 8.1 we conclude that

$$|J_{z_B} F_B - J_{z_{B'}} F_{B'}|_{z_B, \text{diam}(B)} \leq \overline{C} \text{ whenever } B, B' \in \mathcal{W}_0, \ (\frac{6}{5}) \cdot B \cap (\frac{6}{5}) \cdot B' \neq \emptyset. \tag{49}$$

Let $\{\theta_B\}_{B \in \mathcal{W}_0}$ be a partition of unity on B_0 subordinate to the cover $\mathcal{W}_0$, as in Lemma 2.15. Define

$$F = \sum_{B \in \mathcal{W}_0} F_B \theta_B \text{ on } B_0.$$

By Lemma 2.16 (and the conditions (48) and (49)), $F \in C^{m-1,1}(B_0)$ satisfies $\|F\|_{C^{m-1,1}(B_0)} \leq C' \cdot C_{\text{old}}$ and $F = f$ on $E \cap B_0$. Recall the points $\{z_B\}_{B \in \mathcal{W}}$ possess the additional property that $z_B = x_0$ if $x_0 \in \frac{6}{5}B$, and the polynomials $\{P_B\}_{B \in \mathcal{W}}$ possess the additional property that $P_B = P_0$ if $x_0 \in \frac{6}{5}B$ (see condition (e) in Lemma 7.1). Thus, $J_{x_0} F_B = P_0$ whenever $x_0 \in \frac{6}{5}B$. Therefore,

$$\begin{aligned} J_{x_0} F &= \sum_{B \in \mathcal{W}_0 : x_0 \in \frac{6}{5}B} J_{x_0}(F_B \theta_B) = \sum_{B \in \mathcal{W}_0 : x_0 \in \frac{6}{5}B} J_{x_0} F_B \odot_{x_0} J_{x_0} \theta_B \\ &= \sum_{B \in \mathcal{W}_0 : x_0 \in \frac{6}{5}B} P_0 \odot_{x_0} J_{x_0} \theta_B = P_0 \odot_{x_0} 1 = P_0. \end{aligned}$$

We now extend the function F to all of $\mathbb{R}^n$ by a classical extension technique (e.g., Stein's extension theorem). This gives a function $\widehat{F} \in C^{m-1,1}(\mathbb{R}^n)$ with $\|\widehat{F}\| \leq C\|F\|_{C^{m-1,1}(B_0)} \leq C'' \cdot C_{\text{old}}$ and $\widehat{F} = F$ on B_0. In particular, $\widehat{F} = f$ on $E \cap B_0$ and $J_{x_0}\widehat{F} = P_0$ (since $x_0 \in B_0$). Thus, $P_0 \in \Gamma_{E \cap B_0}(x_0, f, C')$. We finally define $C^\# = C'' \cdot C_{\text{old}}$. Since $P_0 \in \Gamma_{\ell^\#}(x_0, f, 1)$ was arbitrary, this finishes the proof of the containment (37).

9.1 The dependence of constants on complexity

In order to obtain the explicit dependence of the constants in Theorem 1.2 on the complexity of E, we will need to track the dependence on K of the constants $\ell^\# = \ell^\#(K)$ and $C^\# = C^\#(K)$ in the Local Main Lemma for K; see Remark 5.7.

It is clear that the constant $C^\# = C^\#(K)$ has the form $C^\# = \text{Const}^K$, for a universal constant Const. Indeed, when we pass from the Local Main Lemma for $K - 1$ to the Local

Main Lemma for K, the constant $C^{\#} = C^{\#}(K)$ takes the form $C^{\#} = C'' \cdot C_{\text{old}}$, where $C_{\text{old}} = C^{\#}(K-1)$ and C'' is a universal constant.

In order to determine the dependence of $\ell^{\#} = \ell^{\#}(K)$ on K, we need to determine how the constant $\overline{\ell}$ in Lemma 8.1 is chosen. In fact, in section 8.2.2 we see that $\overline{\ell}$ is defined to be $\ell(\epsilon^*)$ for a particular choice of ϵ^* determined by m and n. By inspection of the proof of Lemma 8.7, we have $\ell(\epsilon^*) := \ell_{\text{old}} + \lceil \frac{\log(D \cdot 180^n (\epsilon^*)^{-\frac{n}{2}} + 1)}{\log(D+1)} \rceil$. Since we defined $\ell^{\#} = \overline{\ell}$ in the previous section, and since $\ell_{\text{old}} = \ell^{\#}(K-1)$, we learn that $\ell^{\#}$ depends linearly on K.

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