

# Bourgain's slicing problem and KLS isoperimetry up to polylog

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## Abstract

We prove that Bourgain's hyperplane conjecture and the Kannan-Lovász-Simonovits (KLS) isoperimetric conjecture hold true up to a factor that is polylogarithmic in the dimension.

## 1 Introduction

One of the central questions in high-dimensional convex geometry is Bourgain's slicing problem. In its simplest formulation, it asks whether for any convex body  $K \subseteq \mathbb{R}^n$  of volume one, there exists a hyperplane  $H \subseteq \mathbb{R}^n$  such that

$$\text{Vol}_{n-1}(K \cap H) > c,$$

where  $c > 0$  is a universal constant, and  $\text{Vol}_{n-1}$  stands for  $(n-1)$ -dimensional volume. The question originated in Bourgain's work [7, 8], and we refer the reader to [34] and references therein for background on this problem, its implications for convexity and its equivalent formulations. For  $n \geq 1$  define

$$\frac{1}{L_n} := \inf_{K \subseteq \mathbb{R}^n} \sup_{H \subseteq \mathbb{R}^n} \text{Vol}_{n-1}(K \cap H),$$

where the infimum runs over all convex bodies  $K \subseteq \mathbb{R}^n$  of volume one, and the supremum runs over all hyperplanes  $H \subseteq \mathbb{R}^n$ . Thus, a convex body of volume one in  $\mathbb{R}^n$  has a hyperplane section whose  $(n-1)$ -dimensional volume is at least  $1/L_n$ . For decades, the best estimate for  $L_n$  has been the bound  $L_n \leq Cn^{1/4}$  where  $C > 0$  is a universal constant, as proven in Bourgain [9, 10] (up to a logarithmic factor) and in [29]. A recent breakthrough by Chen [15] has led to the bound  $L_n \leq C_\varepsilon n^\varepsilon$  for any  $\varepsilon > 0$ , or more precisely

$$L_n \leq C_1 \exp\left(C_2 \sqrt{\log n} \cdot \sqrt{\log \log(3n)}\right), \quad (1)$$

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where  $C_1, C_2 > 0$  are universal constants. Chen arrives at (1) by exploiting the relation between the slicing problem and the thin shell problem, due to Eldan and Klartag [23]. It is proven in [23] that

$$L_n \leq C\sigma_n, \quad (2)$$

where  $C > 0$  is a universal constant, and where  $\sigma_n$  is the thin-shell constant which we will describe shortly. A probability density  $\rho : \mathbb{R}^n \rightarrow [0, \infty)$  is *log-concave* if the set  $\{\rho > 0\} = \{x \in \mathbb{R}^n ; \rho(x) > 0\}$  is convex, and  $\log \rho$  is concave in  $\{\rho > 0\}$ . A probability measure in  $\mathbb{R}^n$  (or a random vector in  $\mathbb{R}^n$ ) is log-concave if it is supported in an affine subspace of  $\mathbb{R}^n$  and it has a log-concave density in this subspace. For instance, the uniform probability measure on any compact, convex set is log-concave, as well as all Gaussian measures. We say that a probability measure  $\mu$  on  $\mathbb{R}^n$  with finite second moments is *isotropic* if

$$\int_{\mathbb{R}^n} x_i d\mu(x) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} x_i x_j d\mu(x) = \delta_{ij} \quad (i, j = 1, \dots, n), \quad (3)$$

where  $\delta_{ij}$  is Kronecker's delta. Thus  $\mu$  is isotropic when it is centered and its covariance matrix is the identity matrix. A log-concave probability measure has finite moments of all orders (e.g. [11, Lemma 2.2.1]). The convolution of two log-concave probability measures is again log-concave, as follows from the Prékopa-Leindler inequality [11, Theorem 1.2.3] or from the earlier work by Davidovič, Korenbljum and Hacet [17]. The relevance of the class of log-concave distributions to the slicing problem was realized by Ball [3]. The thin-shell constant  $\sigma_\mu > 0$  of an isotropic, log-concave probability measure  $\mu$  in  $\mathbb{R}^n$  is defined via

$$n\sigma_\mu^2 = \text{Var}_\mu(|x|^2), \quad (4)$$

where  $\text{Var}_\mu(f) = \int f^2 d\mu - (\int f d\mu)^2$ . It may be shown that most of the mass of the measure  $\mu$  is located in a spherical shell whose width is at most  $C\sigma_\mu$ , and this estimate for the width is always tight, hence the name *thin-shell constant*, see Anttila, Ball and Perissinaki [1] and Bobkov and Koldobsky [6]. The thin-shell constant is crucial for establishing the Central Limit Theorem for Convex Sets [30, 32], as put forth in [1] following Sudakov [40] and Diaconis and Freedman [19]. The parameter  $\sigma_n$  mentioned above is defined as

$$\sigma_n = \sup_{\mu} \sigma_\mu$$

where the supremum runs over all isotropic, log-concave probability measures  $\mu$  in  $\mathbb{R}^n$ . Earlier bounds for  $\sigma_n$  utilized the Concentration of Measure Phenomenon, see [31], Fleury [25] and Guédon and Milman [26], following Paouris' large deviation principle [39]. More recent advances, due to Eldan [21], Lee and Vempala [37] and Chen [15], deal with the Poincaré constant. The Poincaré constant  $C_P(\mu)$  of a Borel probability measure  $\mu$  in  $\mathbb{R}^n$  is defined as the smallest constant  $C \geq 0$  such that for any locally-Lipschitz function  $f \in L^2(\mu)$ ,

$$\text{Var}_\mu(f) \leq C \cdot \int_{\mathbb{R}^n} |\nabla f|^2 d\mu. \quad (5)$$

The fact that  $\sigma_\mu^2 \leq 4C_P(\mu)$  when  $\mu$  is isotropic is easily proven:

$$n\sigma_\mu^2 = \text{Var}_\mu(|x|^2) \leq C_P(\mu) \int_{\mathbb{R}^n} |2x|^2 d\mu(x) = 4n \cdot C_P(\mu). \quad (6)$$

The Poincaré constant is closely related to the *isoperimetric constant* or the *Cheeger constant* of  $\mu$ . Given a probability measure  $\mu$  in  $\mathbb{R}^n$  with log-concave density  $\rho$ , its isoperimetric constant is

$$\frac{1}{\psi_\mu} = \inf_{A \subseteq \mathbb{R}^n} \left\{ \frac{\int_{\partial A} \rho}{\min\{\mu(A), 1 - \mu(A)\}} \right\}$$

where the infimum runs over all open sets  $A \subseteq \mathbb{R}^n$  with smooth boundary for which  $0 < \mu(A) < 1$ . By the Cheeger inequality [14] and the Buser-Ledoux inequality [12, 36], for any absolutely-continuous, log-concave probability measure  $\mu$  in  $\mathbb{R}^n$ ,

$$\frac{1}{4} \leq \frac{\psi_\mu^2}{C_P(\mu)} \leq 9, \quad (7)$$

where the inequality on the left, Cheeger's inequality, is rather general and does not require log-concavity. Define

$$\psi_n := \sup_{\mu} \psi_\mu \quad (8)$$

where the supremum runs over all isotropic, log-concave probability measures  $\mu$  in  $\mathbb{R}^n$ . Our convention follows Lee and Vempala [37]; the quantity we denote by  $\psi_n$  is denoted by  $G_n$  in Eldan [21] and by  $\psi_n^{-1}$  in Chen [15]. The Kannan-Lovász-Simonovits (KLS) conjecture [28] suggests that  $\psi_n$  is bounded by a universal constant. Thanks to (2), (6) and (7) we have the chain of inequalities

$$L_n \leq C\sigma_n \leq \tilde{C}\psi_n, \quad (9)$$

where  $C, \tilde{C} > 0$  are universal constants. The right-hand side inequality in (9) may be reversed, up to a logarithmic factor. A deep theorem by Eldan [21] indeed states that

$$\psi_n \leq \tilde{C}\sigma_n \cdot \log n. \quad (10)$$

In [15], Chen uses *Eldan's stochastic localization* [21] and the analysis of Lee and Vempala [37] in order to show that

$$\psi_n \leq C_1 \exp\left(C_2 \sqrt{\log n} \cdot \sqrt{\log \log(3n)}\right), \quad (11)$$

where  $C_1, C_2 > 0$  are universal constants. This bound implies (1), in view of (9). The bound in (11) grows slower than any power law, and it is natural to expect that this bound for  $\psi_n$  may be improved to a polylogarithmic one. This is indeed true, as we show in this paper:

**Theorem 1.1.** *For any  $n \geq 2$ ,*

$$\psi_n \leq C(\log n)^\alpha$$

*for some universal constants  $C, \alpha > 0$ . Our proof yields  $\alpha = 5$ .*

More precisely, we actually prove that  $\sigma_n \leq C(\log n)^4$ . This implies Theorem 1.1, according to (10). Moreover, thanks to (2) we conclude from our bound for  $\sigma_n$  that

$$L_n \leq C'(\log n)^4. \quad (12)$$

As in Lee-Vempala [37] and Chen [15], our argument relies on Eldan's stochastic localization, with the new ingredients being the functional-analytic approach from Klartag and

Putterman [35], as well as an  $H^{-1}$ -inequality from [32]. In our proof we analyze the evolution of a log-concave measure  $\mu$  along the heat flow  $(P_s)_{s \geq 0}$  in  $\mathbb{R}^n$  using the functional-analytic formalism from [35]. Eldan's stochastic localization enters the picture through the time reversal  $t := 1/s$ , and it allows us to exploit the isotropicity of  $\mu$ .

Throughout this note, the letters  $c, C, \tilde{c}, C_1, c_2$  etc. denote positive universal constants, whose value may change from one line to the next. We usually use lower-case  $c, \tilde{c}$  to denote universal constants that we view as sufficiently small, while  $C, \hat{C}$  etc. usually denote constants which we view as sufficiently large. Since this paper is concerned with asymptotics in the dimension  $n$ , we may assume in the proof that  $n$  exceeds a certain universal constant. We write  $x \cdot y = \langle x, y \rangle = \sum_i x_i y_i$  for the standard scalar product between  $x, y \in \mathbb{R}^n$  and  $|x| = \sqrt{\langle x, x \rangle}$ . For symmetric matrices  $A, B \in \mathbb{R}^{n \times n}$  we write  $A \leq B$  if  $B - A$  is positive semi-definite. By a smooth function we mean  $C^\infty$ -smooth, and  $\log$  is the natural logarithm.

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## 2 Log-concave measures along the heat flow

Let  $\mu$  be an isotropic, log-concave probability measure in  $\mathbb{R}^n$  with a smooth, positive density. We begin by recalling the setting of Klartag and Putterman [35] as well as the basic properties of the Laplace operator that is associated with  $\mu$ . For  $s > 0$ , we write  $\gamma_s$  for the density of a Gaussian random vector of mean zero and covariance  $s \cdot \text{Id}$  in  $\mathbb{R}^n$ . Denote

$$\mu_s = \mu * \gamma_s,$$

the convolution of  $\mu$  and  $\gamma_s$ , with  $\mu_0 = \mu$ . The heat operator  $P_s f = f * \gamma_s$  is a contraction from  $L^2(\mu_s)$  to  $L^2(\mu)$ . The adjoint operator  $Q_s = P_s^* : L^2(\mu) \rightarrow L^2(\mu_s)$  satisfies

$$Q_s \varphi = \frac{P_s(\varphi \rho)}{P_s \rho} \tag{13}$$

where  $\rho$  is the log-concave density of  $\mu$ . We define  $Q_s \varphi$  via formula (13) for all  $s > 0$  and  $\varphi \in L^1(\mu)$ . We set  $P_0 = \text{Id}$  and  $Q_0 = \text{Id}$ . The Laplace operator associated with  $\mu$  is the operator  $L = L_\mu$ , initially defined for compactly-supported smooth functions via the formula

$$Lu = \Delta u + \nabla(\log \rho) \cdot \nabla u.$$

By integration by parts, it follows that for any two smooth functions  $u, v : \mathbb{R}^n \rightarrow \mathbb{R}$ , if one of them is compactly-supported then

$$\int_{\mathbb{R}^n} (Lu)v d\mu = - \int_{\mathbb{R}^n} (\nabla u \cdot \nabla v) d\mu. \tag{14}$$

It is proven e.g. in [2, Corollary 3.2.2] that  $L$  is essentially self-adjoint in  $L^2(\mu)$  and negative semi-definite. We may thus extend the domain of definition of  $L$ , and from now on we denote by  $L$  the closure in  $L^2(\mu)$  of the operator previously denoted by  $L$ . Note that the self-adjoint

operator  $L$  has a simple eigenvalue at 0 corresponding to the constant eigenfunction. We refer the reader to [2, 18, 20] for standard background on spectral theory. By the spectral theorem (e.g., [2, Theorem A.4.2] or [20, Theorem 11.5.1]) we may write

$$-L = \int_{-\infty}^{\infty} \lambda dE_\lambda \quad (15)$$

for a certain increasing, right-continuous family of orthogonal projections  $(E_\lambda)_{\lambda \in \mathbb{R}}$  with  $\lim_{\lambda \rightarrow \infty} E_\lambda = \text{Id}$  and  $\lim_{\lambda \rightarrow -\infty} E_\lambda = 0$  in the sense of strong convergence of operators. In particular,

$$E_0 f(x) = \int_{\mathbb{R}^n} f d\mu \quad \text{for all } x \in \mathbb{R}^n. \quad (16)$$

For  $f \in L^2(\mu)$  we write  $\nu_f$  for the Borel measure on  $\mathbb{R}$  that satisfies  $\nu_f((a, b]) = \langle E_b f, f \rangle - \langle E_a f, f \rangle$  for all  $a < b$ , i.e., the spectral measure of  $f$ . Thus

$$\nu_f(\mathbb{R}) = \|f\|_{L^2(\mu)}^2.$$

The Poincaré constant  $C_P(\mu)$  of a log-concave probability measure in  $\mathbb{R}^n$  is always finite, by (7) and by Bobkov [5] or Kannan, Lovász and Simonovits [28]. By the definition of the Poincaré constant,

$$\lambda_1 := \frac{1}{C_P(\mu)}$$

is the *spectral gap* of  $L$ , in the sense that  $E_\lambda = E_0$  for  $\lambda < \lambda_1$ . In other words, if  $f \in L^2(\mu)$  satisfies  $\int f d\mu = 0$  then

$$\nu_f([0, \lambda_1)) = 0. \quad (17)$$

The following proposition provides an upper bound for the spectral mass of a given function  $f$  below a certain level in terms of the  $L^2(\mu_s)$ -norm of  $Q_s f$ .

**Proposition 2.1.** *Let  $\mu$  be a probability measure with a smooth, positive density in  $\mathbb{R}^n$ . Let  $f \in L^2(\mu)$  satisfy  $\int_{\mathbb{R}^n} f d\mu = 0$  and  $\|f\|_{L^2(\mu)} = 1$ . Then for  $s, \lambda > 0$ ,*

$$\langle E_\lambda f, f \rangle_{L^2(\mu)} \leq C (\|Q_s f\|_{L^2(\mu_s)} + s\lambda),$$

where  $C > 0$  is a universal constant. Our proof gives  $C = 4$ .

We prove Proposition 2.1 in Section 3. In order to estimate  $\|Q_s f\|_{L^2(\mu_s)}^2$  from above it is convenient to set  $t := 1/s$  and to use the framework of Eldan's stochastic localization. We refer the reader to [15, 21, 37] for background on this subject and for further explanations, and we use the notation from [35, Section 4]. For  $t \geq 0$  and  $\theta \in \mathbb{R}^n$  consider the probability density

$$p_{t, \theta}(x) = \frac{1}{Z(t, \theta)} e^{(\theta, x) - t|x|^2/2} \rho(x) \quad (x \in \mathbb{R}^n)$$

where  $Z(t, \theta) = \int_{\mathbb{R}^n} e^{(\theta, x) - t|x|^2/2} \rho(x) dx$  and we recall that  $\rho$  is the log-concave density of the measure  $\mu$ . The barycenter of  $p_{t, \theta}$  is

$$a(t, \theta) = \int_{\mathbb{R}^n} x p_{t, \theta}(x) dx \in \mathbb{R}^n.$$

We consider the ‘‘tilt process’’, the stochastic process  $(\theta_t)_{t \geq 0}$  in  $\mathbb{R}^n$  that satisfies the stochastic differential equation

$$\theta_0 = 0, \quad d\theta_t = dW_t + a(t, \theta_t)dt, \quad (18)$$

where  $(W_t)_{t \geq 0}$  is a standard Brownian motion in  $\mathbb{R}^n$ . The existence and uniqueness of a strong solution to (18) are standard. Setting  $p_t(x) = p_{t, \theta_t}(x)$  and  $a_t = a(t, \theta_t)$  we obtain the equation of *Eldan’s stochastic localization* in the Lee-Vempala formulation:

$$dp_t(x) = p_t(x) \langle x - a_t, dW_t \rangle \quad (x \in \mathbb{R}^n) \quad (19)$$

with  $p_0(x) = \rho(x)$ .

**Lemma 2.2.** *Let  $\varphi \in L^1(\mu)$  and  $s > 0$ . Consider the stochastic process  $M_t = \int_{\mathbb{R}^n} \varphi p_t$  defined for  $t \geq 0$ . Then with  $t = 1/s$ ,*

$$\mathbb{E}M_t^2 = \|Q_s \varphi\|_{L^2(\mu_s)}^2.$$

*Proof.* Let  $y \in \mathbb{R}^n$  and set  $\theta = ty$ . It follows from [35, Lemma 2.1] that

$$Q_s \varphi(y) = \int_{\mathbb{R}^n} \varphi p_{t, \theta} =: M(t, \theta). \quad (20)$$

Moreover, the law of the random vector  $\theta_t/t$  is the probability measure  $\mu_s$ , as explained in [35, Section 4]. Therefore, by (20),

$$\mathbb{E}M_t^2 = \mathbb{E} [M(t, \theta_t)^2] = \mathbb{E} \left[ (Q_s \varphi)^2 \left( \frac{\theta_t}{t} \right) \right] = \int_{\mathbb{R}^n} (Q_s \varphi)^2 d\mu_s,$$

completing the proof. □

For a vector-valued function  $f = (f_1, \dots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  we define  $Q_s f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  coordinate-wise, and denote  $\|f\|_{L^2(\mu)}^2 = \sum_i \|f_i\|_{L^2(\mu)}^2$ . Since

$$a_t = \int_{\mathbb{R}^n} x p_t(x) dx,$$

we conclude from Lemma 2.2 that for any  $s > 0$ , with  $t = 1/s$ ,

$$\mathbb{E}|a_t|^2 = \|Q_s x\|_{L^2(\mu_s)}^2. \quad (21)$$

For  $t > 0$  and  $\theta \in \mathbb{R}^n$  denote

$$A(t, \theta) = \text{Cov}(p_{t, \theta}) = \int_{\mathbb{R}^n} [x \otimes x] p_{t, \theta}(x) dx - a(t, \theta) \otimes a(t, \theta) \in \mathbb{R}^{n \times n},$$

the covariance matrix of  $p_{t, \theta}$ . Here  $x \otimes x = (x_i x_j)_{i, j=1, \dots, n} \in \mathbb{R}^{n \times n}$  for  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Set  $A_t = A(t, \theta_t)$ . It follows from (19) that

$$da_t = d \left[ \int_{\mathbb{R}^n} x p_t(x) dx \right] = A_t dW_t$$

and consequently,

$$\frac{d}{dt} \mathbb{E} |a_t|^2 = \mathbb{E} \|A_t\|_2^2, \quad (22)$$

where we write  $\|A_t\|_q = \text{Tr}[A_t^q]^{1/q}$  for the  $q$ -Schatten norm of the matrix  $A_t$ . Note that the random matrix  $A_t$  is symmetric and positive definite, since it is the covariance matrix of an absolutely-continuous probability measure in  $\mathbb{R}^n$ .

In addition to the parameters  $\sigma_n$  and  $\psi_n$  mentioned above, we shall also need the quantity  $\kappa_n > 0$  defined via

$$\kappa_n^2 := \sup_X \sup_{\theta \in S^{n-1}} \left\{ \|\mathbb{E} \langle X, \theta \rangle (X \otimes X)\|_2^2 \right\}, \quad (23)$$

where the first supremum runs over all isotropic log-concave random vectors  $X$  in  $\mathbb{R}^n$ , and where  $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$  is the unit sphere. The relation between  $\psi_n, \kappa_n$  and  $\sigma_n$  proven by Eldan [21] is

$$\psi_n^2 \leq C \log n \cdot \kappa_n^2 \leq \tilde{C} \log^2 n \cdot \sigma_n^2, \quad (24)$$

where  $C, \tilde{C} > 0$  are universal constants. The core of Eldan's argument is the first inequality of (24), whereas the second inequality is relatively easy. The following lemma essentially follows from Eldan's ideas [21]. However, the stochastic localization used in [21] is slightly different, and the focus there is on tail probabilities rather than on the expectation of  $\|A_t\|_q^q$ . We thus provide a detailed proof of the following lemma in Section 5.

**Lemma 2.3.** *For  $t \leq (C\kappa_n^2 \cdot \log n)^{-1}$  and  $q \geq 1$  we have*

$$\mathbb{E} \|A_t\|_q^q \leq C_q n,$$

where  $C > 0$  is a universal constant and  $C_q$  depends only on  $q$ .

Applying this lemma with  $q = 2$  we see that  $\mathbb{E} \|A_t\|_2^2$  has the order of magnitude of  $n = \|A_0\|_2^2$  for all values of  $t$  up to time  $t_1 = (C\kappa_n^2 \cdot \log n)^{-1}$ . We would like to show that this quantity cannot increase too rapidly also beyond time  $t_1$ . This is the content of the next lemma, which is rather similar to Chen's lemma [15, Lemma 8]. Chen's lemma is the key ingredient of his sub-polynomial bound for the KLS constant.

**Lemma 2.4.** *For any  $0 \leq t_1 \leq t_2$ , we have*

$$\mathbb{E} \|A_{t_2}\|_2^2 \leq \left( \frac{t_2}{t_1} \right)^3 \mathbb{E} \|A_{t_1}\|_2^2.$$

In [15, Lemma 8], Chen proves an analogous inequality for  $\|A_t\|_q$  for all  $q \geq 3$ . We were not able to deduce Lemma 2.4 from the analysis in [15], which seems to break down for  $q < 3$ . We prove Lemma 2.4 in Section 4. Combining Lemma 2.3 with Lemma 2.4 we arrive at the following.

**Corollary 2.5.** *Setting  $t_1 = (C\kappa_n^2 \cdot \log n)^{-1}$  we have*

$$\mathbb{E} |a_t|^2 \leq C_1 n \cdot t \cdot \max \left\{ 1, \frac{t^3}{t_1^3} \right\}, \quad \forall t > 0,$$

where  $C, C_1 > 0$  are universal constants.

*Proof.* Recall (22) and observe that  $a_0$  is the barycenter of  $\mu$ , which is assumed to be 0. Thus

$$\mathbb{E}|a_t|^2 = \int_0^t \mathbb{E}\|A_r\|_2^2 dr.$$

For  $r \in [0, t_1]$  we have  $\mathbb{E}\|A_r\|_2^2 \leq Cn$  according to Lemma 2.3. For  $r \geq t_1$ ,

$$\mathbb{E}\|A_r\|_2^2 \leq \mathbb{E}\|A_{t_1}\|_2^2 \cdot \frac{r^3}{t_1^3} \leq C_1 n \cdot \frac{r^3}{t_1^3},$$

by Lemma 2.4. The result follows by integrating these inequalities from 0 to  $t$ .  $\square$

*Remark 2.6.* It might be possible and it could be interesting to remove all stochastic processes from the argument, and perform our analysis, as well as that of Lee-Vempala [37] and Chen [15], using differentiations along the heat semi-group and integrations by parts in place of Itô calculus. The advantage of the stochastic point of view, is that the evolution equation (18) provides a convenient coupling between the probability measures  $\mu_{s_1}$  and  $\mu_{s_2}$ , which enables us to analyze the  $s$ -derivatives of certain integrals with respect to the measure  $\mu_s$ . We remark that an alternative coupling to (18), which has so far been less useful for our analysis, is provided by the deterministic evolution equation

$$d\theta_t = \frac{1}{2} \left( a(t, \theta_t) + \frac{\theta_t}{t} \right) dt, \quad (25)$$

which is referred to as the Kim-Milman map in [35]. For any  $0 < t_1 < t_2$ , if  $\theta_{t_1}/t_1$  is a random vector with law  $\mu_{s_1}$  for  $s_1 = 1/t_1$ , and we run either the evolution (18) or else the evolution (25) for  $t \in [t_1, t_2]$ , then  $\theta_{t_2}/t_2$  is a random vector with law  $\mu_{s_2}$  for  $s_2 = 1/t_2$ .

The next three sections are dedicated to the proofs of Proposition 2.1, Lemma 2.4 and Lemma 2.3. Finally in Section 6 we explain how all pieces fit together and prove Theorem 1.1. The basic idea is that Corollary 2.5 provides a bound for  $\mathbb{E}|a_t|^2 = \|Q_s x\|_{L^2(\mu_s)}$ , with  $s = 1/t$ , which together with Proposition 2.1 and the  $H^{-1}$ -inequality of [32] yields a bound on the thin-shell parameter of  $\mu$ .

### 3 Spectral measures and heat flow

In this section we prove Proposition 2.1. Assume that  $\mu$  is a log-concave probability measure with a smooth, positive density  $\rho$  in  $\mathbb{R}^n$ . Recall the spectral decomposition (15) of the Laplace operator associated to  $\mu$ , and recall that for a given function  $f \in L^2(\mu)$  the notation  $\nu_f$  stands for its spectral measure. Write  $H^1(\mu)$  for the space of all functions  $f \in L^2(\mu)$  whose weak derivatives  $\partial^1 f, \dots, \partial^n f$  exist and belong to  $L^2(\mu)$ . For  $f \in H^1(\mu)$  we define

$$\|f\|_{H^1(\mu)}^2 = \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$$

and  $\|f\|_{H^1(\mu)} = \sqrt{\|f\|_{L^2(\mu)}^2 + \|f\|_{H^1(\mu)}^2}$ . It is proven in the appendix of [4] that compactly-supported smooth functions are dense in  $H^1(\mu)$ , with respect to the  $H^1(\mu)$ -norm. It is



explained in [18, Chapter 6] that since  $L$  is the Friedrich extension of the operator initially defined on smooth, compactly-supported functions, for any  $f \in H^1(\mu)$ ,

$$\|f\|_{H^1(\mu)}^2 = \int_0^\infty \lambda d\nu_f(\lambda). \quad (26)$$

Moreover,  $H^1(\mu)$  is the space of all functions in  $L^2(\mu)$  for which the integral on the right-hand side of (26) converges. The next lemma expresses the fact proven in [35] that the function  $s \mapsto \log \|Q_s g\|_{L^2(\mu_s)}^2$  is convex, and hence it lies above its tangent at zero. The lemma implies that  $Q_s$  does not reduce the norm of a low-energy function by much.

**Lemma 3.1.** *Let  $g \in H^1(\mu)$  satisfy  $\int g^2 d\mu = 1$ . Denote  $E = \int_{\mathbb{R}^n} |\nabla g|^2 d\mu$ . Then for all  $s > 0$ ,*

$$\|Q_s g\|_{L^2(\mu_s)}^2 \geq \exp(-sE). \quad (27)$$

*Proof.* Since compactly-supported smooth functions are dense in  $H^1(\mu)$ , by [35, Lemma 2.5] it suffices to prove (27) for a compactly-supported, smooth function  $g$ . According to [35, Section 2], for any  $s > 0$ , the  $s$ -derivative of the function  $-\log \|Q_s g\|_{L^2(\mu_s)}^2$  is the Rayleigh quotient

$$R_g(s) = \|Q_s g\|_{H^1(\mu_s)}^2 / \|Q_s g\|_{L^2(\mu_s)}^2,$$

which is continuous and non-increasing in  $s \in [0, \infty)$ , with  $E = R_g(0)$ . Hence,

$$\log \|Q_s g\|_{L^2(\mu_s)}^2 = - \int_0^s R_g(x) dx \geq -sR_g(0) = -sE,$$

proving (27). □

Lemma 3.1 yields a lower bound for  $\langle Ag, g \rangle_{L^2(\mu)}$  with  $A = P_s Q_s = Q_s^* Q_s$  when  $g$  is a low energy function. The next lemma shows that if in addition  $g$  is an orthogonal projection of a certain function  $f$  then we can upgrade this to a lower bound for  $\langle Af, f \rangle_{L^2(\mu)}$ .

**Lemma 3.2.** *Let  $A$  be a self-adjoint, positive semi-definite operator in  $L^2(\mu)$  whose operator norm is at most one. Let  $f \in L^2(\mu)$  satisfy  $\|f\|_{L^2(\mu)} = 1$ . Let  $\varepsilon, \beta \in (0, 1]$ , and assume that  $g \in L^2(\mu)$  satisfies  $\langle Ag, g \rangle_{L^2(\mu)} \geq (1 - \varepsilon)\beta$  and  $\beta = \|g\|_{L^2(\mu)}^2 = \langle f, g \rangle_{L^2(\mu)}$ . Then,*

$$\langle Af, f \rangle_{L^2(\mu)} \geq \frac{\beta}{4} - \varepsilon.$$

*Proof.* The norm and scalar product in this proof are those of  $L^2(\mu)$ . Consider the spectral decomposition

$$A = \int_0^1 \lambda dF_\lambda$$

for a certain increasing family of orthogonal projections  $(F_\lambda)_{\lambda \in [0,1]}$ . For any  $0 \leq r < 1$ ,

$$(1 - \varepsilon)\beta \leq \langle Ag, g \rangle \leq r \|F_r g\|^2 + (\|g\|^2 - \|F_r g\|^2) = \beta + (r - 1) \|F_r g\|^2.$$

Setting  $r = 1 - 4\varepsilon/\beta$  we obtain

$$\|F_r g\| \leq \sqrt{\frac{\varepsilon\beta}{1-r}} = \beta/2.$$

Consider the orthogonal projection  $P = \text{Id} - F_r$ . Since  $\|F_r f\| \leq \|f\| = 1$ ,

$$\beta = \langle f, g \rangle = \langle F_r f, F_r g \rangle + \langle P f, P g \rangle \leq \|F_r g\| + \langle P f, P g \rangle \leq \frac{\beta}{2} + \langle P f, P g \rangle. \quad (28)$$

Since  $\|g\|^2 = \beta$  we have  $\|P g\| \leq \|g\| \leq \sqrt{\beta}$ . By the Cauchy-Schwartz inequality and (28),

$$\|P f\| \geq \frac{\langle P f, P g \rangle}{\|P g\|} \geq \frac{\beta}{2\|P g\|} \geq \frac{\sqrt{\beta}}{2}.$$

Hence,

$$\langle A f, f \rangle \geq r \|P f\|^2 \geq \frac{\beta r}{4} = \frac{\beta}{4} - \varepsilon,$$

which is the conclusion of the lemma.  $\square$

*Proof of Proposition 2.1.* The norm and scalar product in this proof are those of  $L^2(\mu)$ , unless stated otherwise. Fix  $\lambda \geq 0$  and set  $g = E_\lambda f$ . Then

$$\beta := \|g\|^2 = \langle f, g \rangle = \langle f, E_\lambda f \rangle = \nu_f([0, \lambda]).$$

Additionally  $E_0 g = E_0 f = 0$ . Moreover, since  $\nu_g$  is supported on  $[0, \lambda]$

$$E := \int_0^\infty x d\nu_g(x) \leq \lambda \nu_g([0, \lambda]) = \lambda \nu_f([0, \lambda]) = \lambda \beta. \quad (29)$$

Hence  $g \in H^1(\mu)$  and  $E = \int_{\mathbb{R}^n} |\nabla g|^2 d\mu$ . By Lemma 3.1 and (29),

$$\langle P_s Q_s g, g \rangle = \|Q_s g\|_{L^2(\mu_s)}^2 \geq \beta \exp(-sE/\beta) \geq \beta \exp(-s\lambda) \geq \beta(1 - \varepsilon), \quad (30)$$

for  $\varepsilon = s\lambda$ . The operator  $Q_s : L^2(\mu) \rightarrow L^2(\mu_s)$  is a contraction with  $Q_s(1) = 1$ , according to [35]. Hence the operator  $A = P_s Q_s = Q_s^* Q_s$  is a positive semi-definite, self-adjoint operator of norm one in  $L^2(\mu)$ . By (30) we have  $\langle A g, g \rangle \geq (1 - \varepsilon)\beta$ . Lemma 3.2 implies that

$$\|Q_s f\|_{L^2(\mu_s)}^2 = \langle A f, f \rangle \geq \frac{\beta}{4} - \varepsilon = \frac{1}{4} \cdot \langle E_\lambda f, f \rangle - s\lambda,$$

which is the desired inequality.  $\square$

*Remark 3.3.* Proposition 2.1 would have been improved upon had we known that for any  $s > 0$ ,

$$P_s Q_s \geq e^{sL}. \quad (31)$$

Currently we do not have a counterexample to (31). In fact, (31) holds true in a weak sense, since for any continuous, increasing test function  $\varphi : [0, 1] \rightarrow [0, \infty)$  that vanishes in a neighborhood of zero,

$$\text{Tr } \varphi(P_s Q_s) \geq \text{Tr } \varphi(e^{sL}). \quad (32)$$

Indeed, inequality (32) follows from some spectral theory, combined with the fact that by Lemma 3.1, for any  $f \in \text{Dom}(L) \subseteq H^1(\mu)$  with  $\|f\|_{L^2(\mu)} = 1$ ,

$$\langle P_s Q_s f, f \rangle_{L^2(\mu)} = \|Q_s f\|_{L^2(\mu_s)}^2 \geq \exp(-s\|f\|_{H^1(\mu)}^2) = \exp(s\langle L f, f \rangle_{L^2(\mu)}).$$

## 4 Controlling the growth of the covariance matrix

In this section we prove Lemma 2.4. Given  $t > 0$  we say that an absolutely-continuous probability measure  $\mu$  on  $\mathbb{R}^n$  (or a random vector  $X$  in  $\mathbb{R}^n$ ) is  $t$ -uniformly log-concave if it has a density of the form

$$x \mapsto \exp\left(-\phi(x) - \frac{t}{2}|x|^2\right),$$

for some convex function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ . In other words  $\mu$  is more log-concave than the Gaussian measure with covariance  $\frac{1}{t}\text{Id}$ .

**Lemma 4.1.** *Suppose that  $\mu$  is a centered, 1-uniformly log-concave probability measure on  $\mathbb{R}^n$  and let  $(B_t)$  be a standard Brownian motion in  $\mathbb{R}^n$  with  $B_0 = 0$ . Then there exists an adapted process  $(Q_t)$  of symmetric matrices such that  $\int_0^1 Q_t dB_t$  has law  $\mu$ , and such that almost surely,*

$$0 \leq Q_t \leq \text{Id}, \quad \forall t \in [0, 1]. \quad (33)$$

*Proof.* One possibility is introduce the Brenier map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  pushing forward the standard Gaussian measure to  $\mu$ . This map is the gradient of a convex function and it satisfies that  $T(B_1)$  has law  $\mu$ . See e.g. Villani's book [41] for information about the Brenier map. Since  $\mu$  is 1-uniformly log-concave the map  $T$  is a contraction, according to the Caffarelli theorem. By elementary properties of the heat kernel, for any  $t > 0$ , the smooth map  $P_{1-t}(T) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is still the gradient of a convex function, and

$$M_t(x) := \nabla P_{1-t}(T)(x) \in \mathbb{R}^{n \times n}$$

is a symmetric matrix with  $0 \leq M_t(x) \leq \text{Id}$  for any  $x \in \mathbb{R}^n$  and any  $0 < t < 1$ . Since  $(P_{1-t}T)(B_t)$  is a martingale, we have with  $Q_t = M_t(B_t)$ ,

$$T(B_1) = (P_1T)(B_0) + \int_0^1 Q_t dB_t = \int_0^1 Q_t dB_t,$$

where  $(P_1T)(0) = \mathbb{E}T(B_1) = 0$  since  $\mu$  is centered. This proves the lemma. Another possibility is to note that it is shown in Eldan and Lehec [24] by using stochastic localization that  $\int_0^\infty P_t dB_t$  has law  $\mu$  for some adapted process  $(P_t)$  of symmetric matrices with  $0 \leq P_t \leq e^{-t/2}$ . This also implies the required result, via a time change.  $\square$

The main step in the proof of Lemma 2.4 is the following lemma, whose proof is loosely inspired by the article of Jiang, Lee and Vempala [27] in which they prove a related inequality, but with a different set of hypotheses.

**Lemma 4.2.** *Let  $t > 0$ , let  $\mu$  be a probability measure on  $\mathbb{R}^n$  which is  $t$ -uniformly log-concave and centered, and let  $X, Y$  be two independent random vectors with law  $\mu$ . Then*

$$\mathbb{E}\langle X, Y \rangle^3 \leq \frac{3}{t} \cdot \mathbb{E}\langle X, Y \rangle^2 = \frac{3}{t} \cdot \text{Tr}(A^2),$$

where  $A$  is the covariance matrix of  $\mu$ .

*Remark 4.3.* The value of the constant in Lemma 4.2 directly controls the exponent of the logarithm in our main result. The value 3 is the best we could do, but it is probably not optimal. For instance, the inequality of Lemma 4.2 holds true with constant 2 in dimension one.

*Proof of Lemma 4.2.* By homogeneity it is enough to prove the statement for  $t = 1$  (observe that if  $X$  is  $t$ -uniformly log concave then  $\frac{1}{\sqrt{t}} \cdot X$  is 1-uniformly log-concave). We apply Lemma 4.1 and let  $(X_t)$  be the martingale given by

$$X_t = \int_0^t Q_s dB_s.$$

Set  $X = X_1$  and let  $Y$  be a copy of  $X$  that is independent of the Brownian motion  $(B_t)$ . In particular  $Y$  is independent of  $X$ . By Itô's formula

$$\mathbb{E}\langle X, Y \rangle^3 = 3 \int_0^1 \mathbb{E} [\langle X_t, Y \rangle \cdot |Q_t Y|^2] dt. \quad (34)$$

Fix  $x \in \mathbb{R}^n$  and a symmetric matrix  $Q$ . Recall that  $A$  is the covariance matrix of  $Y$ . Since  $Y$  is centered we have

$$\mathbb{E} [\langle x, Y \rangle \cdot |QY|^2] \leq \mathbb{E} [\langle x, Y \rangle^2]^{1/2} \cdot \text{Var}(|QY|^2)^{1/2} = \langle Ax, x \rangle^{1/2} \cdot \text{Var}(|QY|^2)^{1/2}.$$

Let us apply the Poincaré inequality in order to bound the variance of  $|QY|^2$ . It is well known that a 1-uniformly log-concave random vector satisfies the Poincaré inequality with constant 1. However, according to Cordero-Erausquin, Fradelizi and Maurey [16, Lemma 2], under the 1-uniform log-concavity assumption, functions whose gradient is centered satisfy the Poincaré inequality with constant  $\frac{1}{2}$  (rather than 1). The gradient of  $|QY|^2$  is  $2Q^2Y$ , which is indeed centered, and we obtain

$$\text{Var}(|QY|^2) \leq \frac{1}{2} \cdot \mathbb{E}|2Q^2Y|^2 = 2 \cdot \text{Tr}(Q^4 A).$$

Plugging this back in (34), using the Cauchy-Schwarz inequality, and pulling the power  $1/2$  outside of the integral on  $[0, 1]$ , we get

$$\mathbb{E}[\langle X, Y \rangle^3] \leq 3\sqrt{2} \cdot \left( \int_0^1 \mathbb{E}\langle AX_t, X_t \rangle \cdot \mathbb{E}\text{Tr}(Q_t^4 A) dt \right)^{1/2}. \quad (35)$$

For  $t \in [0, 1]$  set  $M_t = \mathbb{E}Q_t^2$ . By Itô's formula the covariance matrix of  $X_t$  is  $\int_0^t M_s ds$ . In particular  $\int_0^1 M_s ds$  is the covariance matrix of  $X$ , namely  $A$ . Additionally, by (33) we have  $Q_t^4 \leq Q_t^2$  almost surely, hence  $\mathbb{E}\text{Tr}(Q_t^4 A) \leq \text{Tr}(M_t A)$ , since  $A$  is a positive semi-definite matrix. Thus

$$\begin{aligned} \int_0^1 \mathbb{E}\langle AX_t, X_t \rangle \cdot \mathbb{E}\text{Tr}(Q_t^4 A) dt &\leq \int_0^1 \left( \int_0^t \text{Tr}(M_s A) ds \right) \cdot \text{Tr}(M_t A) dt \\ &= \frac{1}{2} \left( \int_0^1 \text{Tr}(M_t A) dt \right)^2 = \frac{1}{2} \text{Tr}(A^2)^2. \end{aligned} \quad (36)$$

The result follows from (35) and (36).  $\square$

We can now prove the Chen type bound for  $q = 2$ .

*Proof of Lemma 2.4.* The matrix  $A_t$  satisfies

$$dA_t = \left\langle \int_{\mathbb{R}^n} (x - a_t)^{\otimes 3} p_t(x) dx, dB_t \right\rangle - A_t^2 dt.$$

See for instance [37] where this computation is explained in detail. Itô's formula yields

$$d\text{Tr}(A_t^2) = 2 \sum_{i,j=1}^n A_{ij,t} \langle H_{ij,t}, dB_t \rangle + \sum_{i,j=1}^n |H_{ij,t}|^2 dt - 2\text{Tr}(A_t^3) dt, \quad (37)$$

where  $A_t = (A_{ij,t})_{i,j=1,\dots,n} \in \mathbb{R}^{n \times n}$  and

$$H_{ij,t} = \int_{\mathbb{R}^n} (x - a_t)_i (x - a_t)_j (x - a_t) p_t(x) dx.$$

We claim that with probability one

$$\sum_{i,j=1}^n |H_{ij,t}|^2 \leq \frac{3}{t} \cdot \text{Tr}(A_t^2). \quad (38)$$

Indeed, conditioning on  $(B_t)$ , we observe that the term on the left-hand side of (38) is precisely  $\mathbb{E}\langle X, Y \rangle^3$  where  $X, Y$  are two independent random vectors having density  $x \mapsto p_t(x + a_t)$ . Since the density  $p_t$  is  $t$ -uniformly log-concave, the random vectors  $X, Y$  are  $t$ -uniformly log-concave, and they are centered by the definition of  $a_t$ . Hence (38) follows from Lemma 4.2. Consequently, by taking expectation in (37) we obtain

$$\frac{d}{dt} \mathbb{E}\text{Tr}(A_t^2) \leq \frac{3}{t} \cdot \mathbb{E}\text{Tr}(A_t^2) - 2 \cdot \mathbb{E}\text{Tr}(A_t^3) \leq \frac{3}{t} \cdot \mathbb{E}\text{Tr}(A_t^2).$$

Integrating this differential inequality yields the conclusion of the lemma.  $\square$

*Remark 4.4.* By using the inequality of Cordero-Erausquin, Fradelizi and Maurey [16] one can improve the exponent in Chen's lemma [15, Lemma 8] from  $2q$  to  $q - 1$ , but still only for  $q \geq 3$ . With this improved exponent, it is possible to adapt our proof of Theorem 1.1 and replace the usage of Lemma 2.4 by the case  $q = 3$  of Lemma 8 from [15].

## 5 Bounding the covariance matrix for small times

In this section we prove Lemma 2.3. For a centered, log-concave random vector  $X$  in  $\mathbb{R}^n$  we set

$$\kappa_X^2 = \sup_{\theta \in S^{n-1}} \left\{ \|\mathbb{E}\langle X, \theta \rangle (X \otimes X)\|_2^2 \right\},$$

and recall that  $\kappa_n^2$  is the supremum of  $\kappa_X^2$  over all isotropic log-concave random vectors  $X$  in  $\mathbb{R}^n$ . The following lemma is elementary.

**Lemma 5.1.** *If  $X$  is a centered, log-concave random vector in  $\mathbb{R}^n$  then*

$$\kappa_X^2 \leq \|\text{Cov}(X)\|_{op}^3 \cdot \kappa_n^2,$$

where  $\text{Cov}(X)$  is the covariance matrix of  $X$ , and the norm here is the operator norm.

*Proof.* Both terms of the inequality are homogeneous of degree 6 in  $X$  so we can assume that  $\text{Cov}(X)$  has operator norm 1. Therefore  $\text{Cov}(X) \leq \text{Id}$ , and in particular  $\text{Id} - \text{Cov}(X)$  is a positive semi-definite matrix. Let  $Y$  be a log-concave random vector independent of  $X$ , having covariance  $\text{Id} - \text{Cov}(X)$ , such that  $Y$  coincides in law with  $-Y$ , i.e.  $Y$  is symmetric. For instance  $Y$  could be a centered Gaussian random vector with the appropriate covariance matrix. Under these conditions

$$\mathbb{E}[(X + Y)^{\otimes 3}] = \mathbb{E}[X^{\otimes 3}].$$

Indeed, all of the other terms in the expansion of the tensor  $(X + Y)^{\otimes 3}$  have zero expectation. Thus  $\kappa_X^2 = \kappa_{X+Y}^2$  and the latter quantity is at most  $\kappa_n^2$ , since  $X + Y$  is log-concave and isotropic.  $\square$

Recall that  $\mu$  is an isotropic, log-concave probability measure in  $\mathbb{R}^n$  with density  $p_0$ , and that  $(p_t)$  is the associated stochastic localization process. As above we denote by  $(a_t)$  and  $(A_t)$  the corresponding processes of barycenters and covariance matrices, respectively.

**Lemma 5.2.** *For every  $t \leq (C\kappa_n^2 \cdot \log n)^{-1}$  we have*

$$\mathbb{P}(\|A_t\|_{op} \geq 2) \leq \exp(-(Ct)^{-1}),$$

where  $C > 0$  is a universal constant.

*Proof.* Recall that the matrix process  $(A_t)_{t \geq 0}$  satisfies

$$dA_t = \left\langle \int_{\mathbb{R}^n} (x - a_t)^{\otimes 3} p_t(x) dx, dB_t \right\rangle - A_t^2 dt.$$

We will apply Itô's formula to a smooth approximation of  $\|A_t\|_{op}$ . Let  $\beta > 0$  be a parameter to be determined soon. Set

$$\Phi_t = \frac{1}{\beta} \log \text{Tr} \left( e^{\beta A_t} \right),$$

and note that

$$\|A_t\|_{op} \leq \Phi_t \leq \|A_t\|_{op} + \frac{\log n}{\beta}.$$

Write  $0 \leq \lambda_{1,t} \leq \dots \leq \lambda_{n,t}$  for the eigenvalues of  $A_t$ , repeated according to their multiplicity, and let  $e_{1,t}, \dots, e_{n,t} \in \mathbb{R}^n$  be a corresponding orthonormal basis of eigenvectors. For easing the computation of  $d\Phi_t$ , we first consider the case where the eigenvalues  $\lambda_{1,t} \leq \dots \leq \lambda_{n,t}$  are almost surely distinct for all positive time. Then Itô's formula gives (see e.g. [22])

$$d\lambda_{i,t} = \langle u_{ii,t}, dB_t \rangle - \lambda_{i,t}^2 dt + \sum_{j: j \neq i} \frac{|u_{ij,t}|^2}{\lambda_{i,t} - \lambda_{j,t}} dt,$$

where

$$u_{ij,t} = \int_{\mathbb{R}^n} \langle x - a_t, e_{i,t} \rangle \langle x - a_t, e_{j,t} \rangle (x - a_t) p_t(x) dx.$$

(The choice of the orthonormal basis of eigenvectors, which are determined only up to a sign, does not affect the above expression for the Itô derivative of  $\lambda_{i,t}$ ). Next we apply the Itô formula to the smooth function

$$f(\lambda_1, \dots, \lambda_n) = \frac{1}{\beta} \log \left( \sum_{i=1}^n e^{\beta \lambda_i} \right)$$

and obtain

$$\begin{aligned} d\Phi_t &= \sum_i \alpha_i \langle u_{ii}, dB_t \rangle - \sum_i \alpha_i \lambda_i^2 dt + \sum_{i \neq j} \frac{\alpha_i |u_{ij}|^2}{\lambda_i - \lambda_j} dt \\ &\quad + \frac{\beta}{2} \sum_i \alpha_i |u_{ii}|^2 dt - \frac{\beta}{2} \left| \sum_i \alpha_i u_{ii} \right|^2 dt, \end{aligned} \quad (39)$$

where we dropped the dependence in  $t$  to lighten notations and where

$$\alpha_i = \frac{e^{\beta \lambda_i}}{\sum_{j=1}^n e^{\beta \lambda_j}}.$$

Symmetrizing the third term of (39), namely replacing  $\alpha_i$  by  $\frac{1}{2}(\alpha_i - \alpha_j)$ , we observe that the expression for  $d\Phi_t$  still makes sense by continuity when the eigenvalues are not necessarily distinct. This suggests that this expression for  $d\Phi_t$  remains valid if we drop this assumption. As a matter of fact, applying Itô's formula directly to the function  $A \mapsto \frac{1}{\beta} \log \text{Tr}(e^{\beta A})$  does lead to the same expression for  $d\Phi_t$ . Using the inequality

$$\frac{e^x - e^y}{x - y} \leq \frac{e^x + e^y}{2}$$

to upper bound the third term of (39), and dropping the non-positive terms, we arrive at the inequality

$$d\Phi_t \leq \sum_i \alpha_i \langle u_{ii}, dB_t \rangle + \frac{\beta}{2} \sum_{i,j} \alpha_i |u_{ij}|^2 dt.$$

By the Cauchy-Schwartz inequality and the reverse Hölder inequalities for log-concave measures (e.g. [11, Theorem 2.4.6]),

$$\begin{aligned} |u_{ii}| &\leq \sup_{\theta \in S^{n-1}} \int_{\mathbb{R}^n} \langle x - a_t, e_i \rangle^2 \langle x - a_t, \theta \rangle |p_t(x)| dx \\ &\leq \left( \int_{\mathbb{R}^n} \langle x - a_t, e_i \rangle^4 p_t(x) dx \right)^{1/2} \cdot \|A_t\|_{op}^{1/2} \leq C \|A_t\|_{op}^{3/2}, \end{aligned}$$

for some universal constant  $C > 0$ . According to Lemma 5.1, for any fixed  $i$ ,

$$\sum_j |u_{ij}|^2 \leq \|A_t\|_{op}^3 \cdot \kappa_n^2.$$

Since the  $\alpha_i$ 's are positive and add up to 1, we finally obtain

$$d\Phi_t = \langle v_t, dB_t \rangle + c_t dt,$$

where

$$|v_t|^2 \leq C \cdot \|A_t\|_{op}^3 \quad \text{and} \quad c_t \leq \frac{1}{2}\beta \cdot \|A_t\|_{op}^3 \cdot \kappa_n^2. \quad (40)$$

Let  $\tau$  be the following stopping time:

$$\tau = \inf\{t \geq 0; \|A_t\|_{op} \geq 2\}.$$

Choose  $\beta = 2 \log n$  and suppose that  $t \leq (32 \cdot \kappa_n^2 \cdot \log n)^{-1}$ . Note that  $\Phi_0 = 3/2$  since  $\mu$  is isotropic. Because of (40) and by the definition of  $\tau$  we have

$$\Phi_{t \wedge \tau} \leq \Phi_0 + M_t + 8t \cdot \kappa_n^2 \cdot \log n \leq \frac{3}{2} + M_t + \frac{1}{4},$$

where  $t \wedge \tau = \min\{t, \tau\}$  and  $(M_t)$  is the martingale

$$M_t = \int_0^{t \wedge \tau} \langle v_s, dB_s \rangle.$$

If  $\|A_t\|_{op} \geq 2$  then  $\tau \leq t$ ,  $\|A_{t \wedge \tau}\|_{op} \geq 2$  and also  $\Phi_{t \wedge \tau} \geq 2$ . In view of the preceding inequality this implies that

$$\mathbb{P}(\|A_t\|_{op} \geq 2) \leq \mathbb{P}\left(M_t \geq \frac{1}{4}\right).$$

The martingale  $(M_s)$  satisfies  $M_0 = 0$  and by (40) its quadratic variation at time  $t$  satisfies

$$[M]_t = \int_0^{t \wedge \tau} |v_s|^2 ds \leq C't, \quad \text{almost surely.}$$

The martingale lemma spelled out below thus implies

$$\mathbb{P}\left(M_t \geq \frac{1}{4}\right) \leq e^{-(C''t)^{-1}},$$

which concludes the proof. □

The following deviation inequality for martingales with bounded quadratic variation is folklore. We provide a short proof for completeness.

**Lemma 5.3.** *Let  $(M_t)_{t \geq 0}$  be a continuous martingale satisfying  $M_0 = 0$  and  $[M]_t \leq \sigma^2$  almost surely for some fixed time  $t > 0$  and some constant  $\sigma > 0$ . Then*

$$\mathbb{P}(M_t \geq u) \leq e^{-u^2/2\sigma^2}, \quad \forall u \geq 0.$$



*Proof.* Let  $\lambda > 0$ . By Itô's formula the process  $(D_s)$  given by

$$D_s = \exp\left(\lambda M_s - \frac{\lambda^2}{2}[M]_s\right)$$

is a positive local martingale, hence a super-martingale by Fatou's lemma. In particular  $\mathbb{E}[D_t] \leq \mathbb{E}[D_0] = 1$ . In view of the hypothesis, this yields

$$\mathbb{E}[\exp(\lambda M_t)] \leq \exp(\lambda^2 \sigma^2 / 2).$$

Now apply the Markov inequality and optimize in  $\lambda$ . □

**Corollary 5.4.** *Let  $t \leq (C\kappa_n^2 \cdot \log n)^{-1}$  and let  $p \geq 1$ . Then*

$$\mathbb{E}[\|A_t\|_{op}^p] \leq C_p,$$

where  $C > 0$  is a universal constant and  $C_p > 0$  is a constant depending only on  $p$ .

*Proof.* Since  $A_t$  is the covariance matrix of a measure which is more log-concave than the Gaussian measure with covariance  $\frac{1}{t}\text{Id}$  we have  $\|A_t\|_{op} \leq \frac{1}{t}$ , almost surely. Applying Lemma 5.2 we thus get for  $t \leq (C\kappa_n^2 \cdot \log n)^{-1}$ ,

$$\mathbb{E}[\|A_t\|_{op}^p] \leq 2^p + \frac{1}{t^p} \mathbb{P}(\|A_t\|_{op} \geq 2) \leq 2^p + \frac{1}{t^p} e^{-1/(Ct)} \leq 2^p + C^p p!,$$

and the corollary is proven. □

Clearly  $\|A_t\|_p^p \leq n \|A_t\|_{op}^p$ , and hence Corollary 5.4 yields the desired Lemma 2.3.

*Remark 5.5.* Corollary 5.4 for  $p = 1$  recovers Eldan's theorem. Indeed it implies that setting  $t_1 := (C\kappa_n^2 \cdot \log n)^{-1}$ , we have

$$\mathbb{E} \left[ \int_0^{t_1} \|A_t\|_{op} dt \right] \leq C' t_1 \leq \hat{C} \cdot \frac{1}{\log n} = o(1).$$

This is well known to imply  $\psi_\mu^2 \leq \tilde{C} \cdot t_1^{-1} = C_1 \kappa_n^2 \cdot \log n$ , see for instance [33, page 9]. We thus get

$$\tilde{\psi}_n^2 \leq C_1 \kappa_n^2 \cdot \log n,$$

which is (the hard part of) Eldan's inequality (24).

## 6 Proof of the main result

In this section we prove Theorem 1.1. Let  $\mu$  be an isotropic, log-concave probability measure in  $\mathbb{R}^n$  with a smooth positive density. Let us furthermore assume that

$$\sigma_\mu > \sigma_n / 2. \tag{41}$$

Thus the thin-shell parameter of  $\mu$  is nearly as large as possible. The requirement (41) is consistent with the assumption that  $\mu$  has a smooth, positive density. Indeed by convolving  $\mu$  with a tiny Gaussian and normalizing back to isotropicity, we obtain a smooth positive

density, and the change in  $\sigma_\mu$  can be made arbitrarily small. For  $f \in L^2(\mu)$  with  $\int f d\mu = 0$  write

$$\|f\|_{H^{-1}(\mu)} = \sup \left\{ \int_{\mathbb{R}^n} f u d\mu; u \in L^2(\mu) \text{ is locally-Lipschitz with } \int_{\mathbb{R}^n} |\nabla u|^2 d\mu \leq 1 \right\}. \quad (42)$$

Recall that for  $f \in L^2(\mu)$  we write  $\nu_f$  for the spectral measure of  $f$  relative to the Laplace operator associated with  $\mu$ . According to (17), setting

$$\lambda_1 = \frac{1}{C_P(\mu)}$$

we have that  $\nu_f([0, \lambda_1)) = 0$  for any  $f \in L^2(\mu)$  with  $\int f d\mu = 0$  (i.e.,  $f$  is centered). From (26) and (42) we can deduce that when  $f \in L^2(\mu)$  is centered,

$$\|f\|_{H^{-1}(\mu)}^2 = \int_0^\infty \frac{d\nu_f(\lambda)}{\lambda} = \int_{\lambda_1}^\infty \frac{d\nu_f(\lambda)}{\lambda}. \quad (43)$$

The following inequality was proven in the case of the uniform measure on a convex body in [32] and for a general log-concave measure in Barthe and Klartag [4]. According to Proposition 10 in [4], for any smooth function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $f, \nabla f \in L^2(\mu)$  such that  $\int_{\mathbb{R}^n} \partial^i f d\mu = 0$  for all  $i$ ,

$$\text{Var}_\mu(f) \leq \sum_{i=1}^n \|\partial^i f\|_{H^{-1}(\mu)}^2. \quad (44)$$

Specializing to the case where  $f(x) = |x|^2$  in (44), which is the main case used already in [32], we obtain

$$n\sigma_\mu^2 = \text{Var}_\mu(|x|^2) \leq 4 \sum_{i=1}^n \|x_i\|_{H^{-1}(\mu)}^2.$$

Since  $\mu$  is centered, we may use (43) and rewrite this as

$$\sigma_\mu^2 \leq \frac{4}{n} \sum_{i=1}^n \int_{\lambda_1}^\infty \frac{d\nu_{x_i}(\lambda)}{\lambda} = 4 \int_{\lambda_1}^\infty \frac{F(\lambda)}{\lambda^2} d\lambda, \quad (45)$$

where the last equality follows by integration by parts, and

$$F(\lambda) := \frac{1}{n} \sum_{i=1}^n \nu_{x_i}([0, \lambda]) \in [0, 1]$$

is the average spectral mass of the coordinate functions below level  $\lambda$ . We are now in a position to prove our main result.

*Proof of Theorem 1.1.* Applying Proposition 2.1 with the coordinate function  $x_i$  and summing over  $i$  yields

$$F(\lambda) \leq C \left( \frac{1}{n} \cdot \|Q_s x\|_{L^2(\mu_s)}^2 + \lambda s \right), \quad (46)$$

for every positive  $\lambda$  and  $s$ . Let us translate this to the normalization  $t = 1/s$  of Eldan's stochastic localization. By (21) we can rewrite (46) as

$$F(\lambda) \leq C \left( \frac{1}{n} \cdot \mathbb{E}|a_t|^2 + \frac{\lambda}{t} \right). \quad (47)$$

According to Corollary 2.5, for all  $t > 0$  we have

$$\mathbb{E}|a_t|^2 \leq C_1 t \cdot \max \{1, (t/t_1)^3\} \cdot n, \quad (48)$$

where  $t_1 = (C\kappa_n^2 \cdot \log n)^{-1}$ . From (7), (8) and Eldan's Theorem in the form of inequality (24) above,

$$t_1 = \frac{c}{\kappa_n^2 \log n} \leq \frac{c'}{\psi_n^2} \leq \frac{\tilde{c}}{C_P(\mu)} = \tilde{c} \cdot \lambda_1. \quad (49)$$

Given  $\lambda \geq \lambda_1$ , combining (47) and (48) and choosing  $t = \lambda^{1/5} \cdot t_1^{3/5} \geq c \cdot t_1$  yields

$$F(\lambda) \leq C t^4 \cdot t_1^{-3} + C' \lambda \cdot t^{-1} \leq \tilde{C} \lambda^{4/5} \cdot t_1^{-3/5}. \quad (50)$$

From (45), (49) and (50),

$$\sigma_\mu^2 \leq 4 \int_{\lambda_1}^{\infty} \frac{F(\lambda)}{\lambda^2} d\lambda \leq C t_1^{-3/5} \cdot \int_{\lambda_1}^{\infty} \lambda^{-6/5} d\lambda = C' t_1^{-3/5} \cdot \lambda_1^{-1/5} \leq C' t_1^{-4/5}. \quad (51)$$

Now recall the definition of  $t_1$ , the fact that  $\mu$  has a nearly maximal thin-shell constant and (the easy part of) Eldan's inequality (24). We thus obtain from (41), (49) and (51) that

$$\sigma_n^2 \leq C \sigma_\mu^2 \leq C' (\kappa_n^2 \log n)^{4/5} \leq \tilde{C} (\sigma_n^2 \log^2 n)^{4/5}.$$

Therefore  $\sigma_n \leq C(\log n)^4$  and thus  $\psi_n \leq C'(\log n)^5$  by one last application of Eldan's theorem.  $\square$

*Remark 6.1.* The exponent of the logarithmic factor in Theorem 1.1 does not seem optimal. As we already mentioned, one way to decrease it could be to improve the constant in Lemma 2.4. Another option is to either lower the gap between  $\kappa_n$  and  $\sigma_n$  or to replace  $\kappa_n$  in Lemma 2.3 by something smaller, maybe an averaged version of  $\kappa_n$ . A matrix-valued version of Lemma 2.4 could be useful too. Lastly, a stronger version of Proposition 2.1 could hold true, see Remark 3.3, and this would improve our main result.

*Remark 6.2.* From (43) one can obtain the following alternative expression for the  $H^{-1}$ -norm of a centered function  $f \in L^2(\mu)$ :

$$\|f\|_{H^{-1}(\mu)}^2 = \int_0^\infty \langle e^{sL} f, f \rangle_{L^2(\mu)} ds.$$

This shows that the  $H^{-1}$ -inequality (44) can be reformulated as follows

$$\text{Var}_\mu(f) \leq \int_0^\infty \langle e^{sL} \nabla f, \nabla f \rangle_{L^2(\mu)} ds = 2 \int_0^\infty \|e^{sL} \nabla f\|_{L^2(\mu)}^2 ds. \quad (52)$$

Here the semi-group  $e^{sL}$  is applied coordinate-wise to the vector field  $\nabla f$ , and the second equality follows from the change of variable  $s \rightarrow 2s$  and the fact that  $e^{sL}$  is self-adjoint in

$L^2(\mu)$ . This should be compared with the expression one gets by differentiating the variance of  $f$  along the semi-group  $e^{sL}$ , namely

$$\text{Var}_\mu(f) = 2 \int_0^\infty \|\nabla(e^{sL} f)\|_{L^2(\mu)}^2 ds, \quad (53)$$

see for instance [2, section 4.2]. Moreover, in the log-concave case, the Bakry-Émery machinery yields the commutation rule

$$|\nabla(e^{sL} f)|^2 \leq e^{sL} |\nabla f|^2,$$

pointwise (in both space and time), see [2, Theorem 3.2.3]. However, it does *not* imply that

$$|\nabla(e^{sL} f)|^2 \leq |e^{sL} \nabla f|^2$$

and we suspect that this inequality is not true in general. Nevertheless, after integrating in both space and time, this becomes a valid inequality. Indeed, the  $H^{-1}$ -inequality (52) and the equality (53) show that when  $\mu$  is log-concave, for any  $f \in L^2(\mu)$ ,

$$\int_0^\infty \|\nabla(e^{sL} f)\|_{L^2(\mu)}^2 ds \leq \int_0^\infty \|e^{sL} \nabla f\|_{L^2(\mu)}^2 ds. \quad (54)$$

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