COMPLEX LEGENDRE DUALITY

BO BERNDTSSON, DARIO CORDERO-ERAUSQUIN, BO'AZ KLARTAG, YANIR A. RUBINSTEIN

ABSTRACT. We introduce complex generalizations of the classical Legendre transform, operating on Kähler metrics on a compact complex manifold. These Legendre transforms give explicit local isometric symmetries for the Mabuchi metric on the space of Kähler metrics around any real analytic Kähler metric, answering a question originating in Semmes' work.

1. INTRODUCTION

For a function $\psi : \mathbb{R}^N \to \mathbb{R}$ its classical Legendre transform is defined as [9, 16]

$$\psi^*(y) = \sup_x [x \cdot y - \psi(x)]. \tag{1.1}$$

This transformation plays an important role in several parts of mathematics, notably in classical mechanics and convex geometry. Being the supremum of affine functions (of y), ψ^* is always convex and in case ψ is convex it equals the Legendre transform of its Legendre transform. One easily verifies (see Section 2) that $\psi^* = \psi$ if and only if $\psi(x) = |x|^2/2$, so the Legendre transformation is a symmetry on the space of convex functions around its fixed point $|x|^2/2$.

In particular this applies when $\mathbb{R}^N = \mathbb{C}^n$. In this case we change the definition slightly and put

$$\hat{\psi}(w) = \sup_z [2\mathrm{Re}\,(z \cdot \bar{w}) - \psi(z)], \tag{1.2}$$

where $a \cdot b := \sum_{i=1}^n a_i b_i$. The reason for this change is that while the supremum in (1.1) is attained if $y = \partial\psi(x)/\partial x$, the supremum in (1.2) is attained if $w = \partial\psi(z)/\partial\bar{z}$, and one verifies that the unique fixed point is now $\psi(z) = |z|^2$.

In this connection a very interesting observation was made by Lempert, [13]: Put $\omega_\psi := i\partial\bar{\partial}\psi$. Assuming that ψ is smooth and strictly convex, $g^*\omega_{\hat{\psi}} = \omega_\psi$, if $g(z) = \partial\psi(z)/\partial\bar{z}$. It follows from this that

$$g^*\omega_{\hat{\psi}}^n = \omega_\psi^n. \tag{1.3}$$

The measure $\omega_\psi^n/n! =: \mathrm{MA}_{\mathbb{C}}(\psi)$ is the complex Monge–Ampère measure associated to ψ, and Lempert's theorem thus implies that $\mathrm{MA}_{\mathbb{C}}(\psi)$ and $\mathrm{MA}_{\mathbb{C}}(\hat{\psi})$ are related under the gradient map g. This may be compared and perhaps contrasted to the way real Monge–Ampère measures transform under gradient maps, see Section 2.

At this point we recall the definition of the Mabuchi metric in the somewhat nonstandard setting of smooth, strictly plurisubharmonic functions on $\mathbb{C}^n$. The idea is to view this space as an infinite dimensional manifold; an open subset of the space of all smooth functions. Its tangent space at a point ϕ should be the collection of smooth functions χ such that $\phi + t\chi$ remains strictly plurisubharmonic for t close enough to zero. This collection of course depends on the particular ϕ we have chosen, but at any rate the tangent space will always contain smooth functions of compact support, so we take by definition the space of such functions as our tangent space. The Mabuchi norm of a tangent vector χ at a point ϕ is now defined by

$$\|\chi\|_\phi^2 := \int_{\mathbb{C}^n} |\chi|^2 \omega_\phi^n/n!. \tag{1.4}$$

We will interpret (1.3) as saying that the Legendre transformation is an isometry for the Mabuchi metric on the space of convex functions (see section 4). It follows, at least formally, that the Legendre transformation maps geodesics for the Mabuchi metric to geodesics, which reflects the so called duality principle for the complex method of interpolation [3].

This is only a special case of Lempert's result, which implies that a much more general class of gradient-like maps are isometries for the Mabuchi metric. In this note we will develop this scheme and define 'Legendre tranforms' for Kähler potentials over a manifold M, usually compact.

For this we note first that the usual Legendre transformation is not involutive on all plurisubharmonic functions, but just on convex functions, hence in particular on functions that are close to its fixed point $|z|^2$. Imitating this, we start with a (local) Kähler potential ϕ on M, and define a 'Legendre tranform' depending on ϕ that is defined for potentials close to ϕ, fixes ϕ and is an isometry for the Mabuchi metric. For this to work, we need to assume that ϕ is real analytic. (*Added in revision:* Lempert, [14], has shown very recently that this condition is also necessary.) The definition of the ϕ-Legendre transformation involves a polarization of our real analytic potential, which is locally a function $\phi_{\mathbb{C}}(z, w)$ defined near the diagonal in $M \times M$. $\phi_{\mathbb{C}}$ is holomorphic in z, antiholomorphic in w and coincides with ϕ on the diagonal (these properties determine $\phi_{\mathbb{C}}$ uniquely). Roughly speaking, the idea is then to replace $z \cdot \bar{w}$ by $\phi_{\mathbb{C}}$ and define our transform as

$$(\mathcal{L}_\phi \psi)(w) := \sup_z [2\mathrm{Re}\, \phi_{\mathbb{C}}(z, w) - \psi(z)]. \tag{1.5}$$

When $\phi(z) = |z|^2$ this gives us back the Legendre transformation of (1.2). Let us first examine this transformation in the case of a linear space, the cradle of the classical Legendre transform. Write $\Delta_{\mathbb{C}^n} = \{(z, z)\,;\, z \in \mathbb{C}^n\}$ for the diagonal. We say that a smooth function ϕ on $\mathbb{C}^n$ is strongly plurisubharmonic if its complex Hessian is bounded from below by a positive constant, uniformly at all points of $\mathbb{C}^n$.

Theorem 1.1. *Let $\phi : \mathbb{C}^n \to \mathbb{R}$ be a real-analytic, strongly plurisubharmonic function. Then there are an open set $V_\phi \subseteq \mathbb{C}^n \times \mathbb{C}^n$ containing $\Delta_{\mathbb{C}^n}$ and a neighborhood U of ϕ in the C^2-norm on $\mathbb{C}^n$ with the following properties:*

0. For $u \in U$ the function $\mathcal{L}_\phi(u) : \mathbb{C}^n \to \mathbb{R} \cup \{+\infty\}$ is well-defined, where the supremum in (1.5) runs over all z with $(z, w) \in V_\phi$.

1. $\mathcal{L}_\phi(u) = u$ if and only if $u = \phi$.

2. For u in a smaller neighbourhood $U' \subset U$, the function $\mathcal{L}_\phi(u)$ lies in U and $L_\phi^2(u) = u$.

3. $\mathcal{L}_\phi$ is an isometry for the Mabuchi metric restricted to U.

The transformation in (1.5) works fine if ϕ, ψ and $\phi_{\mathbb{C}}$ are defined on all of $\mathbb{C}^n$ or $\mathbb{C}^{2n}$, but for functions that are only locally well defined we need to find a variant of the definition that has a global meaning on a manifold. For this it turns out to be very convenient to use a remarkable idea of Calabi, [5]. The Calabi *diastasis function* is defined as

$$D_\phi(z,w) = \phi(z) + \phi(w) - 2\mathrm{Re}\,\phi_{\mathbb{C}}(z,w).$$

We then change the above definition by applying it to $\psi + \phi$ instead of ψ, and then subtract ϕ afterwards. This way we arrive at the equivalent transform

$$L_\phi(\psi)(w) := \mathcal{L}_\phi(\psi+\phi)(w) - \phi(w) = \sup_z(-D_\phi(z,w) - \psi(z)).$$

Notice that in the classical case when $\phi(z) = |z|^2$, $D_\phi(z,w) = |z-w|^2$ and the transform becomes the familiar variant of the Legendre transform

$$\sup_z -(|z-w|^2 + \psi(z)).$$

The point of this is that, as is well known from the work of Calabi, D_ϕ only depends on $\omega_\phi = i\partial\bar\partial\phi$, i e it does not change if we add a pluriharmonic function to ϕ. As we shall see this implies that our construction of $L_\phi = L_{\omega_\phi}$ globalizes and becomes well defined on functions ψ on a manifold M that are close to 0 in the C^2-norm. Following the ideas, but not the precise proof, of Lempert, we can then verify that L_{ω_ϕ} is an isometry for the Mabuchi metric on $\mathcal{U}_{\omega_\phi}$. Our main result is as follows:

Theorem 1.2. *Let M be a compact Kähler manifold, and let ω be a real analytic Kähler form on M. Let $\mathcal{H}_\omega := \{u \in C^\infty(M); i\partial\bar\partial u + \omega > 0\}$. Then the generalized Legendre transform, L_ω (defined in section 4) is defined on a neigbourhood U of 0 in $\mathcal{H}_\omega$ in the C^2-toplogy and*

1. *$L_\omega(u) = u$ if and only if $u = 0$.*
2. *For u in a smaller neighbourhood $U' \subset U$, $L_\omega(u)$ lies in U and $L^2_\omega(u) = u$.*
3. *L_ω is an isometry for the Mabuchi metric on $\mathcal{H}_\omega$ restricted to U.*

2. The classical Legendre transform

As a warm up and for comparison we first briefly look at the classical Legendre transform, (1.1). If ψ is differentiable, and if the supremum in the right hand side is attained in a point x, then $y = \partial\psi/\partial x =: g_\psi(x)$. Hence we have that

$$x \cdot y \leq \psi(x) + \psi^*(y) \tag{2.1}$$

with equality if and only if $y = g_\psi(x)$, or in other words $x \cdot g_\psi(x) - \psi(x) = \psi^*(g_\psi(x))$. If moreover ψ is assumed smooth and strongly convex, g_ψ is invertible. It follows that ψ^* is also smooth, and by the symmetry of (2.1) that the inverse of g_ψ is g_{ψ^*}. Recall that the (real) Monge–Ampère measure of a (smooth) convex function is $\mathrm{MA}_{\mathbb{R}}(\psi) := \det(\psi_{j,k}(x))dx$. It follows from the above that

$$g^*_\psi(dy) = \mathrm{MA}_{\mathbb{R}}(\psi), \quad \text{and} \quad g^*_{\psi^*}(dx) = \mathrm{MA}_{\mathbb{R}}(\psi^*).$$

We next turn to the Legendre transformation of functions on $\mathbb{C}^n$ and its relation to complex Monge–Ampère measures. We then redefine the Legendre transformation by (1.2). Equality now occurs when $w = \partial\psi(z)/\partial\bar z =: g_\psi$, where we have also redefined the gradient map g to fit better with complex notation. We now give the first case of Lempert's theorem; it should be compared to how *real* Monge–Ampère measures transform.

Theorem 2.1. (Lempert) *With the above notation*

$$g_\psi^*(\omega_{\hat\psi}) = \omega_\psi, \tag{2.2}$$

so the complex Monge–Ampère measures of ψ *and* $\hat\psi$ *are related by* $g_\psi^*(MA_{\mathbb{C}}(\hat\psi)) = MA_{\mathbb{C}}(\psi)$.

Proof. Let $\Lambda = \{(z,w); w = g_\psi(z)\}$ be the graph of the gradient map g_ψ considered as a submanifold of $\mathbb{C}^{2n}$. On Λ

$$d(z\cdot\bar w) = \partial\psi(z) + \bar\partial\hat\psi(w).$$

(This is simply because when (z,w) lie on Λ, then $\partial\psi(z) = \sum \bar w_j dz_j$ and $\bar\partial\hat\psi(w) = \sum z_j d\bar w_j$.) Since the left hand side is a closed form, it follows that

$$\bar\partial\partial\psi(z) = d\partial\psi(z) = -d\bar\partial\hat\psi(w) = -\partial\bar\partial\hat\psi(w).$$

If we pull back this equation under the map $z \to (z, g_\psi(z))$ we get

$$\partial\bar\partial\psi = g_\psi^*(\partial\bar\partial\hat\psi),$$

which proves the theorem. □

We remark that the apparent discrepancy between how the gradient map transforms the real versus the complex Monge–Ampère measures can be rectified as follows. First, since $[\psi^{ij}] := [\psi_{ij}]^{-1} = [\psi^*_{ij}]$ under appropriate regularity assumptions, the Riemannian metric $\psi_{ij}dx^i \otimes dx^j$ is the pull-back of $\psi^*_{ij}dy^i \otimes dy^j$ via the gradient map $\nabla\psi : \mathbb{R}^n \to \mathbb{R}^n$. Therefore the gradient map pulls back the measure $\sqrt{\det[\psi^*_{ij}]}dy^1\wedge\cdots\wedge dy^n$ to the measure $\sqrt{\det[\psi_{ij}]}dx^1\wedge\cdots\wedge dx^n$. When M has toric symmetry, Theorem 2.1 precisely produces this observation via a careful translation between the real notation and the complex notation (cf. the proof of [6, Proposition 2.1]).

3. Complex Legendre transforms

Let Ω be a domain in $\mathbb{C}^n$ and let ϕ be a real analytic function in Ω. Then ϕ can be extended in a unique way to a function $\phi_{\mathbb{C}}(z,w)$, defined in a neighbourhood, W_ϕ, of the diagonal in $\Omega\times\Omega$, which is holomorphic in z and antiholomorphic in w. Explicitly, if in local coordinates

$$\phi(z) = \sum c_{\alpha,\beta} z^\alpha \bar z^\beta,$$

then

$$\phi_{\mathbb{C}}(z,w) = \sum c_{\alpha,\beta} z^\alpha \bar w^\beta.$$

The *Calabi diastasis function* associated to a real-analytic strongly psh function ϕ on $\Omega\subset\mathbb{C}^n$ is the function

$$D_\phi(p,q) := \phi(p) + \phi(q) - \phi_{\mathbb{C}}(p,q) - \phi_{\mathbb{C}}(q,p) = \phi(p) + \phi(q) - 2\mathrm{Re}\,\phi_{\mathbb{C}}(p,q). \tag{3.1}$$

defined on $W_\phi \subset \Omega\times\Omega$. Clearly $D_\phi(p,q) = D_\phi(q,p)$ with $D_\phi(p,p) = 0$. In the local coordinates,

$$D_\phi(p,q) = \sum c_{\alpha,\beta}(p^\alpha - q^\alpha)\overline{(p^\beta - q^\beta)}.$$

Note that the second order term in the series is non-negative as ϕ is psh. Moreover, denote by $\pi_i : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n,\ i = 1, 2,$ the natural projections, i.e., $\pi_1(z, w) = z, \pi_2(z, w) = w$. Calabi proves the following [5].

Lemma 3.1. *There exists an open neighborhood V_ϕ of Δ_Ω contained in W_ϕ on whose slices $D_\phi(\,\cdot\,, q)$ is strongly convex with*

$$D_\phi(z, q) \geq C|z - q|^2 \quad \text{on } \pi_1(V_\phi \cap \Omega \times \{q\}). \tag{3.2}$$

We can now define a Legendre type transform associated to ϕ. For simplicity, whenever we refer to a function in our discussions below, we do not allow the constant function $+\infty$. We denote by usc$f$ the upper semi-continuous (usc) regularization of a function $f : X \to \mathbb{R}$,

$$\mathrm{usc} f(x) := \lim_{\delta \to 0} \sup_{\substack{y \in X \\ |y - x| < \delta}} f(y).$$

It is the smallest usc function majorizing f.

Definition 3.2. *The complex Legendre transformation $\mathcal{L}_\phi$ is a mapping taking a function $\psi : \Omega \to \mathbb{R} \cup \{\infty\}$ to*

$$\mathcal{L}_\phi(\psi)(q) := \mathrm{usc} \sup_{\substack{p \in \Omega \\ (p,q) \in V_\phi}} [2\mathrm{Re}\, \phi_{\mathbb{C}}(p, q) - \psi(p)].$$

Since $\mathrm{Re}\, \phi_{\mathbb{C}}(p, q)$ is pluriharmonic in q, $\mathcal{L}_\phi$ is psh. The definition depends on V_ϕ, and we discuss that dependence later.

When $\phi(z) = |z|^2$, then $\phi_{\mathbb{C}}(z, w) = z \cdot \overline{w}$, while $D_\phi(z, w) = |z - w|^2$, $C = 1$ and $W_\phi = V_\phi = \mathbb{C}^n \times \mathbb{C}^n$; we recover, up to a factor of 2, the Legendre transformation on $\mathbb{R}^{2n}$.

Lemma 3.3. *Let $\psi : \Omega \to \mathbb{R} \cup \{\infty\}$. Then $\mathcal{L}_\phi(\psi) = \psi$ if and only if $\psi = \phi$.*

Proof. According to Lemma 3.1, whenever $(z, w) \in V_\phi$,

$$2\mathrm{Re}\, \phi_{\mathbb{C}}(z, w) \leq \phi(z) + \phi(w),$$

with equality iff $z = w$. Thus $\mathcal{L}_\phi \phi = \phi$. Conversely, suppose $\mathcal{L}_\phi \psi = \psi$. Then, whenever $(z, w) \in V_\phi$,

$$\psi(z) + \mathcal{L}_\phi \psi(w) = \psi(z) + \psi(w) \geq 2\mathrm{Re}\, \phi_{\mathbb{C}}(z, w).$$

Setting $z = w$ gives $\psi \geq \phi$. Since the complex Legendre transform is order-reversing then also $\psi \leq \phi$. ∎

Definition 3.4. *Say that a function $\psi : \Omega \to \mathbb{R} \cup \{\infty\}$ is ϕ-convex if $\psi = \mathcal{L}_\phi \eta$ for some usc function $\eta : \Omega \to \mathbb{R} \cup \{\infty\}$.*

Lemma 3.5. *Let $\eta : \Omega \to \mathbb{R} \cup \{\infty\}$ be usc. Then $\mathcal{L}_\phi^2 \eta \leq \eta$ with equality iff η is ϕ-convex.*

Proof. Whenever $(z, w) \in V_\phi$,

$$\eta(z) + \mathcal{L}_\phi \eta(w) \geq 2\mathrm{Re}\, \phi_{\mathbb{C}}(z, w).$$

Thus,

$$\mathcal{L}_\phi^2 \eta(z) = \mathrm{usc} \sup_w [2\mathrm{Re}\, \phi_{\mathbb{C}}(w, z) - \mathcal{L}_\phi \eta(w)] \leq \mathrm{usc}\, \eta(z) = \eta(z). \tag{3.3}$$

Next, if $\mathcal{L}_\phi^2\eta = \eta$, then by definition η is ϕ-convex. It remains therefore to show the converse, and for this it suffices to show that $\mathcal{L}_\phi^3\nu = \mathcal{L}_\phi\nu$ for any usc function $\nu : \Omega \to \mathbb{R} \cup \{\infty\}$. By (3.3), $\mathcal{L}_\phi^3\nu \le \mathcal{L}_\phi\nu$. However, $\mathcal{L}_\phi^2\nu \le \nu$ by (3.3), thus $\mathcal{L}_\phi^3\nu \ge \mathcal{L}_\phi\nu$.

□

4. Legendre duality on compact manifolds

Remarkably, a variant of these transforms can be defined on any (compact) Kähler manifold M. Let ω be a closed strictly positive real-analytic $(1,1)$-form on M. Locally then ω equals $\sqrt{-1}\partial\bar{\partial}u$ for some strongly psh real-analytic function u, and we define $u_{\mathbb{C}}$ and subsequently

$$D_\omega := D_u,$$

locally. To check that these definitions are actually consistent globally and give rise to a diastasis function on a non-empty neighborhood of Δ_M it suffices to observe (see [5]) that whenever h is a real-valued function on a ball in $\mathbb{C}^n$ that is pluriharmonic, i.e., $\sqrt{-1}\partial\bar{\partial}h = 0$, then $h = h_1(p) + \overline{h_1(p)}$ with h_1 holomorphic. Thus, $h_{\mathbb{C}}(p,\bar{q}) = h_1(p) + \overline{h_1(q)}$, so $D_h \equiv 0$. Once again, by a variant of Lemma 3.1 [5, Proposition 5] we obtain an open neighborhood V_ω of the diagonal on which D_ω is nonnegative and strongly convex with respect to local coordinates in each variable and on which ω admits local real analytic Kähler potential u for which $u_{\mathbb{C}}$ exists.

Now, we fix a real-analytic Kähler form ω on M and define a Legendre transformation with respect to ω.

Definition 4.1. *The complex Legendre transformation L_ω maps a function $\psi : M \to \mathbb{R} \cup \{\infty\}$ to*

$$L_\omega(\psi)(q) := \operatorname{usc} \sup_{\substack{p \in M \\ (p,q) \in V_{\omega_\varphi}}} [-D_\omega(p,q) - \psi(p)].$$

As in the setting of $\Omega \subset \mathbb{C}^n$, the transformation also depends on V_{ω_φ}.

Definition 4.2. *Say that ψ is ω-convex if $\psi = L_\omega\eta$ for some usc function $\eta : M \to \mathbb{R} \cup \{\infty\}$.*

The following lemmas follow in the same manner as Lemmas 3.3 and 3.5. In fact, intuitively, L_ω is locally given by

$$L_\omega(\psi)(q) = \mathcal{L}_u(u + \psi) - u,$$

where u is a local Kähler potential for ω.

Lemma 4.3. *Let $\psi : M \to \mathbb{R} \cup \{\infty\}$. Then $L_\omega(\psi) = \psi$ if and only if $\psi = 0$.*

Lemma 4.4. *Let $\eta : M \to \mathbb{R} \cup \{\infty\}$. Then $L_\omega^2\eta \le \eta$ with equality iff η is ω-convex.*

5. A generalized gradient map

The fact that Calabi's diastasis D_ω is locally uniformly convex in each variable on a neighborhood of the diagonal should, intuitively, ensure the supremum in the definition of L_ω is attained in a unique point. In this section we make this intuition rigorous by giving a condition that ensures the supremum is attained. We will discuss only the case of compact manifolds. The case of $\mathbb{C}^n$, leading up to Theorem 1.1, is proved similarly.

Theorem 5.1. *Let (M,ω) be a compact closed real-analytic Kähler manifold. There exists $\epsilon = \epsilon(\omega) > 0$ such that for every function η satisfying $||\eta||_{C^2(M)} < \epsilon$, the supremum in Definition 4.1 is for any q in M attained at a unique point, $z = G(\eta)(q)$.*

1. *If η is of class C^k then $G(\eta)$ is a diffeomorphism of M of class C^{k-1}.*
2. *$L_\omega(\eta)$ is of class C^k and the map $\eta \to L_\omega(\eta)$ is continuous for the C^k-topology.*
3. *$L^2_\omega(\eta) = \eta$.*
4. *$G(L_\omega(\eta)) = G(\eta)^{-1}$.*

Proof. Because M is compact, the neighborhood V_ω contains a ball of fixed size, call it $\delta > 0$ (with respect to the the distance function d of the reference metric ω, say), around every point on the diagonal. Fix $q \in M$. Let

$$f_q(z) = -D_\omega(z,q) - \eta(z), \quad z \in \pi_1(V_\omega \cap (M \times \{q\})). \tag{5.1}$$

We claim that f_q attains a unique maximum in $\pi_1(V_\omega \cap (M \times \{q\}))$. First, Lemma 3.1 implies that

$$f_q(z) \le -Cd(z,q)^2 - \eta(z), \quad z \in \pi_1(V_\omega \cap (M \times \{q\})),$$

and if $||\eta||_{C^2(M)}$ is sufficiently small, f_q is uniformly concave on $\pi_1(V_\omega \cap (M \times \{q\}))$.

If $||\eta||_{C^0(M)} < \epsilon$,

$$f_q(q) \ge -\epsilon,$$

while,

$$f_q(z) \le -Cd(z,q)^2 + \epsilon, \quad z \in \pi_1(V_\varphi \cap (M \times \{q\})),$$

So, if ϵ is small enough,

$$f_q(z) \le -2\epsilon, \quad z \in \pi_1(V_\varphi \cap (M \times \{q\})) \setminus B_{\delta/2}(q),$$

Thus we see that the maximum of f_q over $\pi_1(V_\omega \cap (M \times \{q\}))$ must be attained at a point in $B_{\delta/2}(q)$, which moreover is unique by the strict concavity of f_q.

This maximum point is the unique solution z of

$$F_q(z) := \nabla_z f_q(z) = 0 \tag{5.2}$$

in $B_{\delta/2}(q) \subset \pi_1(V_\phi \cap (M \times \{q\}))$. We denote this unique solution by $z = G(\eta)(q)$. Thus,

$$L_\omega(\eta)(q) = f_q(G(\eta)(q)) \tag{5.3}$$

Since f_q is uniformly concave in $B_{\delta/2}(q)$, the Implicit Function Theorem (IFT) implies that $G(\eta)(q)$ is of class C^{k-1} in q whenever $\eta \in C^k$. Thus, by (5.3) it follows that $L_\omega \eta \in C^{k-1}$.

Next, we claim that $G(\eta)$ is invertible. To see that, let

$$F_{t,q}(z) := \nabla f_{t,q}(z), \quad f_{t,q}(z) := -D_\omega(z,q) - t\eta(z). \tag{5.4}$$

The IFT, applied to $F_{t,q}$, implies that

$$\nabla G(t\eta) = -(\nabla_z F_{t,q})^{-1} \nabla_q F_{t,q} = (\nabla^2_z D_\omega(z,q) - t\nabla^2_z \eta)^{-1} \nabla_q F_q. \tag{5.5}$$

When $t = 0$, Lemma 4.3 implies that $G(0)(q) = q$, so

$$I = \nabla G_\varphi(0) = -(\nabla_z F_q)^{-1} \nabla_q F_q = (\nabla^2_z D(z,q))^{-1} \nabla_q F_q. \tag{5.6}$$

Combining (5.5) and (5.6) we see that whenever $||\eta||_{C^2}$ is sufficiently small, the Jacobian of $G(t\eta)$ is positive definite for all $t \in [0,1]$, hence the Jacobian of $G(\eta)$ is invertible. This means that $G(\eta)$ is locally injective, i e that if $q \neq q'$ and $d(q,q')$ is sufficiently small, then $G(\eta)(q) \neq G(\eta)(q')$. Since we moreover know that $G(\eta)$ is uniformly close to the identity, this gives that $G(\eta)$ is globally injective. Since it is also open, it is a diffeomorphism onto its image.

That the supremum in Definition 4.1 is obtained for $z = G(\eta)(q)$ means that

$$L_\omega(\eta)(q) = -D_\omega(G(\eta)(q), q) - \eta(G((\eta)(q)). \tag{5.7}$$

Since, for $(z,q) \in V_\omega$ we always have

$$\eta(z) \geq -D_\omega(z,q) - L_\omega(\eta)(q) \tag{5.8}$$

it follows that

$$\nabla L_\omega(\eta)(q) = -\nabla_q D_\omega(z,q). \tag{5.9}$$

when $G(\eta)(q) = z$, so

$$\nabla L_\omega(\eta)(q) = -(\nabla_q D_\omega)(G(\eta)(q), q). \tag{5.10}$$

But D_ω is smooth, in fact real analytic, so we get, since $G(\eta)$ is of class C^{k-1} that $\nabla L_\omega(\eta)$ is of class C^{k-1} too. In other words $L_\omega(\eta)$ is of class C^k.

On the other hand, if η is close to zero, we know that $G(\eta)$ is close to the identity, which is $G(0)$. Hence, by (5.9), $\nabla L_\omega(\eta)$ is close to $\nabla L_\omega(0) = 0$. Since it follows directly from the definition that the C^0-norm of $L_\omega(\eta)$ is small if the C^0-norm of η is small, it follows that $L_\omega(\eta)$ is close to zero in the C^k-norm if η is close to zero in the C^k-norm. In particular, with $k = 2$, this implies that we can apply the arguments in the beginning of this proof to $L_\omega(\eta)$. Then (5.6) implies that $L^2_\omega(\eta) = \eta$ and that $G(L_\omega(\eta)) = G(\eta)^{-1}$.

This completes the proof. □

6. The inverse gradient map and the complex Monge–Ampère operator

As we shall comment later on, our next result can be viewed as a variant of a result of Lempert.

Theorem 6.1. *Fix a real-analytic Kähler form ω. Then, for each smooth function ψ such that $\sqrt{-1}\partial\bar\partial\psi + \omega > 0$, we let $\omega_\psi := \omega + \sqrt{-1}\partial\bar\partial\psi$. Then if ψ satisfies the assumptions on η of Theorem 5.1*

$$G(\psi)^\star \omega_\psi = \omega_{L_\omega \psi}.$$

Therefore, the complex Monge–Ampère measure of ψ, ω_ψ^n, is pulled-back under $G(\psi)$ to the complex Monge–Ampère measure of $L_\omega\psi$.

As pointed out in the introduction and section 2, this should be compared with the contrasting fact that the real Monge–Ampère operator is pulled-back under the inverse gradient map to the Euclidean measure.

Proof. Since the statement is local we look at a neighborhood $W \subset M$ where we have a real-analytic Kähler potential ϕ of ω. By definition, for $(z,w) \in V_\omega$

$$-D_\omega(z,w) \leq \psi(z) + L_\omega\psi(w)$$

with equality precisely when $z = G(\psi)(w)$. Let Λ be the set where this holds. Then, when $(z, w) \in \Lambda$,

$$-\partial_z D_\omega(z, w) = \partial_z \psi(z),$$

or, equivalently,

$$-\bar{\partial}_w D_\omega(z, w) = \bar{\partial}_w L_\omega \psi(w).$$

In other words (since $D_\omega(z, w) = \phi(z) + \phi(w) - \phi_{\mathbb{C}}(z, w) - \bar{\phi}_{\mathbb{C}}(z, w)$),

$$\partial_z \phi_{\mathbb{C}}(z, w) - \partial_z \phi(z) = \partial_z \psi(z), \tag{6.1}$$

and

$$\bar{\partial}_w \phi_{\mathbb{C}}(z, w) - \bar{\partial}_w \phi(w) = \bar{\partial}_w L_\omega \psi(w). \tag{6.2}$$

This means that the identity holds when both sides are considered as forms on $\mathbb{C}^{2n}$ and (z, w) lies on Λ. Since Λ is the graph of $G(\psi)$, Λ is a manifold of real dimension $2n$. Let p_1 and p_2 be the projections of $W \times W$ to the first and second factors, and let π_1 and π_2 be their restrictions to Λ. By Theorem 5.1, π_1 and π_2 are invertible maps and

$$G(\psi) = \pi_1 \circ \pi_2^{-1}. \tag{6.3}$$

By (6.1) and (6.2),

$$d\phi_{\mathbb{C}}(z, w) = \pi_1^\star \partial(\phi + \psi)(z) + \pi_2^\star \bar{\partial}(\phi + L_\omega \psi)(w), \quad \text{when } (z, w) \in \Lambda.$$

Hence the same identity holds when we restrict both sides to Λ as differential forms. Since the left hand side is a closed form, it follows that

$$\pi_1^* d\partial(\phi + \psi)(z) + \pi_2^* d\bar{\partial}(\phi + L_\omega \psi)(w) = 0, \quad \text{on } \Lambda.$$

If we apply $(\pi_2^{-1})^*$ to this equation we get

$$(\pi_2^{-1})^* \pi_1^* \bar{\partial}\partial(\phi + \psi) + \partial\bar{\partial}(\phi + L_\omega \psi) = 0.$$

By (6.3) it follows that

$$G(\psi)^\star \omega_\psi = G(\psi)^\star(\sqrt{-1}\partial\bar{\partial}(\phi + \psi)) = \sqrt{-1}\partial\bar{\partial}(\phi + L_\omega \psi) = \omega_{L_\omega \psi},$$

so we are done. □

7. The Mabuchi metric

Let M be a closed compact Kähler manifold. Recall that if ω is a Kähler form on M, the space of ω-plurisubharmonic functions, $\mathcal{H}_\omega$ is the space of smooth functions on M such that $\omega_\psi := \sqrt{-1}\partial\bar{\partial}\psi + \omega > 0$. This is an open subset of the space of smooth functions and inherits a structure as a differentiable manifold from the one on $C^\infty(M)$. The tangent space to $\mathcal{H}_\omega$ is the space of smooth functions on M and one defines a weak Riemannian metric on $\mathcal{H}_\omega$ by

$$g_M(\nu, \chi)_\psi = \int_M \nu \chi \omega_\psi^n,$$

for every $\nu, \chi \in T_\psi \mathcal{H}_\omega \cong C^\infty(M)$.

Proposition 7.1. *Let ω be real-analytic. There exists a neighborhood $\mathcal{U}_\omega$ of 0 in $\mathcal{H}_\omega$ in the C^2 topology such that L_ω defines a Fréchet differentiable map from $\mathcal{U}_\omega$ to $\mathcal{U}_\omega$. Its differential is*

$$dL_\omega(\eta).\chi = -\chi \circ G(\eta), \quad \forall \eta \in \mathcal{U}_\omega.$$

Proof. Let $q \in M$. Define (cf. (5.1)),

$$f_q(z, \eta) := -D_\omega(z, q) - \eta(z), \quad z \in \pi_1(V_\omega \cap (M \times \{q\})),$$

and let $F_q(z, \eta) := \nabla_z f_q(z, \eta)$. Then F_q is of class C^{k-1} if η is of class C^k. By the implicit function theorem the equation

$$F_q(z, \eta) = 0$$

defines z as a function of η, $z = z(\eta)$, and since $z(\eta)$ is the point maximizing $f_q(z, \eta)$ for given η we have that $z(\eta) = G(\eta)(q)$ (which we now regard as a function of η, while q is fixed). Hence we see that $z(\eta) = G(\eta)(q)$ is of class C^{k-1}. Moreover

$$L_\omega(\eta)(q) = f_q(z(\eta), \eta).$$

Hence, by the chain rule

$$d/dt|_{t=0} L_\omega(\eta + t\chi) = d/dt|_{t=0} f_q(z(\eta), \eta + t\chi),$$

since $\nabla_z f_q(z, \eta) = 0$ for $z = z(\eta)$. Since

$$d/dt|_{t=0} f_q(z(\eta), \eta + t\chi) = -\chi(z(\eta)) = -\chi(G(\eta)(q))$$

we are done. □

Theorem 7.2. *Let M be a closed compact Kähler manifold and let ω be a real analytic Kähler form. There exists a C^2 neighborhood $\mathcal{U}_\omega$ of 0 in $\mathcal{H}_\omega$ such that L_ω defines a Fréchet differentiable map from $\mathcal{U}_\omega$ to itself with the following properties:*

(i) L_ω is an isometry for the Mabuchi metric on $\mathcal{U}_\omega$.

(ii) $L_\omega^2 \psi = \psi$ for $\psi \in \mathcal{U}_\omega$.

(iii) $L_\omega \psi = \psi$ if and only if $\psi = 0$.

Proof. Properties (ii) and (iii) are the content of Lemma 4.3 and Theorem 5.1. We turn to proving (i). Indeed, by Theorem 5.1 and Proposition 7.1

$$\begin{aligned} g_M(dL_\omega\psi(\chi), dL_\omega\psi(\nu))|_{L_\omega\psi} &= \int_M \chi\nu \circ G(\psi)\omega^n_{L_\omega\psi} \\ &= \int_M \chi\nu \circ G(\psi) G(\psi)^\star(\omega^n_\psi) \\ &= \int_M \chi\nu\omega^n_\psi = g_M(\chi, \nu)|_\psi, \end{aligned}$$

proving (i).

Finally, if $\mathcal{U}_\omega$ is a neighborhood enjoying properties (i)–(iii), then replacing $\mathcal{U}_\omega$ by $\mathcal{U}_\omega \cap \mathcal{L}(\mathcal{U}_\omega)$, we may assume $\mathcal{L}$ maps $\mathcal{U}_\omega$ to itself. □

This theorem should be seen in the light of the picture of $\mathcal{H}_\omega$ as a symmetric space, put forward by Mabuchi, Semmes and Donaldson, [15, 22, 7]. In these works $\mathcal{H}_\omega$ is first studied as a Riemannian manifold, its curvature tensor is computed and is found to be covariantly constant. In the finite dimensional case, this implies the existence of symmetries around any point in the space. As described in §8.1, Semmes has also found symmetries for the Mabuchi metric, but to our knowledge the ω_φ-Legendre transforms are the first examples of explicit symmetries for $\mathcal{H}_\omega$. It would be interesting to generalize the theorems of Artstein-Avidan–Milman [1] and Böröczky–Schneider [4] to this setting and investigate whether these are *all* the symmetries of $\mathcal{H}_\omega$ under some reasonable regularity assumptions.

From Theorem 7.2 it follows in particular that the ω-Legendre transformation maps geodesics in $\mathcal{H}_\omega$ to geodesics. By the work of Semmes [22], geodesics in $\mathcal{H}_\omega$ are precisely given by solutions of the homogenous complex Monge–Ampère equation, so that a curve $t \to \psi_t(z) = \psi(t,z)$, where t lies in a strip $0 < \mathrm{Re}\, t < 1$ and ψ depends only on the real part of t, is a geodesic in $\mathcal{H}_\omega$ if and only if

$$(\sqrt{-1}\partial\bar{\partial}_{t,z}\psi + \omega)^{n+1} = 0.$$

One main motivation for Lempert's work was to find symmetries of the inhomogenous complex Monge–Ampère equation. Here we find a somewhat different kind of symmetries for the homogenous complex Monge–Ampère equation (HCMA). The applicability of this may be somewhat limited by the absence of positive existence results for geodesics, but if we change the setup slightly and consider functions ψ defined for t in a disk instead of a strip, there is at least one setting in which our theorem applies. Considering boundary data $s \to \psi_s$ on the unit circle that happen to extend to a smooth solution of the HCMA, then the same thing holds for sufficiently small perturbations of the data [8], see also [17]. Taking the given boundary data to be identically equal to 0, for which trivially an extension exists, we see that any boundary data that are sufficiently small can be extended to a solution of the HCMA, ψ_t with t in the disk Δ. Theorem 7.2 shows that then $L_\omega(\psi_t)$ also solves the HCMA. Indeed, solutions of the HCMA are critical points of the energy functional induced by the Mabuchi metric

$$E(\psi) = \int_{\Delta\times M} \partial_t\psi \wedge \bar{\partial}_t\psi \wedge \omega^n_{\psi_t},$$

and as shown above the energy functional is preserved under the φ-Legendre transform.

8. Relations with Lempert's and Semmes' work

8.1. **Comparison to Lempert's theorem.** Lempert starts with a complex manifold M and its holomorphic cotangent bundle $T^*(M)$. If z are local coordinates on M it induces local coordinates (z,ξ) on $T^*(M)$, so that a one-form, i e a point in $T^*(M)$ can be written $\sum \xi_j dz_j$. There is a standard holomorphic symplectic form Ω on $T^*(M)$ that in such coordinates is written $\Omega = \sum d\xi_j \wedge dz_j$. A (local) holomorphic map from $T^*(M)$ to itself, F, is symplectic if $F^*(\Omega) = \Omega$, and Lempert's construction depends on the choice of such a symplectic map. Another ingredient is a differentiable real valued function ψ on M. From ψ we get a gradient map

$$z \to (z, \partial\psi) =: \nabla\psi \tag{8.1}$$

which is a section of $T^*(M)$. Lempert's generalized gradient map is the map from M to itself

$$G_\psi = \pi \circ F \circ \nabla\psi,$$

where π is the projection from $T^*(M)$ to M. He then defines a generalized Legendre transformation by

$$L_F(\psi)(G_\psi(z)) = \psi(z) + 2\mathrm{Re}\,\Sigma(\nabla\psi),$$

where Σ is a *generating function* of the symplectic tranformation F. This means that Σ is holomorphic on $T^*(M)$ and satisfies

$$d\Sigma = \xi \cdot dz - F^*(\xi \cdot dz).$$

Such a generating function exists at least locally since the right hand side is a closed form if F is symplectic.

We indicate briefly how this translates to our set up. First, there is a minor difference that we work with a symplectic form and generating function that is holomorphic in z and antiholomorphic in ξ, but the major difference is that we chose a different kind of generating function. The symplectic transformation F gives a map from $T^*(M)$ to M by $w = \pi(F(z))$. For special symplectic maps (sometimes called *free canonical transformations*) one can choose (z, w) as coordinates on $T^*(M)$ and express the generating function in terms of these coordinates instead. Locally, our construction amounts to choosing $\phi_{\mathbb{C}}(z, w)$ as such a generating function. If we define a symplectic transformation using $\phi_{\mathbb{C}}$ as a generating function one can check that our Legendre transformation coincides with Lempert's.

8.2. **Semmes' work.** Another major motivation for our work is Semmes' work [22] and we now relate the previous theorem to his work. Semmes starts by endowing the holomorphic cotangent bundle $(T^*)^{1,0}M$ with the complex structure $\hat{J}$ induced by pulling back the standard complex structure (induced by the complex strucutre J on M) under the (locally defined) maps [22, p. 530]

$$(z, \lambda) \mapsto (z, \lambda + \partial_z\phi).$$

This is well-defined and independent of the choice of local potential ϕ for ω since $\partial_z\phi - \partial_z\phi'$ is holomorphic whenever ϕ' is another such choice. To any smooth Kähler potential ψ, Semmes then associates the submanifold Λ_ψ, the graph of $\partial\psi$ in $(T^*)^{1,0}M$. Under the biholomorphism between $((T^*)^{1,0}M, \hat{J})$ and $((T^*)^{1,0}M, J)$ the standard tautological 1-form $\alpha = \sum \lambda_i dz_i$ and holomorphic symplectic form $\Omega = \sum dz_i \wedge d\lambda_i$ on the latter are pulled back to forms that we denote by $\hat{\alpha}$ and $\hat{\Omega}$. Then $\sqrt{-1}\hat{\Omega}|_{\Lambda_\psi} = \sqrt{-1}\partial\bar{\partial}(\phi + \psi) = \omega_\psi$. Semmes goes on to observe that whenever φ is real-analytic, there exists an involutive anti-biholomorphism of a neighborhood of Λ_φ in $((T^*)^{1,0}M, \hat{J})$ whose fixed-point set equals Λ_φ. Thus, if ψ is sufficiently close to φ in C^2 then Λ_ψ is mapped to another submanifold that must be of the form Λ_η for some η. Theorem 6.1 precisely establishes that this involution is given by our generalized gradient map $G_\varphi(\psi)$, so $G_\varphi(\psi)(\Lambda_{L_\omega\psi}) = \Lambda_\psi$.

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References

[1] S. Artstein-Avidan, V. Milman, Hidden structures in the class of convex functions and a new duality transform, J. Euro. Math. Soc. 13 (2011), 975–1004.

[2] E. Bedford, B.A. Taylor, The Dirichlet problem for a complex Monge–Ampère equation, Invent. Math. 37 (1976), 1–44.

[3] B. Berndtsson, D. Cordero-Erausquin, B. Klartag, Y.A. Rubinstein, Complex interpolation of real norms, duality, and foliations, preprint, 2016.

[4] K. J., Böröczky, R., Schneider, A characterization of the duality mapping for convex bodies. Geom. Funct. Anal. 18 (2008), no. 3, 657–667.

[5] E. Calabi, Isometric imbeddings of complex submanifolds, Ann. of Math. 58 (1953), 1–23.

[6] B. Clarke, Y.A. Rubinstein, Ricci flow and the metric completion of the space of Kähler metrics, Amer. J. Math. 135 (2013), 1477–1505.

[7] S.K. Donaldson, Symmetric spaces, Kähler geometry and Hamiltonian dynamics, in: Northern California Symplectic Geometry Seminar (Ya. Eliashberg et al., Eds.), Amer. Math. Soc., 1999, pp. 13–33.

[8] S.K. Donaldson, Holomorphic discs and the complex Monge–Ampère equation, J. Symplectic Geom. 1 (2002), 171–196.

[9] W. Fenchel, On conjugate convex functions, Canad. J. Math. 1 (1949), 73–77.

[10] A. Hatcher, Algebraic topology, Cambridge University Press, 2002.

[11] J.-B. Hiriart-Urruty, C. Lemaréchal, Convex analysis and minimization algorithms II, Springer, 1993.

[12] C.O. Kiselman, The partial Legendre transformation for plurisubharmonic functions, Invent. Math. 49

[13] L. Lempert, Symmetries and other transformations of the complex Monge–Ampère equation, Duke Math. J. 52 (1985), 869–885.

[14] L. Lempert, On complex Legendre duality, preprint 2017, available at Arxiv 1703.01811.

[15] T. Mabuchi, Some symplectic geometry on compact Kähler manifolds I, Osaka J. Math. 24, 1987, 227–252.

[16] S. Mandelbrojt, Sur les fonctiones convexes, C. R. Acad. Sci. Paris 209 (1939), 977–978.

[17] R. Moriyon, Regularity of the Dirichlet problem for the complex Monge–Ampère equation, Proc. Natl. Acad. Sci. USA 76 (1979), 1022–1023; Comm. Pure Appl. Math. 35 (1982), 1–27.

[18] J. Rauch, B.A. Taylor, The Dirichlet problem for the multidimensional Monge–Ampère equation, Rocky Mountain J. Math. 7 (1977), 345–364.

[19] R. Rochberg, Interpolation of Banach spaces and negatively curved vector bundles, Pacific J. Math. 110 (1984), 355–376.

[20] R.T. Rockafellar, Convex analysis, Princeton University Press, 1970.

[21] S. Semmes, Interpolation of spaces, differential geometry and differential equations, Rev. Mat. Iberoamericana 4 (1988), 155–176.

[22] ____, Complex Monge–Ampère and symplectic manifolds, Amer. J. Math. 114 (1992), 495–550.

DEPARTMENT OF MATHEMATICS, CTH, 412 96 GÖTEBORG, SWEDEN
and
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GOTHENBURG, 412 96 GÖTEBORG, SWEDEN
bob@chalmers.se

INSTITUT DE MATHÉMATIQUES DE JUSSIEU, ÉQUIPE D'ANALYSE FONCTIONNELLE, UNIVERSITÉ PIERRE ET MARIE CURIE, 4, PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE
cordero@math.jussieu.fr

DEPARTMENT OF MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100 ISRAEL,
and
SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY, TEL AVIV 69978 ISRAEL.
klartagb@post.tau.ac.il

UNIVERSITY OF MARYLAND, DEPARTMENT OF MATHEMATICS, 4176 CAMPUS DRIVE, COLLEGE PARK, MD 20742-4015, USA
yanir@umd.edu

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