# Functional affine-isoperimetry and an inverse logarithmic Sobolev inequality *

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**Abstract**

We give a functional version of the affine isoperimetric inequality for log-concave functions which may be interpreted as an inverse form of a logarithmic Sobolev inequality inequality for entropy. A linearization of this inequality gives an inverse inequality to the Poincaré inequality for the Gaussian measure.

## 1 Introduction

There is a general approach to extend invariants of convex bodies to the corresponding invariants of functions [1, 7, 10, 17]. We investigate here the affine surface area and the affine isoperimetric inequality and their corresponding invariants for log-concave functions. The affine isoperimetric inequality corresponds to an inequality that may be viewed as an inverse logarithmic Sobolev inequality for entropy. A linearization of this inequality yields an inverse inequality to a Poincaré inequality.

Logarithmic Sobolev inequalities provide upper bounds for the entropy. There is a vast amount of literature on logarithmic Sobolev inequalities and related topics, e.g. [2, 5, 6, 9, 11, 15, 21]. We quote only the sharp logarithmic Sobolev inequality for the Lebesgue measure on $\mathbb{R}^n$ (see, e.g., [4])

$$\int_{\text{supp}(f)} |f|^2 \ln(|f|^2) dx - \left( \int_{\mathbb{R}^n} |f|^2 dx \right) \ln \left( \int_{\mathbb{R}^n} |f|^2 dx \right) \leq \frac{n}{2} \ln \left( \frac{2}{\pi e n} \int_{\mathbb{R}^n} \|\nabla f\|^2 dx \right), \quad (1)$$

with equality if and only if $f(x) = (2\pi)^{-(n/4)} \exp(-\|x-b\|^2/4)$ for a vector $b \in \mathbb{R}^n$. Here, and throughout the paper, $\|\cdot\|$ denotes the standard Euclidean norm and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^n$. This inequality is directly equivalent to the Gross logarithmic Sobolev inequality [4, 9]

$$\int_{\text{supp}(h)} |h|^2 \ln \left( \frac{|h|}{\|h\|_{L^2(\gamma_n)}} \right) d\gamma_n \leq \int_{\mathbb{R}^n} \|\nabla h\|^2 d\gamma_n, \quad (2)$$

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where $\gamma_n$ is the normalized Gauss measure on $\mathbb{R}^n$, $d\gamma_n = (2\pi)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{2}} dx$. Equation (2) becomes an equality if and only if $h(x) = ce^{\langle a,x\rangle}$ with $c > 0$ and $a \in \mathbb{R}^n$.

We will now integrate by parts, and rewrite the logarithmic Sobolev inequality as an upper bound for the entropy in terms of the Laplacian of the function. The main result in this note shall be a lower bound for entropy in terms of the Laplacian, the difference between the two bounds being an interchange between integration and logarithm and replacement of the arithmetic mean of the eigenvalues of the Hessian by the geometric mean.

We shall need some more notation. Let $(X, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a measurable function. Denote the support of $f$ by $\mathrm{supp}(f) = \{x : f(x) \neq 0\}$. Then the *entropy* of $f$, $\mathrm{Ent}(f)$, is defined (whenever it makes sense) by

$$\mathrm{Ent}(f) = \int_{\mathrm{supp}(f)} |f| \ln(|f|) d\mu - \|f\|_{L^1(X,\mu)} \ln \|f\|_{L^1(X,\mu)} = \int_{\mathrm{supp}(f)} f \ln\left(\frac{|f|}{\|f\|_{L^1(X,\mu)}}\right) d\mu, \tag{3}$$

where $\|f\|_{L^1(X,\mu)} = \|f\|_{L^1(\mu)} = \int_X |f| d\mu$. In particular, if $\|f\|_{L^1(X,\mu)} = 1$,

$$\mathrm{Ent}(f) = \int_{\mathrm{supp}(f)} |f| \ln(|f|) d\mu.$$

If $f$ is a positive function, we get in (1)

$$\begin{aligned} \mathrm{Ent}(f) &= \int_{\mathbb{R}^n} f \ln(f) dx - \left(\int_{\mathbb{R}^n} f dx\right) \ln\left(\int_{\mathbb{R}^n} f dx\right) \\ &\leq \frac{n}{2} \ln\left(\frac{2}{\pi e n} \int_{\mathbb{R}^n} \|\nabla \sqrt{f}\|^2 dx\right) = \frac{n}{2} \ln\left(\frac{1}{2\pi e n} \int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f} dx\right). \end{aligned} \tag{4}$$

For a sufficiently smooth function $f$ defined on $\mathbb{R}^n$, we denote the Hessian of $f$ by $\nabla^2(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\ldots,n}$. Note that

$$\int_{\mathrm{supp}(f)} \frac{\|\nabla f\|^2}{f} dx = \int_{\mathrm{supp}(f)} f \left(\mathrm{tr}\left(\nabla^2(-\ln f)\right)\right) dx. \tag{5}$$

For $f \geq 0$ with $\int f dx = 1$, this is the *Fisher information*. Equation (5) is easily verified using integration by parts.

The logarithmic Sobolev inequality (4), together with (5), becomes

$$\mathrm{Ent}(f) + \ln((2\pi e)^{\frac{n}{2}}) \leq \frac{n}{2} \ln\left[\frac{1}{n} \int_{\mathrm{supp}(f)} f \left(\mathrm{tr}\left(\nabla^2(-\ln f)\right)\right) dx\right]. \tag{6}$$

The main goal in this paper is to prove, for log-concave functions, a converse of inequality (6). A function $f : \mathbb{R}^n \to \mathbb{R}$ is called log-concave if it takes the form $\exp(-\Psi)$ for a convex function $\Psi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. We shall usually assume also that the function is upper semi-continuous.

This converse log Sobolev inequality is stated in the following theorem. It relates entropy to a new expression, which can be thought of as an affine invariant version of Fisher information.

The inequality is obtained by suitably applying and analysing the affine isoperimetric inequality, which, for convex bodies $K$ in $\mathbb{R}^n$, gives an upper bound for the affine surface area. Affine surface area measures and their related inequalities (see below for the definition and statements) have attracted considerable attention recently e.g. [8, 12, 14, 20, 23].

**Theorem 1.** *Let $f : \mathbb{R}^n \to [0, \infty)$ be an upper semi-continuous log-concave function which belongs to $C^2(supp(f)) \cap L^1(\mathbb{R}^n, dx)$ and such that $f \ln f$ and $f \ln \det \Big(\nabla^2 \left(-\ln f\right)\Big) \in L^1(supp(f), dx)$. Then*

$$\int_{supp(f)} f \;\ln \Big( \det \left(\nabla^2 \left(-\ln f\right)\right) \Big) dx \leq 2 \Big[ \mathrm{Ent}(f) + \|f\|_{L^1(dx)} \ln(2\pi e)^{\frac{n}{2}} \Big].$$

*There is equality for $f(x) = Ce^{-\langle Ax, x\rangle}$, where $C > 0$ and $A$ is an $n \times n$ positive-definite matrix of determinant one.*

It is important to note the affine invariant nature of Theorem 1. Both the left-hand side and the right-hand side are invariant under volume-preserving linear transformations. This is not the case with the logarithmic Sobolev inequality. The expression on the right-hand side of (6) involves the arithmetic mean $\frac{1}{n} \; \left(\mathrm{tr}\left(\nabla^2 \left(-\ln f\right)\right)\right)$ of the eigenvalues of $\nabla^2 \left(-\ln f\right)$. The expression on the left-hand side of Theorem 1 can be written as $n \;\ln \Big(\det \left(\nabla^2 \left(-\ln f\right)\right)\Big)^{\frac{1}{n}}$ and involves the geometric mean of the eigenvalues of $\nabla^2 \left(-\ln f\right)$. Thus, we get from an upper bound for the entropy to a lower bound for the entropy by interchanging integration and logarithm and by replacing the arithmetic mean of the eigenvalues of the Hessian by its geometric mean.

As the entropy for the Gaussian random variable $g(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{\|x\|^2}{2}}$ is $\mathrm{Ent}(g) = -\ln(2\pi e)^{\frac{n}{2}}$, Theorem 1 immediately implies the following corollary.

**Corollary 2.** *Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function such that $f \in C^2(\mathbb{R}^n)$, $\|f\|_{L^1(dx)} = 1$ and such that $f \ln f$ and $f \ln \left(\det \nabla^2 \left(-\ln f\right)\right) \in L^1(supp(f), dx)$. Then*

$$\int_{supp(f)} f \ln \Big( \det \left(\nabla^2 \left(-\ln f\right)\right) \Big) dx \leq 2 \big( \mathrm{Ent}(f) - \mathrm{Ent}(g) \big),$$

*with equality for $f(x) = e^{-\pi\langle Ax, x\rangle}$ for a positive-definite matrix $A$ of determinant one.*

The expression $\mathrm{Ent}(f) - \mathrm{Ent}(g)$ is called the entropy gap. The linearization of Theorem 1 yields the following corollary, and alternative proof of which, together with a generalization, is also given below in Section 4.

**Corollary 3.** *For all functions $\varphi \in C^2(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, \gamma_n)$ with $\|\nabla^2 \varphi\|_{HS} \in L^2(\mathbb{R}^n, \gamma_n)$ we have*

$$\int_{\mathbb{R}^n} \left[ \|\nabla \varphi\|^2 - \frac{\|\nabla^2 \varphi\|_{HS}^2}{2} \right] d\gamma_n \leq Var_{\gamma_n}(\varphi). \tag{7}$$

*Here, $\| \; \|_{HS}$ denotes the Hilbert-Schmidt norm and $Var_{\gamma_n}(\varphi) = \int_{\mathbb{R}^n} \varphi^2 d\gamma_n - \left(\int_{\mathbb{R}^n} \varphi d\gamma_n\right)^2$ is the variance. There is equality for all polynomials of degree 2.*

The Poincaré inequality for the Gauss measure is ( see [3])

$$\int_{\mathbb{R}^n} |f|^2 d\gamma_n - \left( \int_{\mathbb{R}^n} f d\gamma_n \right)^2 \leq \int_{\mathbb{R}^n} \|\nabla f\|^2 d\gamma_n.$$

Hence, the inequality of Corollary 3 gives a reverse Poincaré inequality. We shall also give an alternative proof of Corollary 3, which generalizes to the following family of inequalities (which we state only in the one dimensional case for simplicity)

**Theorem 4.** *For all $m$ and all $\varphi \in C^{m,2}(\mathbb{R})$ with $\int \varphi d\gamma = 0$, one has*

$$\int \sum_{j=0}^{m-1} \frac{\left(\varphi^{(2j+1)}\right)^2}{(2j+1)!} d\gamma \leq \int \sum_{j=0}^{m} \frac{\left(\varphi^{(2j)}\right)^2}{(2j)!} d\gamma \leq \int \sum_{j=0}^{m} \frac{\left(\varphi^{(2j+1)}\right)^2}{(2j+1)!} d\gamma.$$

Here $\gamma = \gamma_1$ denotes the one-dimensional standard Gaussian distribution, and $C^{m,2}(\mathbb{R})$ means functions which are $m$ times continuously differentiable whose respective derivatives belong to $L_2$.

Our results are formulated and proved for functions that are sufficiently smooth. However, they can be generalized to functions that are not necessarily satisfying any $C^2$-assumptions. We then need to replace the second derivatives by the generalized second derivatives (compare e.g. [20]).

## 2 Affine isoperimetry for $s$-concave functions

**Definition 5.** *Let $s, n \in \mathbb{N}$. We say that $f : \mathbb{R}^n \to [0, \infty)$ is s-concave, and denote $f \in Conc_s(\mathbb{R}^n)$, if $f$ is upper semi continuous, $\overline{supp(f)}$ is a convex body (convex, compact and with non-empty interior) and $f^{\frac{1}{s}}$ is concave on $\overline{supp(f)}$. The class $Conc_s^{(2)}(\mathbb{R}^n)$ shall consist of such $f \in Conc_s(\mathbb{R}^n)$ which are twice continuously differentiable in the interior of their support.*

Note that for every $f \in Conc_s(\mathbb{R}^n)$ there exists a constant $C > 0$ such that $0 \leq f \leq C$. In particular, such an $f$ is integrable.

As in [1], we associate with a function $f \in Conc_s(\mathbb{R}^n)$ a the convex body $K_s(f)$ in $\mathbb{R}^n \times \mathbb{R}^s$ given by

$$K_s(f) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^s : x \in \text{supp}(f), \|y\| \leq f^{\frac{1}{s}}(x) \right\}. \tag{8}$$

A special function in the class $Conc_s(\mathbb{R}^n)$, which will play the role of the Euclidean ball in convexity, is

$$g_s(x) := (1 - \|x\|^2)_+^{\frac{s}{2}}$$

where, for $a \in \mathbb{R}$, $a_+ = \max\{a, 0\}$. It follows immediately from the definition that $K_s(g_s) = B_2^{n+s}$, the $(n+s)$-dimensional Euclidean unit ball centred at the origin. By Fubini's theorem, we have that for all $f \in Conc_s(\mathbb{R}^n)$

$$\text{vol}_{n+s}\left(K_s(f)\right) = \text{vol}_s(B_2^s) \int_{\mathbb{R}^n} f dx.$$

An important affine invariant quantity in convex geometric analysis is the affine surface area which, for a convex body $K \subset \mathbb{R}^n$ with a smooth boundary is defined by

$$as_1(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} d\mu_K(x). \tag{9}$$

Here, $\kappa(x) = \kappa_K(x)$ is the generalized Gaussian curvature at the point $x$ in $\partial K$, the boundary of $K$, and $\mu = \mu_K$ is the surface area measure on the boundary $\partial K$. See e.g. [13, 16, 19] for extensions of the definition of affine surface area to an arbitrary convex body in $\mathbb{R}^n$. For a function $f \in Conc_s(\mathbb{R}^n)$, we define

$$as_1^{(s)}(f) = as_1\left(K_s(f)\right). \tag{10}$$

Our first goal is to give a precise formula for $as_1^{(s)}(f)$ in terms of derivatives of the function $f$. This is done in the next proposition. There, for $x, y > 0$,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

is the Beta function.

**Proposition 6.** *Let $s \in \mathbb{N}$ and $f \in Conc_s^{(2)}(\mathbb{R}^n)$. Then*

$$as_1^{(s)}(f) = c_s \int_{supp(f)} \left|\det\left(\nabla^2 f^{\frac{1}{s}}\right)\right|^{\frac{1}{n+s+1}} f^{\frac{(s-1)(n+s)}{s(n+s+1)}} \, dx.$$

*Here, $c_s = (s-1)vol_{n-1}(B_2^{s-1})B\left(\frac{s-1}{2}, \frac{1}{2}\right)$ if $s \neq 1$ and $c_1 = 2$.*

In order to derive the formula for $as_1^{(s)}(f)$, we have to compute the affine surface area of the body $K_s(f)$. To this end, we compute the curvature of this body, which is circular in $s$ directions, and is behaving like $f^{1/s}$ in the other directions. We make use of the following well known lemma.

**Lemma 7.** *([22], p. 93, exercise 12.13) Let $h : \mathbb{R}^n \to [0, +\infty)$ be twice continuously differentiable. Let $x = (t, h(t)) \in \mathbb{R}^n \times \mathbb{R}$ be a point on the graph of $h$. Then, with the appropriate orientation, the Gauss curvature $\kappa$ at $x$ is*

$$\kappa(x) = \frac{\det(\nabla^2 h)}{(1 + \|\nabla h\|^2)^{\frac{n+2}{2}}}.$$

We shall apply Lemma 7 to the boundary of a convex body $K$. We consider only the orientation that gives nonnegative curvature. Thus, for a point $x \in \partial K$ whose boundary is described locally by the convex function $h$ we can use the formula

$$\kappa(x) = \left|\frac{\det\left(\nabla^2 h\right)}{(1 + \|\nabla h\|^2)^{\frac{n+2}{2}}}\right|. \tag{11}$$

We shall denote by $N_K(x)$ the outer unit normal vector to $\partial K$ at $x \in \partial K$.

**Lemma 8.** *Let $f \in Conc_s^{(2)}(\mathbb{R}^n)$. Then for all $x = (x_1, \dots, x_{n+s}) \in \partial K_s(f)$ with $(x_1 \dots, x_n) \in int(supp(f))$,*
*(i)* $N_{K_s(f)}(x) = \frac{\left(f^{\frac{1}{s}} \nabla f^{\frac{1}{s}}, -x_{n+1}, \dots, -x_{n+s}\right)}{f^{\frac{1}{s}}\left(1+\|\nabla f^{\frac{1}{s}}\|^2\right)^{\frac{1}{2}}}$,
*(ii)* $\kappa_{K_s(f)}(x) = \left| \frac{\det\left(\nabla^2 \ f^{\frac{1}{s}}\right)}{f^{\frac{s-1}{s}}\left(1+\|\nabla f^{\frac{1}{s}}\|^2\right)^{\frac{n+s+1}{2}}} \right|$.
*Here, $f$ is evaluated at $(x_1, \dots, x_n) \in \mathbb{R}^n$.*

*Proof of Lemma 8.* If $s = 1$, (i) of the lemma follows immediately from elementary calculus and (ii) from Lemma 7.

Therefore, we can assume that $s \geq 2$. Since, by equation (8), the boundary of $K_s(f)$ is given by $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^s : \|y\| = f^{1/s}(x)\}$, the boundary of $K_s(f)$ is the union of the graphs of the two mappings

$$(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+s-1}) \to (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+s-1}, \pm x_{n+s}),$$

where, with $x = (x_1, \dots, x_n)$,

$$x_{n+s} \quad = \quad \left(f^{\frac{2}{s}}(x) - \sum_{i=n+1}^{n+s-1} x_i^2\right)^{\frac{1}{2}}. \tag{12}$$

Because of symmetry, it is enough to consider only the "positive" part of $\partial K_s(f)$, in which the last coordinate is non-negative. We will show that the outer normal and the curvature exist for $(x, y)$ with $x \in \text{supp}(f)$ and $\|y\| = f(x)^{\frac{1}{s}}$ (they may not exist for $x \in \partial\left(\text{supp}(f)\right)$).

Letting $g = f^{1/s}$ we have

$$x_{n+s} = \sqrt{g(x_1, \dots, x_n)^2 - \sum_{i=n+1}^{n+s-1} x_i^2}.$$

As $f^{\frac{1}{s}}$ is everywhere differentiable on its support, we have for $i$ with $1 \leq i \leq n$ and, provided $s \geq 2$, for $j$ with $n+1 \leq j \leq n+s-1$,

$$\frac{\partial x_{n+s}}{\partial x_i} = \frac{g\frac{\partial g}{\partial x_i}}{\sqrt{g^2 - \sum_{i=n+1}^{n+s-1} x_i^2}} = \frac{g\frac{\partial g}{\partial x_i}}{x_{n+s}} \quad \text{and} \quad \frac{\partial x_{n+s}}{\partial x_j} = -\frac{x_j}{\sqrt{g^2 - \sum_{i=n+1}^{n+s-1} x_i^2}} = -\frac{x_j}{x_{n+s}}. \tag{13}$$

(i) Therefore we get for almost all $z \in \partial K_s(f)$ with (12) and (13)

$$N_{K_s(f)}(z) = \frac{(\nabla x_{n+s}, -1)}{(1 + \|\nabla x_{n+s}\|^2)^{\frac{1}{2}}} = \frac{\left(f^{\frac{1}{s}} \nabla f^{\frac{1}{s}}, -x_{n+1}, \dots, -x_{n+s}\right)}{f^{\frac{1}{s}}\left(1 + \|\nabla f^{\frac{1}{s}}\|^2\right)^{\frac{1}{2}}}.$$

(ii) We have for all $i$ with $1 \leq i \leq n$,

$$\frac{\partial^2 x_{n+s}}{\partial x_i^2} = \frac{g\frac{\partial^2 g}{\partial x_i^2} + (\frac{\partial g}{\partial x_i})^2}{x_{n+s}} - \frac{g^2(\frac{\partial g}{\partial x_i})^2}{x_{n+s}^3} = \frac{g\frac{\partial^2 g}{\partial x_i^2}}{x_{n+s}} - \frac{(\frac{\partial g}{\partial x_i})^2 \sum_{j=n+1}^{n+s-1} x_j^2}{x_{n+s}^3}.$$

For $i \neq j$ with $1 \leq i, j \leq n$,

$$\frac{\partial^2 x_{n+s}}{\partial x_i \partial x_j} = \frac{g\frac{\partial^2 g}{\partial x_i \partial x_j} + \frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}}{x_{n+s}} - \frac{g^2 \frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}}{x_{n+s}^3} = \frac{g\frac{\partial^2 g}{\partial x_i \partial x_j}}{x_{n+s}} - \frac{\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}\sum_{j=n+1}^{n+s-1} x_j^2}{x_{n+s}^3}.$$

For $1 \leq i \leq n$ and $n+1 \leq j \leq n+s-1$

$$\frac{\partial^2 x_{n+s}}{\partial x_i \partial x_j} = -\frac{x_j g \frac{\partial g}{\partial x_i}}{x_{n+s}^3}.$$

For $n+1 \leq i \leq n+s-1$,

$$\frac{\partial^2 x_{n+s}}{\partial x_i^2} = -\frac{1}{x_{n+s}} - \frac{x_i^2}{x_{n+s}^3} = -\frac{x_{n+s}^2 + x_i^2}{x_{n+s}^3}.$$

For $i$ and $j$ with $n+1 \leq i, j \leq n+s-1$ and $j \neq i$,

$$\frac{\partial^2 x_{n+s}}{\partial x_i \partial x_j} = -\frac{x_i x_j}{x_{n+s}^3}.$$

We compute now the determinant of the following $[n+(s-1)] \times [n+(s-1)]$ matrix

$$\begin{pmatrix}
\frac{g\frac{\partial^2 g}{\partial x_1^2}}{x_{n+s}} - \frac{(\frac{\partial g}{\partial x_1})^2 \sum_{j=n+1}^{n+s-1} x_j^2}{x_{n+s}^3} & \cdots & \frac{g\frac{\partial^2 g}{\partial x_1 \partial x_n}}{x_{n+s}} - \frac{\frac{\partial g}{\partial x_1}\frac{\partial g}{\partial x_n}\sum_{j=n+1}^{n+s-1} x_j^2}{x_{n+s}^3} & -\frac{x_{n+1} g \frac{\partial g}{\partial x_1}}{x_{n+s}^3} & \cdots & -\frac{x_{n+s-1} g \frac{\partial g}{\partial x_1}}{x_{n+s}^3} \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{g\frac{\partial^2 g}{\partial x_n \partial x_1}}{x_{n+s}} - \frac{\frac{\partial g}{\partial x_n}\frac{\partial g}{\partial x_1}\sum_{j=n+1}^{n+s-1} x_j^2}{x_{n+s}^3} & \cdots & \frac{g\frac{\partial^2 g}{\partial x_n^2}}{x_{n+s}} - \frac{(\frac{\partial g}{\partial x_n})^2 \sum_{j=n+1}^{n+s-1} x_j^2}{x_{n+s}^3} & -\frac{x_{n+1} g \frac{\partial g}{\partial x_n}}{x_{n+s}^3} & \cdots & -\frac{x_{n+s-1} g \frac{\partial g}{\partial x_n}}{x_{n+s}^3} \\
-\frac{x_{n+1} g \frac{\partial g}{\partial x_1}}{x_{n+s}^3} & \cdots & -\frac{x_{n+1} g \frac{\partial g}{\partial x_n}}{x_{n+s}^3} & -\frac{x_{n+s}^2 + x_{n+1}^2}{x_{n+s}^3} & \cdots & -\frac{x_{n+s-1} x_{n+1}}{x_{n+s}^3} \\
\vdots & & \vdots & \vdots & & \vdots \\
-\frac{x_{n+s-1} g \frac{\partial g}{\partial x_1}}{x_{n+s}^3} & \cdots & -\frac{x_{n+s-1} g \frac{\partial g}{\partial x_n}}{x_{n+s}^3} & -\frac{x_{n+s-1} x_{n+1}}{x_{n+s}^3} & \cdots & -\frac{x_{n+s}^2 + x_{n+s-1}^2}{x_{n+s}^3}
\end{pmatrix}$$

For fixed $i$, $1 \leq i \leq n$ we multiply each of the rows $n+1 \leq j \leq n+s-1$ by

$$\frac{x_j}{g}\frac{\partial g}{\partial x_i}$$

and add them up. We obtain the vector

$$\left(-\frac{\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_1}\sum_{j=n+1}^{n+s-1} x_j^2}{x_{n+s}^3}, \ldots, -\frac{\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_n}\sum_{j=n+1}^{n+s-1} x_j^2}{x_{n+s}^3}, -\frac{x_{n+1} g \frac{\partial g}{\partial x_i}}{x_{n+s}^3}, \ldots, -\frac{x_{n+s-1} g \frac{\partial g}{\partial x_i}}{x_{n+s}^3}\right)$$

and subtract it from the $i$-th row. The determinant does not change and we obtain

$$\begin{pmatrix}
\frac{g\frac{\partial^2 g}{\partial x_1^2}}{x_{n+s}} & \cdots & \frac{g\frac{\partial^2 g}{\partial x_1 \partial x_n}}{x_{n+s}} & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{g\frac{\partial^2 g}{\partial x_n \partial x_1}}{x_{n+s}} & \cdots & \frac{g\frac{\partial^2 g}{\partial x_n^2}}{x_{n+s}} & 0 & \cdots & 0 \\
-\frac{x_{n+1} g \frac{\partial g}{\partial x_1}}{x_{n+s}^3} & \cdots & -\frac{x_{n+1} g \frac{\partial g}{\partial x_n}}{x_{n+s}^3} & -\frac{x_{n+s}^2 + x_{n+1}^2}{x_{n+s}^3} & \cdots & -\frac{x_{n+s-1} x_{n+1}}{x_{n+s}^3} \\
\vdots & & \vdots & \vdots & & \vdots \\
-\frac{x_{n+s-1} g \frac{\partial g}{\partial x_1}}{x_{n+s}^3} & \cdots & -\frac{x_{n+s-1} g \frac{\partial g}{\partial x_n}}{x_{n+s}^3} & -\frac{x_{n+s-1} x_{n+1}}{x_{n+s}^3} & \cdots & -\frac{x_{n+s}^2 + x_{n+s-1}^2}{x_{n+s}^3}
\end{pmatrix}.$$

The determinant of this matrix equals, up to a sign, to

$$\frac{g^n}{x_{n+s}^{n+3(s-1)}}\det\begin{pmatrix} \frac{\partial^2 g}{\partial x_1^2} & \cdots & \frac{\partial^2 g}{\partial x_1 \partial x_n} \\ \cdots & & \vdots \\ \frac{\partial^2 g}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 g}{\partial x_n^2} \end{pmatrix} \det\begin{pmatrix} x_{n+s}^2 + x_{n+1}^2 & \cdots & x_{n+s-1}x_{n+1} \\ \vdots & & \vdots \\ x_{n+s-1}x_{n+1} & \cdots & x_{n+s}^2 + x_{n+s-1}^2 \end{pmatrix} \tag{14}$$

It is left to evaluate the second determinant. To that end we use a well-known matrix determinant formula: For any dimension $m$ and $y \in \mathbb{R}^m$,

$$\det(Id + y \otimes y) = 1 + \|y\|^2 \tag{15}$$

where $y \otimes y$ is the matrix whose $(y_i y_j)_{i,j=1,\dots,n}$. Consequently, for the second determinant in (14) we have

$$\det\begin{pmatrix} x_{n+s}^2 + x_{n+1}^2 & \cdots & x_{n+s-1}x_{n+1} \\ \vdots & & \vdots \\ x_{n+s-1}x_{n+1} & \cdots & x_{n+s}^2 + x_{n+s-1}^2 \end{pmatrix} = \left(\sum_{i=n+1}^{n+s} \frac{x_i^2}{x_{n+s}^2}\right) x_{n+s}^{2s} = g^2 x_{n+s}^{2(s-2)}.$$

Therefore we get for the expression (14)

$$\frac{g^{n+2}}{x_{n+s}^{n+s+1}}\det\begin{pmatrix} \frac{\partial^2 g}{\partial x_1^2} & \cdots & \frac{\partial^2 g}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 g}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 g}{\partial x_n^2} \end{pmatrix}.$$

Moreover

$$\begin{aligned} 1 + \sum_{i=1}^{n+s-1} \left| \frac{\partial x_{n+s}}{\partial x_i} \right|^2 &= 1 + \sum_{i=1}^{n} \left| \frac{g \frac{\partial g}{\partial x_i}}{x_{n+s}} \right|^2 + \sum_{i=n+1}^{n+s-1} \left| \frac{x_i}{x_{n+s}} \right|^2 = \sum_{i=1}^{n} \left| \frac{g \frac{\partial g}{\partial x_i}}{x_{n+s}} \right|^2 + \left| \frac{g}{x_{n+s}} \right|^2 \\ &= \left| \frac{g}{x_{n+s}} \right|^2 \left( 1 + \sum_{i=1}^{n} \left| \frac{\partial g}{\partial x_i} \right|^2 \right). \end{aligned}$$

Therefore, we get by (11) for the curvature

$$\begin{aligned} \kappa(z) &= \frac{\frac{g^{n+2}}{x_{n+s}^{n+s+1}}\det\begin{pmatrix} \frac{\partial^2 g}{\partial x_1^2} & \cdots & \frac{\partial^2 g}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 g}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 g}{\partial x_n^2} \end{pmatrix}}{\left( \left| \frac{g}{x_{n+2}} \right|^2 \left( 1 + \sum_{i=1}^{n} \left| \frac{\partial g}{\partial x_i} \right|^2 \right) \right)^{\frac{n+s+1}{2}}} = \frac{\det\begin{pmatrix} \frac{\partial^2 g}{\partial x_1^2} & \cdots & \frac{\partial^2 g}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 g}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 g}{\partial x_n^2} \end{pmatrix}}{g^{s-1} \left( 1 + \sum_{i=1}^{n} \left| \frac{\partial g}{\partial x_i} \right|^2 \right)^{\frac{n+s+1}{2}}} \\ &= \frac{\det\left(\nabla^2 f^{\frac{1}{s}}\right)}{f^{\frac{s-1}{s}} \left(1 + \|\nabla f^{\frac{1}{s}}\|^2\right)^{\frac{n+s+1}{2}}}. \end{aligned}$$

This completes the proof of Lemma 8. □

*Proof of Proposition 6.* Denote by $\tilde{\partial} K_s(f)$ the collection of all points $(x_1, \dots, x_{n+s}) \in \partial K_s(f)$ such that $(x_1, \dots, x_n) \in \mathrm{int}(\mathrm{supp}(f))$. Since there is no contribution to the integral of $as_1(K_s(f))$ from $\partial K_s(f) \setminus \overline{\tilde{\partial} K_s(f)}$ (since the Gauss curvature vanishes on the part with full dimension, if exists) clearly

$$as_1^{(s)}(f) = as_1(K_s(f)) = \int_{\partial K_s(f)} \kappa_{K_s(f)}^{\frac{1}{n+s+1}} \, d\mu_{K_s(f)} = \int_{\tilde{\partial} K_s(f)} \kappa_{K_s(f)}^{\frac{1}{n+s+1}} \, d\mu_{K_s(f)}.$$

By Lemma 8

$$\begin{aligned} as_1^{(s)}(f) &= \int_{\tilde{\partial} K_s(f)} \frac{\left(\det\left(\nabla^2(f^{\frac{1}{s}})\right)\right)^{\frac{1}{n+s+1}}}{\left(1 + \|\nabla f^{\frac{1}{s}}\|\right)^{\frac{1}{2}}} \, f^{-\frac{s-1}{s(n+s+1)}} \, d\mu_{K_s(f)} \\ &= 2 \int_{\mathbb{R}^{n+s-1}} f^{\frac{1}{s}} \left(\frac{\det\left(\nabla^2(f^{\frac{1}{s}})\right)}{f^{\frac{s-1}{s}}}\right)^{\frac{1}{n+s+1}} \frac{dx_1 \dots dx_{n+s-1}}{|x_{n+s}|} \end{aligned} \tag{16}$$

where $f$ is evaluated, of course, at $(x_1, \dots, x_n)$. The last equality follows as the boundary of $K_s(f)$ consists of two, "positive" and "negative", parts. For $s = 1$, we get

$$2 \int_{\mathbb{R}^n} \left(\det\left(\nabla^2 f\right)\right)^{\frac{1}{n+2}} dx_1 \dots dx_n,$$

hence $c_1 = 2$. For $s > 1$,

$$\begin{aligned} \int_{\mathbb{R}^{s-1}} \frac{dx_{n+1} \dots dx_{n+s-1}}{|x_{n+s}|} &= \int_{\mathbb{R}^{s-1}} f^{-\frac{1}{s}} \left(1 - \sum_{i=n+1}^{n+s-1} \left(\frac{x_i}{f^{\frac{1}{s}}}\right)^2\right)^{-\frac{1}{2}} dx_{n+1} \dots dx_{n+s-1} \\ &= \int_{\sum_{i=n+1}^{n+s-1} y_i^2 \le 1} \frac{f^{\frac{s-1}{s}}}{f^{\frac{1}{s}}} \left(1 - \sum_{i=n+1}^{n+s-1} y_i^2\right)^{-\frac{1}{2}} dy_{n+1} \dots dy_{n+s-1} \\ &= \frac{f^{\frac{s-1}{s}}}{f^{\frac{1}{s}}} (s-1)\mathrm{vol}_{s-1}\left(B_2^{s-1}\right) \int_0^1 \frac{r^{s-2} dr}{(1-r^2)^{\frac{1}{2}}} \\ &= \frac{f^{\frac{s-1}{s}}}{f^{\frac{1}{s}}} (s-1)\mathrm{vol}_{s-1}\left(B_2^{s-1}\right) \frac{1}{2} B\left(\frac{s-1}{2}, \frac{1}{2}\right). \end{aligned}$$

Thus (16) becomes

$$as_1^{(s)}(f) = (s-1)\mathrm{vol}_{s-1}\left(B_2^{s-1}\right) B\left(\frac{s-1}{2}, \frac{1}{2}\right) \int_{\mathbb{R}^n} f^{\frac{s-1}{s}} \left(\frac{\det\left(\mathrm{Hess}(f^{\frac{1}{s}})\right)}{f^{\frac{s-1}{s}}}\right)^{\frac{1}{n+s+1}} dx,$$

and the proof of Proposition 6 is complete. □

With the formula for $as_1^{(s)}(f)$ in hand, we may use the affine isoperimetric inequality for convex bodies to obtain the following corollary.

**Corollary 9.** *For all $s \in \mathbb{N}$ and for all $f \in Conc_s^{(2)}(\mathbb{R}^n)$ we have*

$$\int_{supp(f)} \left|\det\left(\nabla^2 f^{\frac{1}{s}}\right)\right|^{\frac{1}{n+s+1}} f^{\frac{s-1}{s}\left(\frac{n+s}{n+s+1}\right)}\, dx \le d(n,s)\left(\int_{supp(f)} f dx\right)^{\frac{n+s-1}{n+s+1}},$$

*where*

$$d(n,s) = \pi^{\frac{n}{n+s+1}}\left(\frac{n+s}{s}\right)^{\frac{n+s-1}{n+s+1}}\left(\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{n+s}{2})}\right)^{\frac{2}{n+s+1}}.$$

*Equality holds if and only if $f = (a + \langle b, x\rangle - \langle Ax, x\rangle)_+^{s/2}$ for $a \in \mathbb{R}, b \in \mathbb{R}^n$ and a positive-definite matrix $A$.*

*Proof of Corollary 9.* The affine isoperimetric inequality for convex bodies $K$ in $\mathbb{R}^n$ (see, e.g., [18]) says that

$$\frac{as_1(K)}{as_1(B_2^n)} \le \left(\frac{\mathrm{vol}_n\,(K)}{\mathrm{vol}_n\,(B_2^n)}\right)^{\frac{n-1}{n+1}}, \tag{17}$$

with equality if and only if $K$ is an ellipsoid. We apply (17) to $K_s(f) \subset \mathbb{R}^{n+s}$ and get

$$\begin{aligned}
\frac{as_1^{(s)}(f)}{as_1^{(s)}(g_s)} &= \frac{as_1\,(K_s(f))}{as_1\,(K_s(g_s))} \\
&= \frac{c_s}{as_1\,(B_2^{n+s})}\int_{\mathrm{supp}(f)}\left(\det\Big(\frac{\partial^2 f^{\frac{1}{s}}}{\partial x_i \partial x_j}\Big)_{i,j=1,\ldots,n}\right)^{\frac{1}{n+s+1}} f^{\frac{(s-1)(n+s)}{s(n+s+1)}}dx \\
&\le \left(\frac{\mathrm{vol}_s\,(B_2^s)\,\int_{\mathrm{supp}(f)} f dx}{\mathrm{vol}_{n+s}\,(B_2^{n+s})}\right)^{\frac{n+s-1}{n+s+1}},
\end{aligned}$$

with equality if and only if $f(x) = (a + \langle b, x\rangle - \langle Ax, x\rangle)_+^{s/2}$ for $a \in \mathbb{R}, b \in \mathbb{R}^n$ and a positive-definite matrix $A$. This is rewritten as

$$\int_{\mathrm{supp}(f)}\left(\det\left(\nabla^2 (f^{\frac{1}{s}})\right)\right)^{\frac{1}{n+s+1}} f^{\frac{(s-1)(n+s)}{s(n+s+1)}} dx \le d(n,s)\left(\int_{\mathrm{supp}(f)} f dx\right)^{\frac{n+s-1}{n+s+1}},$$

where

$$\begin{aligned}
d(n,s) &= \frac{(n+s)\ \mathrm{vol}_{n+s}\,(B_2^{n+s})}{c_s}\left(\frac{\mathrm{vol}_s\,(B_2^s)}{\mathrm{vol}_{n+s}\,(B_2^{n+s})}\right)^{\frac{n+s-1}{n+s+1}} \\
&= \pi^{\frac{n}{n+s+1}}\left(\frac{n+s}{s}\right)^{\frac{n+s-1}{n+s+1}}\left(\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{n+s}{2})}\right)^{\frac{2}{n+s+1}}.
\end{aligned}$$

□

It follows immediately from the definition and from Proposition 6, that $as_1^{(s)}(f)$ is affine invariant and that it is a valuation:

**Corollary 10.** *Let $s \in \mathbb{N}$ and let $f \in Conc_s(\mathbb{R}^n) \cap C^2\,(supp(f))$.*

*(i) For all linear maps $A : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathrm{det}A \neq 0$, and for all $\lambda \in \mathbb{R}$, we have*

$$as_1^{(s)}((\lambda f) \circ A) = \frac{\lambda^{\frac{n+s-1}{n+s+1}}}{|\mathrm{det}A|} as_1^{(s)}(f).$$

*In particular, if $|\mathrm{det}A| = 1$,*

$$as_1^{(s)}(f \circ A) = as_1^{(s)}(f).$$

*(ii) $as_1^{(s)}$ is a "valuation": If $\max(f_1, f_2)$ is $s$-concave, then*

$$as_1^{(s)}(f_1) + as_1^{(s)}(f_2) = as_1^{(s)}(\max(f_1, f_2)) + as_1^{(s)}(\min(f_1, f_2))$$

*Proof of Corollary 10.* (i) By Proposition 6,

$$\begin{aligned}
& as_1^{(s)}((\lambda f) \circ A) \\
&= c_s \int_{\mathrm{supp}(f\circ A)} \left|\det\left(\nabla^2\left((\lambda f)\circ A\right)^{\frac{1}{s}}\right)\right|^{\frac{1}{n+s+1}} \lambda^{\frac{(s-1)(n+s)}{s(n+s+1)}} f(Ax)^{\frac{(s-1)(n+s)}{s(n+s+1)}}\, dx \\
&= c_s \frac{\lambda^{\frac{n+s-1}{n+s+1}}}{|\mathrm{det}A|} \int_{\mathrm{supp}(f)} \left|\det\left(\nabla^2(f^{\frac{1}{s}})\right)\right|^{\frac{1}{n+s+1}} f^{\frac{(s-1)(n+s)}{s(n+s+1)}}\, dy \\
&= \frac{\lambda^{\frac{n+s-1}{n+s+1}}}{|\mathrm{det}A|} as_1^{(s)}(f).
\end{aligned}$$

(ii) By (10) and since the affine surface area for convex bodies is a valuation [**?**],

$$\begin{aligned}
as_1^{(s)}(f_1) + as_1^{(s)}(f_2) &= as_1\left(K_s(f_1)\right) + as_1\left(K_s(f_2)\right) \\
&= as_1\left(K_s(f_1) \cup K_s(f_2)\right) + as_1\left(K_s(f_1) \cap K_s(f_2)\right) \\
&= as_1^{(s)}(\max(f_1, f_2)) + as_1^{(s)}(\min(f_1, f_2)),
\end{aligned}$$

provided that $K_s(f_1) \cup K_s(f_2)$ is convex. □

## 3 log-concave functions

We would like to obtain an inequality corresponding to the one of Corollary 9 not only for $s$-concave functions but, more generally, for log-concave functions on $\mathbb{R}^n$, which are the natural functional extension of convex bodies. The union of all classes of $s$ concave functions over all $s$ is dense within log-concave functions in many natural topologies.

Note that if a function $f$ is $s_0$-concave for some $s_0$, then it is $s$-concave for all $s \geq s_0$. Therefore, by Corollary 9, we get that for any $s_0 \in \mathbb{N}$ and any $f \in Conc_{s_0}(\mathbb{R}^n) \cap C^2\,(\mathrm{supp}(f))$we have for all $s \geq s_0$

$$\int_{\mathrm{supp}(f)} f^{\frac{(s-1)(n+s)}{s(n+s+1)}} \left|\det\left(\nabla^2(f^{\frac{1}{s}})\right)\right|^{\frac{1}{n+s+1}} dx \leq d(n,s) \left(\int_{\mathrm{supp}(f)} f dx\right)^{\frac{n+s-1}{n+s+1}}.$$

Taking the limit as $s \to \infty$ one sees that the limit on both sides is simply $\int_{\text{supp}(f)} f dx$, so that one does not get an interesting inequality. However, we may take the derivative at $s = +\infty$ as in [10] (the details are given in the proof below), and doing so, we obtain the inequality of Theorem 1.

Before we present the proof of Theorem 1, we give an example in which both sides are computable. The computation is straightforward and left for the interested reader.

**Example 11.** *Let $p > 1$ and $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = e^{-\sum_{i=1}^n |x_i|^p}$. Then*

$$\int_{\mathbb{R}^n} f \ \ln\Big(\det\left(\nabla^2\left(-\ln f\right)\right)\Big) dx = n \ \left(\frac{2}{p}\ \Gamma\left(\frac{1}{p}\right)\right)^n \left(\ln\left(p(p-1)\right) + (p-2)\ \frac{\Gamma'(\frac{1}{p})}{\Gamma(\frac{1}{p})}\right)$$

*and*

$$2\Big[\operatorname{Ent}(f) + \|f\|_{L^1(dx)} \ln(2\pi e)^{\frac{n}{2}}\Big] = n \ \left(\frac{2}{p}\ \Gamma\left(\frac{1}{p}\right)\right)^n \left(\ln\left(\frac{\pi e}{2\Gamma(1+\frac{1}{p})^2}\right) - \frac{2}{p}\right).$$

*Both expressions are equal when $p = 2$.*

*Proof of Theorem 1:* One is given a function $f$ which is log-concave and $C^2$-smooth in the interior of its support. In order to apply Corollary 9, we modify $f$ slightly as follows: For $\varepsilon > 0$, set

$$f_\varepsilon(x) = f(x)\exp(-\varepsilon\|x\|^2)\chi_{\{f \geq \varepsilon\}}(x) \qquad (x \in \mathbb{R}^n).$$

By a standard compactness argument, every log-concave function with compact support is $s_0$-concave for some $s_0$. Hence there exists $s_0 > 0$ such that $f_\varepsilon$ is $s$-concave for all $s \geq s_0$ and thus (18) holds for $f_\varepsilon$ and any $s \geq s_0$. We expand the left hand side and the right hand side of the inequality in Corollary 9 in terms of $\frac{1}{s}$. We have

$$\frac{\partial^2 f_\varepsilon^{\frac{1}{s}}}{\partial x_i \partial x_j} = \frac{1}{s}\frac{\partial}{\partial x_j}\left(f_\varepsilon^{\frac{1}{s}-1}\frac{\partial f_\varepsilon}{\partial x_i}\right) = \frac{f_\varepsilon^{\frac{1}{s}-2}}{s}\left(f_\varepsilon\frac{\partial^2 f_\varepsilon}{\partial x_i \partial x_j} - \frac{\partial f_\varepsilon}{\partial x_j}\frac{\partial f_\varepsilon}{\partial x_i} + \frac{1}{s}\ \frac{\partial f_\varepsilon}{\partial x_j}\frac{\partial f_\varepsilon}{\partial x_i}\right)$$

Thus

$$\begin{aligned}
\nabla^2(f_\varepsilon^{1/s}) &= \frac{f_\varepsilon^{\frac{1}{s}}}{s}\left(\frac{f_\varepsilon \nabla^2(f_\varepsilon) - \nabla f_\varepsilon \otimes \nabla f_\varepsilon + \frac{1}{s}\nabla f_\varepsilon \otimes \nabla f_\varepsilon}{f_\varepsilon^2}\right) \\
&= \frac{f_\varepsilon^{\frac{1}{s}}}{s}\ \left(\nabla^2\left(\ln f_\varepsilon\right) + \frac{1}{s}\frac{\nabla f_\varepsilon \otimes \nabla f_\varepsilon}{f_\varepsilon^2}\right)
\end{aligned}$$

and hence

$$\det\left(-\frac{\partial^2 f_\varepsilon^{\frac{1}{s}}}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,n} = \frac{f_\varepsilon^{\frac{n}{s}}}{s^n}\ \det\left(-\left(\nabla^2\left(\ln f_\varepsilon\right) + \frac{1}{s}\frac{\nabla f_\varepsilon \otimes \nabla f_\varepsilon}{f_\varepsilon^2}\right)\right).$$

Thus the inequality of Corollary 9 is equivalent to

$$\begin{aligned}
&\int_{\text{supp}(f_\varepsilon)} \left|\det\left(-\left(\nabla^2\left(\ln f_\varepsilon\right) + \frac{1}{s}\frac{\nabla f_\varepsilon \otimes \nabla f_\varepsilon}{f_\varepsilon^2}\right)\right)\right|^{\frac{1}{n+s+1}} \ f_\varepsilon^{\frac{n+s-1}{n+s+1}} dx \\
&\qquad\qquad \leq d(n,s)\ s^{\frac{n}{n+s+1}}\ \left(\int_{\text{supp}(f_\varepsilon)} f_\varepsilon dx\right)^{\frac{n+s-1}{n+s+1}}. \qquad (18)
\end{aligned}$$

Applying again the formula for the determinant of a rank-one perturbation of a matrix, we have

$$\begin{aligned}
&\det\left(-\left(\nabla^2\left(\ln f_\varepsilon\right)+\frac{1}{s}\frac{\nabla f_\varepsilon\otimes\nabla f_\varepsilon}{f_\varepsilon^2}\right)\right)\\
&=\det\left(-\nabla^2\left(\ln f_\varepsilon\right)\right)\left[1+s^{-2}f_\varepsilon^{-2}\langle\left(\nabla^2\ln f_\varepsilon\right)^{-1}\nabla f_\varepsilon,\nabla f_\varepsilon\rangle\right]\\
&=\det\left(-\nabla^2\left(\ln f_\varepsilon\right)\right)+s^{-2}\alpha_\varepsilon(x),
\end{aligned}\tag{19}$$

where, for a fixed $\varepsilon$, the function $\alpha_\varepsilon(x)$ is defined by (19) and is clearly bounded on the interior of the support of $f_\varepsilon$. We write, for the left hand side of (18),

$$f_\varepsilon^{\frac{n+s-1}{n+s+1}}=f_\varepsilon\ \left(f_\varepsilon^{-2}\right)^{\frac{1}{n+s+1}}$$

and on the right hand side

$$\left(\int_{\operatorname{supp}(f_\varepsilon)}f_\varepsilon dx\right)^{\frac{n+s-1}{n+s+1}}=\left(\int_{\operatorname{supp}(f_\varepsilon)}f_\varepsilon dx\right)\left(\int_{\operatorname{supp}(f_\varepsilon)}f_\varepsilon dx\right)^{\frac{-2}{n+s+1}}.$$

Moreover,

$$\begin{aligned}
d(n,s)\ s^{\frac{n}{n+s+1}}&=(s\pi)^{\frac{n}{n+s+1}}\left(\frac{n+s}{s}\right)^{\frac{n+s-1}{n+s+1}}\left(\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{n+s}{2})}\right)^{\frac{2}{n+s+1}}\\
&\leq(2\pi e)^{\frac{n}{n+s+1}}\left(1+\frac{1}{3s}\right)^{\frac{2}{n+s+1}},
\end{aligned}$$

where we have used that for $x\to\infty$,

$$\Gamma(x)=\sqrt{2\pi}\ x^{x-\frac{1}{2}}\ e^{-x}\ \left[1+\frac{1}{12x}+\frac{1}{288x^2}\pm o(x^{-2})\right],$$

and we make the legitimate assumption that $s$ is sufficiently large. Thus, together with (19), it follows from (18) that

$$\begin{aligned}
&\int_{\operatorname{supp}(f_\varepsilon)}f_\varepsilon\left|f_\varepsilon^{-2}\left(\det\left(-\nabla^2\left(\ln f_\varepsilon\right)\right)+s^{-2}\alpha_\varepsilon(x)\right)\right|^{\frac{1}{n+s+1}}dx\\
&\leq\left(\int_{\operatorname{supp}(f_\varepsilon)}f_\varepsilon dx\right)\left(\left(1+\frac{1}{3s}\right)^2(2\pi e)^n\left(\int_{\operatorname{supp}(f_\varepsilon)}f_\varepsilon dx\right)^{-2}\right)^{\frac{1}{n+s+1}}.
\end{aligned}\tag{20}$$

We estimate the left hand side of (20) from below by

$$\begin{aligned}
&\int_{\operatorname{supp}(f_\varepsilon)}f_\varepsilon\left|f_\varepsilon^{-2}\left(\det\left(-\left(\nabla^2\ln f_\varepsilon\right)\right)+s^{-2}\alpha_\varepsilon(x)\right)\right|^{\frac{1}{n+s+1}}dx\\
&=\int_{\operatorname{supp}(f_\varepsilon)}f_\varepsilon\exp\left(\frac{1}{n+s+1}\ln\left|f_\varepsilon^{-2}\Big(\det\left(-\left(\nabla^2(\ln f_\varepsilon)\right)+s^{-2}\alpha_\varepsilon(x)\right)\Big)\right|\right)dx\\
&\geq\int_{\operatorname{supp}(f_\varepsilon)}f_\varepsilon\left(1+\frac{1}{n+s+1}\ln\left|f_\varepsilon^{-2}\Big(\det\left(-\left(\nabla^2\ln f_\varepsilon\right)\right)+s^{-2}\alpha_\varepsilon(x)\Big)\right|\right)dx.
\end{aligned}$$

We write the right hand side of (20)

$$\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)\left(\left(1+\frac{1}{3s}\right)^2(2\pi e)^n\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)^{-2}\right)^{\frac{1}{n+s+1}}$$
$$=\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)\sum_{j=0}^{\infty}\frac{1}{j!(n+s+1)^j}\left(\ln\left(\frac{\left(1+\frac{1}{3s}\right)^2(2\pi e)^n}{\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)^2}\right)\right)^j.$$

Therefore we get the following inequality

$$\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon\left(1+\frac{1}{n+s+1}\ln\left|f_\varepsilon^{-2}\left(\det\left(-\left(\nabla^2\ln f_\varepsilon\right)\right)+s^{-2}\alpha_\varepsilon(x)\right)\right|\right)dx$$
$$\leq\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)\sum_{j=0}^{\infty}\frac{1}{j!(n+s+1)^j}\left(\ln\left(\frac{\left(1+\frac{1}{3s}\right)^2(2\pi e)^n}{\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)^2}\right)\right)^j. \qquad (21)$$

We subtract the first order term $\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx$ from both side, multiply by $n+s+1$ and take the limit as $s\to\infty$. We get

$$\liminf_{s\to\infty}\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon\left(\ln\left|f_\varepsilon^{-2}\left(\det\left(-\left(\nabla^2\ln f_\varepsilon\right)\right)+s^{-2}\alpha_\varepsilon(x)\right)\right|\right)dx$$
$$\leq\limsup_{s\to\infty}\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon\left(\ln\left|f_\varepsilon^{-2}\left(\det\left(-\left(\nabla^2\ln f_\varepsilon\right)\right)+s^{-2}\alpha_\varepsilon(x)\right)\right|\right)dx$$
$$\leq\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)\limsup_{s\to\infty}\sum_{j=1}^{\infty}\frac{1}{j!(n+s+1)^{j-1}}\left(\ln\left(\frac{\left(1+\frac{1}{3s}\right)^2(2\pi e)^n}{\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)^2}\right)\right)^j$$
$$=\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)\ln\left(\frac{(2\pi e)^n}{\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)^2}\right).$$

In the interior of the support of $f_\varepsilon$, the Hessian of $\nabla^2(\ln f_\varepsilon)$ is greater than $\varepsilon Id$, hence we can apply Fatou's lemma on the left hand side to get

$$\int_{\mathrm{supp}(f_\varepsilon)}\liminf_{s\to\infty} f_\varepsilon\left(\ln\left|f_\varepsilon^{-2}\left(\det\left(-\left(\nabla^2\left(\ln f_\varepsilon\right)\right)_{i,j=1,\ldots,n}\right)+s^{-2}\alpha_\varepsilon(x)\right)\right|\right)dx$$
$$\leq\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)\ln\left(\frac{(2\pi e)^n}{\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)^2}\right),$$

which simplifies to

$$\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon\left(\ln\left(\det\left(-\left(\nabla^2\ln f_\varepsilon\right)\right)\right)\right)dx$$
$$\leq\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)\ln\left(\frac{(2\pi e)^n}{\left(\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon dx\right)^2}\right)+2\int_{\mathrm{supp}(f_\varepsilon)} f_\varepsilon\ln f_\varepsilon dx. \qquad (22)$$

Now we pass to the limit $\varepsilon \to 0$ on both sides of (22). We deal with each of the three terms separately. For the first term, since $-\ln f_\varepsilon = -\ln f + \varepsilon \|\cdot\|^2/2$, we have

$$\int_{\{f\geq\varepsilon\}} f_\varepsilon \Big( \ln \Big( \det \left( -\nabla^2 (\ln f) + \varepsilon Id \right) \Big) dx \geq \int_{\{f\geq\varepsilon\}} f_\varepsilon \Big( \ln \Big( \det \left( -\nabla^2 \ln f \right) \Big) dx.$$

Since the integral $f \ln(\det(\nabla^2 \ln f))$ is assumed to belong to $L_1$ and $f_\varepsilon$ increases monotonously to $f$ as $\varepsilon \to 0$, the integrand is bounded by $f \left|\ln(\det(\nabla^2 \ln f))\right|$ and by the dominated convergence theorem

$$\lim_{\varepsilon\to 0} \int_{\{f\geq\varepsilon\}} f_\varepsilon \Big( \ln \Big( \det \left( -\nabla^2 \ln f \right) \Big) dx = \int_{\text{supp} f} f \Big( \ln \Big( \det \left( -\nabla^2 \ln f \right) \Big) dx.$$

Similarly, monotone convergence theorem ensures that

$$\lim_{\varepsilon\to 0} \left( \int_{\text{supp}(f_\varepsilon)} f_\varepsilon dx \right) \ln \left( \frac{(2\pi e)^n}{\left( \int_{\text{supp}(f_\varepsilon)} f_\varepsilon dx \right)^2} \right) = \left( \int_{\text{supp}(f)} f dx \right) \ln \left( \frac{(2\pi e)^n}{\left( \int_{\text{supp}(f)} f dx \right)^2} \right).$$

We are left with showing that for the entropy function

$$\lim_{\varepsilon\to 0} \int f_\varepsilon \ln f_\varepsilon = \int f \ln f.$$

This is straightforward from the definition of $f_\varepsilon$ and the assumptions on $f$, as

$$f_\varepsilon \ln f_\varepsilon = \left( e^{-\varepsilon\|x\|^2/2} f \ln f + \varepsilon f e^{-\varepsilon\|x\|^2/2} \|x\|^2/2 \right) \chi_{\{f\geq\varepsilon\}}.$$

For the first term, apply again the dominated convergence theorem, and the second term disappears since the second moment of $f_\varepsilon$ is bounded uniformly by the second moment of $f$. We end up with

$$\begin{aligned} &\int_{\text{supp}(f)} f \Big( \ln \Big( \det \left( - \left( \nabla^2 \ln f \right) \right) \Big) \Big) dx \\ &\leq \left( \int_{\text{supp}(f)} f \right) \ln \left( \frac{(2\pi e)^n}{\left( \int_{\text{supp}(f)} f dx \right)^2} \right) + 2 \int_{\text{supp}(f)} f \ln f. \end{aligned} \tag{23}$$

This completes the proof of the main inequality. The equality case is easily verified, and in particular follows from the affine invariance together with the computation in Example 11. □

## 4 Linearization

In this section we prove Corollary 3, be means of linearization of our main inequality around its equality case. for convenience, we rewrite the inequality of Theorem 1 in terms of a convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $f = e^{-\psi}$. We get

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{-\psi} \ln(\det(\nabla^2(\psi)))dx \leq \\ &2 \left\{ -\int_{\mathbb{R}^n} e^{-\psi} \psi dx - \left( \int_{\mathbb{R}^n} e^{-\psi} dx \right) \ln \left( \int_{\mathbb{R}^n} e^{-\psi} dx \right) + \left( \int_{\mathbb{R}^n} e^{-\psi} dx \right) \ln(2\pi e)^{\frac{n}{2}} \right\}. \end{aligned} \tag{24}$$

Note that the support of $f$ is $\mathbb{R}^n$. We then linearize around the equality case $\psi(x) = \|x\|^2/2$.

*Proof of Corollary 3.* We first prove the corollary for functions with bounded support. Thus, let $\varphi$ be a twice continuously differentiable function with bounded support and let $\psi(x) = \|x\|^2/2 + \varepsilon\varphi(x)$. Note that for sufficiently small $\varepsilon$ the function $\psi$ is convex. Therefore we can plug $\psi$ into inequality (24) and develop in powers of $\varepsilon$. We evaluate first the left hand expression of (24). Since $\nabla^2(\psi) = I + \varepsilon\varphi$, we obtain for the left hand side

$$\int_{\mathbb{R}^n} e^{-\|x\|^2/2-\varepsilon\varphi} \ln(\det(I + \varepsilon\nabla^2\varphi))dx.$$

By Taylor's theorem this equals

$$\int_{\mathbb{R}^n} e^{-\|x\|^2/2} \left(1 - \varepsilon\varphi + \frac{\varepsilon^2}{2}\varphi^2\right) \cdot \ln(\det(I + \varepsilon\nabla^2\varphi))dx + O(\varepsilon^3).$$

For a matrix $A = (a_{i,j})_{i,j=1,\ldots,n}$, let $D(A) = \sum_{i=1}^n \sum_{j\neq i}^n [a_{i,i}a_{j,j} - a_{i,j}^2]$. Note that each $2\times 2$ minor is counted twice. Then

$$\det(I + \varepsilon\nabla^2\varphi) = 1 + \varepsilon\triangle\varphi + \frac{\varepsilon^2}{2}D(\nabla^2\varphi) + O(\varepsilon^3)$$

where $\triangle\varphi = \mathrm{tr}(\nabla^2\varphi)$ is the Laplacian of $\varphi$. Therefore the left hand side equals

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-\|x\|^2/2} \left(1 - \varepsilon\varphi + \frac{\varepsilon^2}{2}\varphi^2\right) \cdot \left(\varepsilon\triangle\varphi + \frac{\varepsilon^2}{2}D(\nabla^2\varphi) - \frac{\varepsilon^2}{2}(\triangle\varphi)^2\right) dx + O(\varepsilon^3) \\
&= \varepsilon \int_{\mathbb{R}^n} e^{-\|x\|^2/2}\triangle\varphi dx + \varepsilon^2 \int_{\mathbb{R}^n} e^{-\|x\|^2/2} \left[-\varphi\triangle\varphi + \frac{D(\nabla^2\varphi) - (\triangle\varphi)^2}{2}\right] dx + O(\varepsilon^3) \\
&= \varepsilon \int_{\mathbb{R}^n} \left(\|x\|^2 - n\right) e^{-\|x\|^2/2}\varphi + \varepsilon^2 \int_{\mathbb{R}^n} e^{-\|x\|^2/2} \left[-\varphi\triangle\varphi - \frac{\|\nabla^2\varphi\|_2^2}{2}\right] + O(\varepsilon^3).
\end{aligned}$$

The last equation follows by twice integration by parts.

Now we evaluate the right hand side expression. First consider

$$\int_{\mathbb{R}^n} e^{-\psi}dx \quad = \quad \int_{\mathbb{R}^n} e^{-\|x\|^2/2}dx - \varepsilon \int_{\mathbb{R}^n} e^{-\|x\|^2/2}\varphi dx + \varepsilon^2 \int_{\mathbb{R}^n} e^{-\|x\|^2/2}\frac{\varphi^2}{2}dx + O(\varepsilon^3).$$

Next,

$$\begin{aligned}
-\int_{\mathbb{R}^n} e^{-\psi}\psi dx \quad &= \quad -\int_{\mathbb{R}^n} e^{-\|x\|^2/2} \left(1 - \varepsilon\varphi + \varepsilon^2\frac{\varphi^2}{2}\right) \cdot \left(\frac{\|x\|^2}{2} + \varepsilon\varphi dx\right) + O(\varepsilon^3) \\
&= \quad -\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\frac{\|x\|^2}{2} + \varepsilon \left(\int_{\mathbb{R}^n} \varphi e^{-\|x\|^2/2} \left(\frac{\|x\|^2}{2} - 1\right) dx\right) \\
&\qquad +\varepsilon^2 \left(\int_{\mathbb{R}^n} \varphi^2 e^{-\|x\|^2/2} \left(1 - \frac{\|x\|^2}{4}\right) dx\right) + O(\varepsilon^3)
\end{aligned}$$

To treat $\left(\int_{\mathbb{R}^n} e^{-\psi}dx\right) \ln\left(\int_{\mathbb{R}^n} e^{-\psi}dx\right)$, we consider the function $g(y) = y\ln y$, which we will apply to $\int e^{-\|x\|^2/2-\varepsilon\varphi}$. We obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-\|x\|^2/2-\varepsilon\varphi} dx \ \ln\left(\int_{\mathbb{R}^n} e^{-\|x\|^2/2-\varepsilon\varphi} dx\right) \\
&= \frac{n}{2}(2\pi)^{n/2}\ln(2\pi) + \varepsilon\left(-\left(\frac{n}{2}\ln(2\pi)+1\right)\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\varphi dx\right) \\
&+\varepsilon^2\left(\left(\frac{n}{2}\ln(2\pi)+1\right)\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\frac{\varphi^2}{2}dx + \frac{1}{2(2\pi)^{n/2}}\left(\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\varphi dx\right)^2\right) + O(\varepsilon^3)
\end{aligned}$$

Altogether, the right hand side equals

$$\begin{aligned}
&2\left\{\int_{\mathbb{R}^n} e^{-\|x\|^2/2-\varepsilon\varphi}(-\|x\|^2/2-\varepsilon\varphi)dx - \left(\int_{\mathbb{R}^n} e^{-\|x\|^2/2-\varepsilon\varphi}dx\right)\ln\left(\int_{\mathbb{R}^n} e^{-\|x\|^2/2-\varepsilon\varphi}dx\right)\right\} \\
&+\left(\int_{\mathbb{R}^n} e^{-\|x\|^2/2-\varepsilon\varphi}dx\right) n\ln(2\pi e) \\
&= -\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\|x\|^2 dx - n(2\pi)^{n/2}\ln(2\pi) + n\ln(2\pi e)\int_{\mathbb{R}^n} e^{-\|x\|^2/2}dx \\
&+\varepsilon\left\{2\left(\int_{\mathbb{R}^n}\varphi e^{-\|x\|^2/2}(\frac{\|x\|^2}{2}-1)dx\right) - 2\left(-(\frac{n}{2}\ln(2\pi)+1)\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\varphi dx\right)\right. \\
&\left.-n\ln(2\pi e)\int e^{-\|x\|^2/2}\varphi\right\} \\
&+\varepsilon^2\left\{\int_{\mathbb{R}^n}\varphi^2 e^{-\|x\|^2/2}(1+\frac{n}{2}-\frac{1}{2}\|x\|^2)dx - \frac{1}{(2\pi)^{n/2}}\left(\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\varphi dx\right)^2\right\} + O(\varepsilon^3).
\end{aligned}$$

Since

$$\int_{\mathbb{R}^n} e^{-\|x\|^2/2}dx = (2\pi)^{n/2} \qquad \text{and} \qquad \int_{\mathbb{R}^n}\|x\|^2 e^{-\|x\|^2/2}dx = n(2\pi)^{n/2},$$

we get for the zeroth order term,

$$\begin{aligned}
&-\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\|x\|^2 dx - n(2\pi)^{n/2}\ln(2\pi) + n\ln(2\pi e)\int_{\mathbb{R}^n} e^{-\|x\|^2/2}dx \\
&= -n(2\pi)^{n/2} - n(2\pi)^{n/2}\ln(2\pi) + n\ln(2\pi e)(2\pi)^{n/2} = 0.
\end{aligned}$$

Therefore, we get for the right hand side

$$\begin{aligned}
&\varepsilon\int_{\mathbb{R}^n}\varphi e^{-\|x\|^2/2}(\|x\|^2-n)dx \\
&+\varepsilon^2\left\{\int_{\mathbb{R}^n}\varphi^2 e^{-\|x\|^2/2}\left(\frac{n+2-\|x\|^2}{2}\right)dx - \frac{1}{(2\pi)^{n/2}}\left(\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\varphi dx\right)^2\right\} + O(\varepsilon^3).
\end{aligned}$$

The coefficients of $\varepsilon$ on the left and right hand side are the same and we disacrd them. We divide both sides by $\varepsilon^2$ and take the limit for $\varepsilon \to 0$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\left[-\varphi\triangle\varphi - \frac{\|\nabla^2\varphi\|_2^2}{2}\right]dx \\
&\qquad \le \int_{\mathbb{R}^n}\varphi^2 e^{-\|x\|^2/2}(\frac{n+2-\|x\|^2}{2})dx - \frac{1}{(2\pi)^{n/2}}\left(\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\varphi dx\right)^2.
\end{aligned}$$

If we want the right hand side to include the variance, we may write the inequality as follows

$$\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\left[-\varphi\triangle\varphi - \frac{\|\nabla^2\varphi\|_2^2}{2}\right]dx$$
$$\leq \int_{\mathbb{R}^n}\varphi^2 e^{-\|x\|^2/2}\left(\frac{n-|x|^2}{2}\right)dx + (2\pi)^{n/2}\left[\int_{\mathbb{R}^n}\varphi^2 d\gamma_n - \left(\int_{\mathbb{R}^n}\varphi d\gamma_n\right)^2\right] \quad (25)$$

Now we integrate on the right by parts twice, noting that $(n-\|x\|^2)e^{-\|x\|^2/2} = \triangle(e^{-\|x\|^2/2})$, so that the first term on the right hand side is

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\varphi^2(\frac{n-\|x\|^2}{2})dx &= -\frac{1}{2}\int_{\mathbb{R}^n} e^{-\|x\|^2/2}\triangle(\varphi^2)dx \\ &= -\int_{\mathbb{R}^n} e^{-\|x\|^2/2}(\varphi\triangle\varphi + \|\nabla\varphi\|^2)dx.\end{aligned}$$

We put that in (25) and one gets

$$\int_{\mathbb{R}^n} e^{-|x|^2/2}\left[\|\nabla\varphi\|^2 - \frac{\|\nabla^2\varphi\|_{HS}^2}{2}\right]dx \leq (2\pi)^{n/2}\left[\int_{\mathbb{R}^n}\varphi^2 d\gamma_n - \left(\int_{\mathbb{R}^n}\varphi d\gamma_n\right)^2\right],$$

which we can rewrite as

$$\int_{\mathbb{R}^n}\|\nabla\varphi\|^2 - \frac{\|\nabla^2\varphi\|_2^2}{2}d\gamma_n \leq \int_{\mathbb{R}^n}\varphi^2 d\gamma_n - \left(\int_{\mathbb{R}^n}\varphi d\gamma_n\right)^2.$$

Thus we have shown that the inequality holds for all twice continuously differentiable functions $\varphi$ with bounded support. One may extend it to all twice continuously differentiable functions $\varphi \in L^2(\mathbb{R}^n,\gamma_n)$ with $\|\nabla^2\varphi\|_{HS} \in L^2(\mathbb{R}^n,\gamma_n)$ by a standard approximation argument, as follows.

Let $\chi_k$ be a twice continuously differentiable function bounded between zero and one such that $\chi_n(x) = 1$ for all $\|x\| \leq k$ and $\chi_n(x) = 0$ for all $\|x\| > k+1$. Then, for all $k \in \mathbb{N}$

$$\int_{\mathbb{R}^n}\left[\|\nabla(\varphi\circ\chi_k)\|^2 - \frac{\|\nabla^2\varphi\|_{HS}^2}{2}\right]d\gamma_n \leq \int_{\mathbb{R}}(\varphi\circ\chi_k)^2 d\gamma - \left(\int_{\mathbb{R}^n}(\varphi\circ\chi_k)d\gamma_n\right)^2,$$

or, equivalently,

$$\int_{\mathbb{R}^n}\|\nabla(\varphi\circ\chi_k)\|^2 d\gamma + \left(\int(\varphi\circ\chi_k)d\gamma_n\right)^2 \leq \int_{\mathbb{R}^n}(\varphi\circ\chi_k)^2 d\gamma + \int_{\mathbb{R}^n}\frac{\|\nabla^2\varphi\|_{HS}^2}{2}d\gamma_n.$$

It follows that

$$\liminf_{k\to\infty}\int_{\mathbb{R}^n}\|\nabla(\varphi\circ\chi_k)\|^2 d\gamma_n + \liminf_{k\to\infty}\left(\int(\varphi\circ\chi_k)d\gamma_n\right)^2$$
$$\leq \limsup_{k\to\infty}\int_{\mathbb{R}^n}(\varphi\circ\chi_k)^2 d\gamma_n + \limsup_{k\to\infty}\int_{\mathbb{R}^n}\frac{\|\nabla^2\varphi\|_{HS}^2}{2}d\gamma_n.$$

By Fatou's lemma and the dominated convergence theorem

$$\int_{\mathbb{R}^n}\liminf_{k\to\infty}\|\nabla(\varphi\circ\chi_k)\|^2 d\gamma_n + \left(\int_{\mathbb{R}^n}\lim_{k\to\infty}(\varphi\circ\chi_k)d\gamma_n\right)^2$$
$$\leq \int_{\mathbb{R}}\lim_{k\to\infty}(\varphi\circ\chi_k)^2 d\gamma + \int_{\mathbb{R}}\limsup_{k\to\infty}\frac{\|\nabla^2\varphi\|_{HS}^2}{2}d\gamma,$$

which gives

$$\int_{\mathbb{R}^n} \|\nabla\varphi\|^2 d\gamma + \left(\int_{\mathbb{R}^n} \varphi d\gamma\right)^2 \le \int_{\mathbb{R}^n} \varphi^2 d\gamma_n + \int_{\mathbb{R}^n} \frac{\|\nabla^2\varphi\|_{HS}^2}{2} d\gamma_n.$$

□

An alternative, direct proof of Corollary 3 may be given by expanding $\varphi \in C^2(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, \gamma_n)$ into Hermite polynomials. That is, denote by $h_0(x), h_1(x), \ldots$ the Hermite polynomials in one variable, normalized so that $\|h_i\|_{L^2(\gamma_1)} = 1$ for all $i$. We may decompose

$$\varphi = \sum_{i_1,\ldots,i_n=0}^{\infty} a_{i_1,\ldots,i_n} \prod_{j=1}^{n} h_{i_j}(x_i)$$

where the convergence is in $L^2(\mathbb{R}^n, \gamma_n)$. Then the right-hand side of (7) equals

$$\sum_{\substack{i_1,\ldots,i_n=0 \\ (i_1,\ldots,i_n)\neq(0,\ldots,0)}}^{\infty} a_{i_1,\ldots,i_n}^2. \tag{26}$$

Using the identity $h_i' = \sqrt{i} \cdot h_{i-1}$, we see that the left-hand side of (7) is

$$\int_{\mathbb{R}^n} \left[\|\nabla\varphi\|^2 - \frac{\|\nabla^2\varphi\|_{HS}^2}{2}\right] d\gamma_n = \sum_{i_1,\ldots,i_n=0}^{\infty} \left[\frac{3}{2}\sum_{j=1}^{n} i_j - \frac{1}{2}\left(\sum_{j=1}^{n} i_j\right)^2\right] a_{i_1,\ldots,i_n}^2. \tag{27}$$

We will use the simple fact that $x(3-x)/2 \le 1$ for any integer $x \ge 1$, for $x = \sum_{j=1}^n i_j$. Glancing at (26) with (27) and using the aforementioned simple fact, we deduce Corollary 3. We also see that equality in (7) holds if and only if $\varphi$ is a polynomial of degree at most 2, because $x(3-x)/2 = 1$ only for $x = 1, 2$.

The proof of Theorem 4 is along the exact same lines, using the all the derivatives are diagonalized by the Hermite polynomials with respect to the Gaussian measure, only that the inequality $x(3-x)/2 \le 1$, which can be rewritten as $(x-1)(x-2) \ge 0$ for integers $x \ge 1$, is replaced by the more general inequality $(x-1)(x-2)\cdots(x-j) \ge 0$ for integers $x \ge 1$, with equality if and only if $x \in \{1, \ldots, j\}$.

We remark that it is desirable to find an alternative, direct proof of Theorem 1, which does not rely on the affine isoperimetric inequality.

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