

# DISCRETE VARIANTS OF BRUNN-MINKOWSKI TYPE INEQUALITIES

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ABSTRACT. We present an alternative, short proof of a recent discrete version of the Brunn-Minkowski inequality due to Lehec and the second named author. Our proof also yields the four functions theorem of Ahlswede and Daykin as well as some new variants.

## 1. INTRODUCTION

Correlation inequalities such as the Fortuin-Kasteleyin-Ginibre (FKG) inequality are of use in the analysis of several models in probability theory and statistical physics (see, e.g., Grimmett [5, 6]). These inequalities are closely related to the following four functions theorem of Ahlswede and Daykin [1]:

**Theorem 1.1.** *Suppose that  $f, g, h, k : \mathbb{Z}^n \rightarrow [0, \infty)$  satisfy*

$$f(x)g(y) \leq h(x \wedge y)k(x \vee y) \quad \forall x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{Z}^n$$

where  $x \wedge y = (\min(x_1, y_1), \dots, \min(x_n, y_n))$ , and  $x \vee y = (\max(x_1, y_1), \dots, \max(x_n, y_n))$ .

Then

$$\left( \sum_{x \in \mathbb{Z}^n} f(x) \right) \left( \sum_{x \in \mathbb{Z}^n} g(x) \right) \leq \left( \sum_{x \in \mathbb{Z}^n} h(x) \right) \left( \sum_{x \in \mathbb{Z}^n} k(x) \right).$$

Theorem 1.1 is usually formulated under the additional assumption that  $f, g, h, k$  are all supported in the discrete cube  $\{0, 1\}^n$ . It was suggested by Gozlan, Roberto, Samson, and Tetali [4] that Theorem 1.1 is connected with a discrete variant of the Brunn-Minkowski inequality, recently proven by Lehec and the second named author [8, Theorem 1.4], which is the case  $\lambda = 1/2, n = 1$  of the following theorem:

**Theorem 1.2.** *Let  $\lambda \in [0, 1]$  and suppose that  $f, g, h, k : \mathbb{Z}^n \rightarrow [0, \infty)$  satisfy*

$$f(x)g(y) \leq h(\lfloor \lambda x + (1 - \lambda)y \rfloor)k(\lceil (1 - \lambda)x + \lambda y \rceil) \quad \forall x, y \in \mathbb{Z}^n$$

where  $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor)$  and  $\lceil x \rceil = (\lceil x_1 \rceil, \dots, \lceil x_n \rceil)$ . Then

$$\left( \sum_{x \in \mathbb{Z}^n} f(x) \right) \left( \sum_{x \in \mathbb{Z}^n} g(x) \right) \leq \left( \sum_{x \in \mathbb{Z}^n} h(x) \right) \left( \sum_{x \in \mathbb{Z}^n} k(x) \right).$$

Here  $\lfloor r \rfloor = \max\{m \in \mathbb{Z}; m \leq r\}$  is the lower integer part of  $r \in \mathbb{R}$  and  $\lceil r \rceil = -\lfloor -r \rfloor$  the upper integer part. A standard limiting argument (see [4, Section 2.3]) leads from the case

$\lambda = 1/2, h = k$  of Theorem 1.2 to the case  $\lambda = 1/2$  of the Brunn-Minkowski inequality in its multiplicative form:

$$\text{Vol}\left(\frac{A+B}{2}\right) \geq \sqrt{\text{Vol}(A)\text{Vol}(B)},$$

where  $A+B = \{x+y; x \in A, y \in B\}$ , where  $A, B \subseteq \mathbb{R}^n$  are any Borel-measurable sets, and  $\text{Vol}(\cdot)$  stands for the  $n$ -dimensional Lebesgue volume. The proof in [8] for the case  $n = 1, \lambda = 1/2$ , which involves stochastic analysis, admits a straightforward generalization to the more general case described above. An alternative argument using ideas from the theory of optimal transport was given by Gozlan, Roberto, Samson and Tetali [4].

Our goal in this note is to provide a unified proof of Theorem 1.1 and Theorem 1.2, which is perhaps as elementary as the original proof of the four functions theorem by Ahlswede and Daykin [1]. The first issue that we would like to address, is the identification of the relevant common features of operations such as

$$(1) \quad x \wedge y, x \vee y, \left\lfloor \frac{x+y}{2} \right\rfloor, \left\lceil \frac{x+y}{2} \right\rceil, \lfloor \lambda x + (1-\lambda)y \rfloor, \lceil \lambda x + (1-\lambda)y \rceil, \dots$$

that are defined for  $x, y \in \mathbb{Z}^n$ , with  $0 < \lambda < 1$ .

Our observation is that these operations  $T : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  satisfy two axioms:

(P1)  $T$  is *translation equivariant*:  $T(x+z, y+z) = T(x, y) + z$  for all  $z \in \mathbb{Z}^n$ .

(P2)  $T$  is *monotone in the sense of Knothe* with respect to some decomposition of  $\mathbb{Z}^n$  into a direct sum of groups  $\mathbb{Z}^n = G_1 \times \dots \times G_k$ . That is,  $T = (T_1, \dots, T_k)$  where for each  $i \in \{1, \dots, k\}$ :

(i)  $T_i : (G_1 \times \dots \times G_i) \times (G_1 \times \dots \times G_i) \rightarrow G_i$ . In other words,  $T_i(x, y)$  depends only on the first  $i$  coordinates of its arguments  $x, y \in G_1 \times \dots \times G_k$ .

(ii) There exists a total additive ordering  $\preceq_i$  on  $G_i$  such that  $T_i^{(a,b)} : G_i \times G_i \rightarrow G_i$  defined by  $T_i^{(a,b)}(x, y) = T_i((a, x), (b, y))$  for  $a, b \in G_1 \times \dots \times G_{i-1}$  satisfies

$$x_1 \preceq_i x_2, y_1 \preceq_i y_2 \quad \implies \quad T_i^{(a,b)}(x_1, y_1) \preceq_i T_i^{(a,b)}(x_2, y_2)$$

for all  $a, b \in G_1 \times \dots \times G_{i-1}$  and  $x_1, x_2, y_1, y_2 \in G_i$ .

Maps that satisfy a condition similar to (P2) were used by Knothe [9] in his proof of the Brunn-Minkowski inequality. In the language of stochastic processes, one could say that the map  $T$  is adapted to the filtration induced by the decomposition  $\mathbb{Z}^n = G_1 \times \dots \times G_k$ , or that the map  $T$  cannot see into the future and it is monotone when conditioning on the past. Recall that a total ordering  $\preceq$  on an abelian group  $G$  is a binary relation which is reflexive,

anti-symmetric and transitive, such that for any distinct  $x, y$ , either  $x \preceq y$  or else  $y \preceq x$ . An ordering  $\preceq$  is additive if for all  $x, y, z$ ,

$$x \preceq y \implies x + z \preceq y + z.$$

We remark that the requirement of existence of a total additive ordering on a finitely-generated abelian group  $G$ , forces  $G$  to be isomorphic to  $\mathbb{Z}^\ell$  for some  $\ell$ .

The standard cartesian decomposition of  $\mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$  into one-dimensional groups, each of which equipped with the standard order on  $\mathbb{Z}$ , attests to the fact that all the examples in (1) satisfy properties (P1) and (P2). In these examples, each  $T_i$  is a function from  $G_i \times G_i$  to  $G_i$ .

Another natural example for an additive, total ordering on  $\mathbb{Z}^n$  (or on  $\mathbb{R}^n$ ) is the standard lexicographic order relation. Given an additive, total ordering  $\preceq$  on  $\mathbb{R}^n$  and an invertible, linear map  $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$  we may construct another additive, total ordering  $\preceq_L$  by requiring that  $x \preceq_L y$  if and only if  $Lx \preceq Ly$ . For an additive, total ordering  $\preceq$  on  $\mathbb{Z}^n$  the operations  $\max(x, y)$  and  $\min(x, y)$  are well-defined, and they satisfy properties (P1) and (P2) with  $k = 1$ .

Yet another example for an operation  $T : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  that satisfies properties (P1) and (P2) is given by  $T = (T_1, \dots, T_n)$  where

$$(2) \quad T_i(x, y) = \begin{cases} x_i, & \#\{j \leq i ; x_j \neq y_j\} \text{ is odd} \\ y_i, & \text{otherwise} \end{cases}.$$

We prove the following:

**Theorem 1.3.** *Let  $T : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  satisfy properties (P1) and (P2). Suppose that  $f, g, h, k : \mathbb{Z}^n \rightarrow [0, \infty)$  satisfy*

$$f(x)g(y) \leq h(T(x, y))k(x + y - T(x, y)) \quad \forall x, y \in \mathbb{Z}^n.$$

*Then*

$$\left( \sum_{x \in \mathbb{Z}^n} f(x) \right) \left( \sum_{x \in \mathbb{Z}^n} g(x) \right) \leq \left( \sum_{x \in \mathbb{Z}^n} h(x) \right) \left( \sum_{x \in \mathbb{Z}^n} k(x) \right).$$

Clearly Theorem 1.1 and Theorem 1.2 follow from Theorem 1.3. See also Borell [2, Theorem 2.1] for Brunn-Minkowski type inequalities for operations other than Minkowski sum with monotonicity properties.

One can relax the monotonicity property (P2) by replacing it with another “exclusion” property, which requires no ordering at all. We formulate this property, as well as our next theorem, in greater generality, with  $\mathbb{Z}^n$  replaced by a finitely generated abelian group  $G$ , and  $T : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  replaced by  $T : G \times G \rightarrow G$ . It is well-known that any such  $G$  is isomorphic to  $\mathbb{Z}^n \times (\mathbb{Z}/p_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_k\mathbb{Z})$  for some powers of primes  $p_1, \dots, p_k$ .

**Definition 1.4.** We say that an operation  $T : G \times G \rightarrow G$  is *exclusive* if for every finite set  $A \subseteq G$  with at least two elements, and all  $z \in G$ , there exist distinct  $x, y \in A$  such that for  $A_1 = A \setminus \{x\}$ ,  $A_2 = A \setminus \{y\}$ , and  $A_3 = A \setminus \{x, y\}$ , the following conditions holds:

- (a)  $\{T(x, z - y), T(y, z - x)\} \not\subseteq T(A_i, z - A_i)$  for  $i \in \{1, 2\}$ ,
  - (b)  $\{T(x, z - y), T(y, z - x)\} \cap T(A_3, z - A_3) = \emptyset$ ,
- where  $T(A_i, z - A_i) = \{T(u, z - v) ; u, v \in A_i\}$ .

In the next theorem, we replace (P2) by the following property:

- (P2') There exists a decomposition of abelian groups  $G = G_1 \times \cdots \times G_k$  such that
  - (i)  $T = (T_1, \dots, T_k)$  with  $T_i : (G_1 \times \cdots \times G_i) \times (G_1 \times \cdots \times G_i) \rightarrow G_i$  for each  $i$ .
  - (ii) For all  $i \in \{1, \dots, k\}$  and  $a, b \in G_1 \times \cdots \times G_{i-1}$  the operation  $T_i^{(a,b)} : G_i \times G_i \rightarrow G_i$  defined by  $T_i^{(a,b)}(x, y) = T_i((a, x), (b, y))$  is exclusive.

We prove the following:

**Theorem 1.5.** *Let  $(G, +)$  be a finitely generated abelian group, and  $T : G \times G \rightarrow G$  satisfy (P1) and (P2'). Suppose that  $f, g, h, k : G \rightarrow [0, \infty)$  satisfy*

$$f(x)g(y) \leq h(T(x, y))k(x + y - T(x, y)) \quad \forall x, y \in G.$$

Then

$$\left( \sum_{x \in G} f(x) \right) \left( \sum_{x \in G} g(x) \right) \leq \left( \sum_{x \in G} h(x) \right) \left( \sum_{x \in G} k(x) \right).$$

The next two sections are devoted to the proofs of the above theorems. We additionally include a final section with commentary on the applicability of this work to related inequalities, such as the ones proven by Cordero-Erausquin and Maurey [3], Iglesias, Yepes Nicolás and Zvavitch [7] and Ollivier and Villani [10]

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## 2. PROOF OF THEOREM 1.5

The core of this paper is the proof of Theorem 1.5 in the particular case where  $T$  itself is exclusive, which is Proposition 2.2 below. We begin with the following elementary fact:

**Fact 2.1.** *Suppose  $a, b, c, d \geq 0$ . If  $ab \leq cd$  and  $\max\{a, b\} \leq \max\{c, d\}$  then  $a + b \leq c + d$ .*

*Proof.* Pick  $A \geq a, B \geq b$  such that  $\max\{A, B\} \leq \max\{c, d\}$  and  $AB = cd = r$ . Then  $(A - B)^2 \leq (c - d)^2$  and so  $(a + b)^2 \leq (A + B)^2 = 4r + (A - B)^2 \leq 4r + (c - d)^2 = (c + d)^2$ .  $\square$

Recall that under the assumptions of Theorem 1.5 we have  $f, g, h, k : G \rightarrow [0, \infty)$  satisfying

$$(3) \quad f(x)g(y) \leq h(T(x, y))k(x + y - T(x, y)) \quad \forall x, y \in G.$$

For  $j, z \in G$  denote  $F_z(j) = f(j)g(z - j)$  and  $H_z(j) = h(j)k(z - j)$ . Note that, by (3),

$$(4) \quad F_z(j) \leq H_z(T(j, z - j)).$$

We claim that for all  $i, j, z \in G$  we have

$$(5) \quad F_z(i)F_z(j) \leq H_z(T(i, z - j))H_z(T(j, z - i)).$$

Indeed, by (3) and (P1) we have

$$\begin{aligned} F_z(i)F_z(j) &= f(i)g(z - i)f(j)g(z - j) = f(i)g(z - j)f(j)g(z - i) \\ &\leq h(T(i, z - j))k(z + (i - j) - T(i, z - j))h(T(j, z - i))k(z + (j - i) - T(j, z - i)) \\ &= h(T(i, z - j))k(z - T(i, z - j))h(T(j, z - i))k(z - T(j, z - i)) \\ &= H_z(T(i, z - j))H_z(T(j, z - i)). \end{aligned}$$

**Proposition 2.2.** *Let  $(G, +)$  be a finitely generated abelian group, and let  $T : G \times G \rightarrow G$  be an exclusive operation that satisfies (P1). Suppose that  $f, g, h, k : G \rightarrow [0, \infty)$  satisfy*

$$f(x)g(y) \leq h(T(x, y))k(x + y - T(x, y)) \quad \forall x, y \in G.$$

*Then*

$$\left( \sum_{x \in G} f(x) \right) \left( \sum_{x \in G} g(x) \right) \leq \left( \sum_{x \in G} h(x) \right) \left( \sum_{x \in G} k(x) \right).$$

*Proof.* We need to prove that

$$\sum_{j, z \in G} F_z(j) \leq \sum_{j, z \in G} H_z(j).$$

Fix  $z \in G$ . It is sufficient to prove that for every finite set  $A \subseteq G$ ,

$$(6) \quad \sum_{j \in A} F_z(j) \leq \sum_{j \in T(A, z-A)} H_z(j).$$

We proceed to prove so by induction on  $n = |A|$ .

**Induction base:** For  $n = 0$  the statement is vacuous, as the empty sum equals zero. For  $n = 1$  the statement holds by (4).

**Induction step:** Assume  $n \geq 2$  and that the statement holds for all  $m \leq n - 1$ . Let  $A \subseteq G$  with  $|A| = n$ . By assumption, there exist distinct  $x, y \in A$  such that assertions (a) and (b) of Definition 1.4 are satisfied. By switching  $x$  with  $y$  if necessary, we may assume that  $F_z(x) \leq F_z(y)$ . By (5) we have

$$(7) \quad F_z(x)F_z(y) \leq H_z(T(x, z - y))H_z(T(y, z - x)).$$

Case 1: Assume  $F_z(y) \geq \max\{H_z(T(x, z - y)), H_z(T(y, z - x))\}$ . Then, it follows from (7) that

$$(8) \quad F_z(x) \leq \min\{H_z(T(x, z - y)), H_z(T(y, z - x))\}.$$

The induction hypothesis for  $A_1 = A \setminus \{x\}$  tells us that

$$(9) \quad \sum_{j \in A_1} F_z(j) \leq \sum_{j \in T(A_1, z-A_1)} H_z(j).$$

By adding inequalities (8) and (9), and using property (a) of Definition 1.4, we obtain the desired inequality (6).

Case 2: Assume  $F_z(y) \leq \max\{H_z(T(x, z - y)), H_z(T(y, z - x))\}$ . Since  $F_z(x) \leq F_z(y)$ , we may apply (7) and Fact 2.1 and obtain

$$(10) \quad F_z(x) + F_z(y) \leq H_z(T(x, z - y)) + H_z(T(y, z - x)).$$

Note that  $T(x, z - y) \neq T(y, z - x)$  as  $T(y, z - x) - T(x, z - y) = y - x \neq 0$ . Therefore, by combining (10) with the induction hypothesis for  $A_3 = A \setminus \{x, y\}$  and property (b), we deduce the inequality (6). This completes our proof.  $\square$

*Proof of Theorem 1.5.* We proceed by induction on  $k$ , the number of groups participating in the decomposition of  $G$  given in (P2'). For  $k = 1$ , the statement is equivalent to that in Proposition 2.2. Assume next that  $k \geq 2$  and that the statement holds true for  $k - 1$ .

Denote  $G' = G_2 \times \cdots \times G_k$ . For  $a, b \in G_1$  and  $x', y' \in G'$  denote

$$f^a(x') = f(a, x'), \quad g^a(x') = g(a, x'), \quad h^a(x') = h(a, x'), \quad k^a(x') = k(a, x').$$

Fix  $a, b \in G_1$ . For  $i \in \{2, \dots, k\}$ , define  $T'_i : (G_2 \times \dots \times G_i) \times (G_2 \times \dots \times G_i) \rightarrow G_i$  by  $T'_i(x, y) = T_i((a, x), (b, y))$ , and  $T' : G' \times G' \rightarrow G'$  by  $T' = (T'_2, \dots, T'_k)$ . Note that  $T'$  satisfies (P1) and (P2') with respect to this decomposition. The assumptions of the theorem tell us that

$$f^a(x')g^b(y') \leq h^{T_1(a,b)}(T'(x', y'))k^{a+b-T_1(a,b)}(x' + y' - T'(x', y')) \quad \forall x', y' \in G'.$$

By the induction hypothesis, it follows that

$$\left( \sum_{x' \in G'} f^a(x') \right) \left( \sum_{x' \in G'} g^b(x') \right) \leq \left( \sum_{x' \in G'} h^{T_1(a,b)}(x') \right) \left( \sum_{x' \in G'} k^{a+b-T_1(a,b)}(x') \right).$$

For every  $a \in G_1$  set

$$F(a) = \sum_{x' \in G'} f^a(x'), \quad G(a) = \sum_{x' \in G'} g^a(x'), \quad H(a) = \sum_{x' \in G'} h^a(x') \quad \text{and} \quad K(a) = \sum_{x' \in G'} k^a(x').$$

Rewriting the previous inequality, we have for all  $a, b \in G_1$ ,

$$F(a)G(b) \leq H(T_1(a, b))K(a + b - T_1(a, b)).$$

Since  $T_1 : G_1 \times G_1 \rightarrow G_1$  is an exclusive map satisfying (P1), we may apply Proposition 2.2 and conclude that

$$\left( \sum_{a \in G_1} F(a) \right) \left( \sum_{a \in G_1} G(a) \right) \leq \left( \sum_{a \in G_1} H(a) \right) \left( \sum_{a \in G_1} K(a) \right).$$

This completes the proof. □

### 3. PROOF OF THEOREM 1.3

Theorem 1.3 is an immediate consequence of Theorem 1.5 due to the following observation:

**Lemma 3.1.** *Suppose  $T : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  satisfies properties (P1) and (P2). Then  $T$  satisfies property (P2').*

*Proof.* We first show that  $T_1$  is exclusive. To this end, let  $m \geq 2$  and  $z \in G_1$ . Suppose  $A = \{x_1, \dots, x_m\} \subseteq G_1$ , where  $x_1 \prec_1 \dots \prec_1 x_m$ . Here  $a \prec_1 b$  means that  $a \preceq_1 b$  and  $a \neq b$ . Set  $x = x_1, y = x_m$ , and recall that  $A_1 = A \setminus \{x\}$ ,  $A_2 = A \setminus \{y\}$ , and  $A_3 = A \setminus \{x, y\}$ . For a finite subset  $S \subseteq G_1$  we write  $\max S$  and  $\min S$  for the maximal and minimal elements with

respect to the total order  $\preceq_1$ . By properties (P1) and (P2) we have

$$\begin{aligned} \max_{v,w \in A_1} T_1(w, z - v) - \min_{v,w \in A_1} T_1(v, z - w) &= T_1(x_m, z - x_2) - T_1(x_2, z - x_m) \\ &= x_m - x_2 \prec_1 x_m - x_1 = T_1(x_m, z - x_1) - T_1(x_1, z - x_m). \end{aligned}$$

Therefore,  $\{T_1(x_1, z - x_m), T_1(x_m, z - x_1)\} \not\subseteq T_1(A_1, z - A_1)$ . A similar argument shows that the same holds for  $A_2$ . This verifies condition (a) of Definition 1.4. To verify condition (b) of Definition 1.4, note that if  $m = 2$  then  $A_3 = \emptyset$ , and hence condition (b) holds trivially. Otherwise, letting  $v = \min\{x_{i+1} - x_i; i \in \{1, \dots, m-1\}\} \succ 0$ , we have

$$\begin{aligned} \max_{v,w \in A_3} T_1(v, z - w) &= T_1(x_{m-1}, z - x_2) \\ &\preceq T_1(x_m - v, z - x_1 - v) = T_1(x_m, z - x_1) - v \prec T_1(x_m, z - x_1). \end{aligned}$$

Therefore,  $T_1(x_m, z - x_1) \notin T_1(A_3, z - A_3)$ . Similarly,  $T_1(x_1, z - x_m) \notin T_1(A_3, z - A_3)$ , which verifies condition (b) of Definition 1.4. Hence  $T_1$  is an exclusive map. We proceed by induction on the number of groups  $k$  in the decomposition of  $G$  given in property (P2'). For  $k = 1$ , we verified above that the statement holds for  $T = T_1$ .

Let  $k \geq 2$  and assume that the statement holds for a decomposition into  $k - 1$  groups. Fix  $a_1, b_1 \in G_1$  and let  $G' = G_2 \times \dots \times G_k$ . For  $i \in \{2, \dots, k\}$ , define  $T'_i : (G_2 \times \dots \times G_i) \times (G_2 \times \dots \times G_i) \rightarrow G_i$  by  $T'_i(x, y) = T_i((a_1, x), (b_1, y))$  and  $T' : G' \times G' \rightarrow G'$  by  $T' = (T'_2, \dots, T'_k)$ . Note that  $T'$  satisfies (P1) and (P2) with respect to this decomposition. By the induction hypothesis,  $T'$  satisfies (P2').

This means that for all  $i \in \{2, \dots, k\}$  and  $a = (a_1, a'), b = (b_1, b') \in G_1 \times \dots \times G_{i-1}$ , the map  $T_i^{(a,b)} = T_i^{(a',b')}$  is exclusive. Since  $a_1$  and  $b_1$  are arbitrary, it follows that  $T_i^{(a,b)}$  is exclusive for all  $i \in \{1, \dots, k\}$  and  $a, b \in G_1 \times \dots \times G_{i-1}$ , and thus  $T$  satisfies (P2').  $\square$

*Remark 3.2.* Using Lemma 3.1, one can show that the operations in (1) do not satisfy (P2) without decomposing  $\mathbb{Z}^n$  into a direct sum of more than one group. To see this, consider e.g.,  $T(x, y) = x \vee y$ , defined for  $x, y \in \mathbb{Z}^2$ . By Lemma 3.1, it is sufficient to show that  $T$  is not exclusive. A direct inspection of the set  $A = \{(-1, -1), (-1, 1), (1, -1), (1, 1)\} \subset \mathbb{Z}^2$  and the point  $z = (0, 0) \in \mathbb{Z}^2$  shows that  $T$  indeed violates the conditions of Definition 1.4.

## 4. RELATED INEQUALITIES

**4.1. Continuous Brunn-Minkowski type inequalities.** The classical Brunn-Minkowski inequality states that for any two non-empty Borel-measurable subsets of  $\mathbb{R}^n$ , one has

$$\text{Vol}(A + B)^{1/n} \geq \text{Vol}(A)^{1/n} + \text{Vol}(B)^{1/n}.$$



In its equivalent dimension-free form, it states that for any  $\lambda \in [0, 1]$ ,

$$\text{Vol}(\lambda A + (1 - \lambda)B) \geq \text{Vol}(A)^\lambda \text{Vol}(B)^{1-\lambda}.$$

A functional form of the Brunn-Minkowski inequality, known as the Prékopa-Leindler inequality, states that for any Borel functions  $f, g, h : \mathbb{R}^n \rightarrow [0, \infty)$  and any  $\lambda \in [0, 1]$  such that  $f(x)^\lambda g(y)^{1-\lambda} \leq h(\lambda x + (1 - \lambda)y)$  for all  $x, y \in \mathbb{R}^n$ , we have

$$(11) \quad \left( \int_{\mathbb{R}^n} f(x) dx \right)^\lambda \left( \int_{\mathbb{R}^n} g(x) dx \right)^{1-\lambda} \leq \int_{\mathbb{R}^n} h(x) dx.$$

See, e.g., the first pages in Pisier [11] for proofs of these inequalities. When  $\lambda = 1/2$  and  $h = k$ , the analogy between Theorem 1.2 and the Prékopa–Leindler inequality is evident, see [4, Section 2.3] for a standard limiting argument that leads from Theorem 1.2 to (11). For  $\lambda \neq 1/2$ , a similar limiting argument leads to a weighted variant of the Prékopa–Leindler inequality due to Cordero-Erausquin and Maurey [3]:

**Theorem 4.1.** *Let  $\lambda \in [0, 1]$ . Suppose  $f, g, h, k : \mathbb{R}^n \rightarrow [0, \infty)$  are measurable functions satisfying*

$$f(x)g(y) \leq h(\lambda x + (1 - \lambda)y)k((1 - \lambda)x + \lambda y) \quad \forall x, y \in \mathbb{R}^n.$$

*Then*

$$\left( \int_{\mathbb{R}^n} f(x) dx \right) \left( \int_{\mathbb{R}^n} g(x) dx \right) \leq \left( \int_{\mathbb{R}^n} h(x) dx \right) \left( \int_{\mathbb{R}^n} k(x) dx \right).$$

Note that for  $\lambda = 1/2$  and  $h = k$ , Theorem 4.1 coincides with (11). We omit the details of the standard limiting argument leading from Theorem 1.2 to Theorem 4.1, as they are almost identical to the argument in [4, Section 2.3]. Another inequality in the spirit of Theorem 4.1 is the following limit case of Theorem 1.1. Again, the limiting argument is standard and it is omitted.

**Theorem 4.2.** *Suppose  $f, g, h, k : \mathbb{R}^n \rightarrow [0, \infty)$  are Borel functions satisfying*

$$(12) \quad f(x)g(y) \leq h(x \wedge y)k(x \vee y) \quad \forall x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$$

*where  $x \wedge y = (\min(x_1, y_1), \dots, \min(x_n, y_n))$ , and  $x \vee y = (\max(x_1, y_1), \dots, \max(x_n, y_n))$ .*

*Then*

$$\left( \int_{\mathbb{R}^n} f(x) dx \right) \left( \int_{\mathbb{R}^n} g(x) dx \right) \leq \left( \int_{\mathbb{R}^n} h(x) dx \right) \left( \int_{\mathbb{R}^n} k(x) dx \right).$$

Another possibility, is to replace the operations  $x \wedge y$  and  $x \vee y$  in (12) by the operations  $\min(x, y)$  and  $\max(x, y)$  with respect to the standard lexicographic order on  $\mathbb{R}^n$ . The conclusion of Theorem 4.2 holds true in this case as well, being a limiting case of Theorem 1.3, as the reader may verify.

**4.2. A discrete Brunn-Minkowski inequality I.** In Ollivier and Villani [10], a Brunn-Minkowski type inequality with curvature terms is proved on the discrete hypercube. A simplified version of their inequality, without curvature, states that for any sets  $A, B \subseteq \{0, 1\}^n$  one has

$$(13) \quad \#M \geq \sqrt{\#A \cdot \#B},$$

where  $M$  is the set of all midpoints of pairs  $(a, b)$  with  $a \in A$  and  $b \in B$ . In the terminology of [10], a point  $m = (m_1, \dots, m_n) \in \{0, 1\}^n$  is a midpoint of two points  $a, b \in \{0, 1\}^n$  if  $m_i = a_i$  whenever  $a_i = b_i$ , and

$$\#\{1 \leq j \leq n; m_j = a_j\} = \#\{1 \leq j \leq n; m_j = b_j\} + \varepsilon$$

with  $\varepsilon \in \{-1, 0, 1\}$ , i.e., essentially half of the remaining bits of  $m$  coincide with those of  $a$  and the other half with those of  $b$ .

We show that inequality (13) holds for a much smaller subset of midpoints. For example, let us use the operation  $T$  given in (2). Recall that  $T(a, b) = (T_1(a, b), \dots, T_n(a, b))$  is defined by:

$$T_i(a, b) = \begin{cases} a_i, & \#\{j \leq i : a_j \neq b_j\} \text{ is odd} \\ b_i, & \text{otherwise} \end{cases}$$

It is clear that  $T(a, b)$  is one of the midpoints of  $a$  and  $b$  in the sense of [10], as well as the point  $a + b - T(a, b)$ . Denote

$$M_1^- = \bigcup_{a \in A, b \in B} T(a, b), \quad M_1^+ = \bigcup_{a \in A, b \in B} (a + b - T(a, b)), \quad M_1 = M_1^- \cup M_1^+,$$

and let  $f = \mathbb{1}_A, g = \mathbb{1}_B, h = k = \mathbb{1}_{M_1}$ . Applying Theorem 1.3 with the above operation  $T$  we obtain

$$\sqrt{\#A \cdot \#B} \leq \sqrt{\#M_1^- \cdot \#M_1^+} \leq \#M_1.$$

This inequality implies (13) since  $M_1 \subset M$ . Our inequality is quite flexible, as there is nothing canonical about the specific definition (2) of the map  $T$ , and moreover the analysis applies for subsets of  $\mathbb{Z}^n$  and not only for subsets of  $\{0, 1\}^n$ .

**4.3. A discrete Brunn-Minkowski inequality II.** Recently, the following inequality was proven by Iglesias, Yepes Nicolás and Zvavitch [7]:

**Theorem 4.3.** *Let  $\lambda \in [0, 1]$ . For any two bounded non-empty sets  $K, L \subseteq \mathbb{R}^n$ , we have*

$$G_n(\lambda K + (1 - \lambda)L + (-1, 1)^n)^{1/n} \geq \lambda G_n(K)^{1/n} + (1 - \lambda)G_n(L)^{1/n},$$

where  $G_n(M)$  denotes the number of lattice points in  $M \subseteq \mathbb{R}^n$ .

We recover a multiplicative version of Theorem 4.3 for  $\lambda = 1/2$ : Let  $f(x) = \mathbb{1}_K(x)$ ,  $g(x) = \mathbb{1}_L(x)$  be the indicator functions of  $K$  and  $L$ , and  $h = \mathbb{1}_{\frac{1}{2}(K+L)+(-1,0]^n}$ ,  $k = \mathbb{1}_{\frac{1}{2}(K+L)+[0,1)^n}$ . Note that for every  $x \in K$  and  $y \in L$ , we have

$$\left\lfloor \frac{x+y}{2} \right\rfloor \in \frac{K+L}{2} - [0, 1)^n \text{ and } \left\lceil \frac{x+y}{2} \right\rceil \in \frac{K+L}{2} + [0, 1)^n,$$

which implies that  $f(x)g(y) \leq h(\lfloor \frac{x+y}{2} \rfloor)k(\lceil \frac{x+y}{2} \rceil)$  for all  $x, y \in \mathbb{Z}^n$ . By Theorem 1.2, we have

$$\sqrt{G_n(K)G_n(L)} \leq \sqrt{G_n\left(\frac{K+L}{2} + [0, 1)^n\right)G_n\left(\frac{K+L}{2} + (-1, 0]^n\right)} \leq G_n\left(\frac{K+L}{2} + (-1, 1)^n\right),$$

as follows also from Theorem 4.3 via the arithmetic/geometric means inequality.

*Remark 4.4.* We conclude this paper with a little remark on the case of a finite abelian group, where  $G = (\mathbb{Z}/p_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_n\mathbb{Z})$ . There are no additive, complete orderings on such groups. Therefore, in order to apply Theorem 1.3 for four functions  $f, g, h, k : G \rightarrow [0, \infty)$ , one option is to define

$$\tilde{f}(x) = \begin{cases} f(\pi(x)) & 0 \leq x_1 < p_1, \dots, 0 \leq x_n < p_n \\ 0 & \text{otherwise} \end{cases}$$

where  $\pi : \mathbb{Z}^n \rightarrow (\mathbb{Z}/p_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_n\mathbb{Z}) = G$  is the projection map, and similarly to define  $\tilde{g}, \tilde{h}, \tilde{k}$ . In the case where the four functions  $\tilde{f}, \tilde{g}, \tilde{h}, \tilde{k} : \mathbb{Z}^n \rightarrow [0, \infty)$  satisfy the assumptions of Theorem 1.3, we obtain the inequality

$$\left(\sum_{x \in G} \tilde{f}(x)\right) \left(\sum_{x \in G} \tilde{g}(x)\right) \leq \left(\sum_{x \in G} \tilde{h}(x)\right) \left(\sum_{x \in G} \tilde{k}(x)\right).$$

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