<u>Last week:</u> S^m is $\boxed{m \in \mathbb{R}}$ 7/6/20

the class of symbol $a(x,\xi)$ $x,\xi \in \mathbb{R}^n$

s.t. $\forall \alpha, \beta$

$$\frac{|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|}{(1+|\xi|)^{m-|\beta|}} \text{ is bdd.}$$

- Main property: $|a(x,\xi)| \lesssim (1+|\xi|)^m$

- S^m is a Frechet space.

<u>Def:</u> We say that a sequence $a_1, a_2, \ldots$ belongs to S^m <u>with uniform estimates</u> if $\forall \alpha, \beta$ $\exists A_{\alpha,\beta}$ s.t.

$$|\partial_x^\alpha \partial_\xi^\beta a_k(x,\xi)| \leq A_{\alpha,\beta} (1+|\xi|)^{m-|\beta|}$$

for all x, ξ and k.

· We associate an operator to $a \in S^m$,

$$a(x,D)\varphi = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot x}\, \hat{\varphi}(\xi)\, d\xi$$

for $\varphi \in \mathcal{S}$

· $a(x,D) = Op[a] = T_a$, different notation

ΨDO of order $\leq m$.

· We proved, for $m=0$ that

$$a(x,D) : L^2 \to L^2 \quad \text{bdd}$$

$$H^k(\mathbb{R}^n) \to H^k(\mathbb{R}^n) \quad \text{bdd.}$$

We promised also in Hölder norm (next week...). Proved $\forall a \in S^\infty$, $Op[a]$ is cont. in $\mathcal{S}$.

<u>Todays</u> Relation between S^m for different m.

· <u>Properties</u>:

1) $S^{m-1} \subseteq S^m$ $\quad S_m \nearrow$

$$S^{\infty} = \bigcup_{m \in \mathbb{Z}} S^m = \text{all symbols whatsoever}$$

$$S^{-\infty} = \bigcap_{m \in \mathbb{Z}} S^m = \text{infinitely smoothing } \Psi DOs.$$

$\bigcup\!|$

compactly-supported (in ξ) symbols

2) $a \in S^m$, $b \in S^l \Rightarrow ab \in S^{m+l}$

as by the Leibnitz rule

$$\left|\partial_x^{\alpha} \partial_\xi^{\beta} (ab)\right| = \left| \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \underbrace{\partial_\xi^{\gamma} a}_{x\text{-deriv}} \; \underbrace{\partial_\xi^{\beta-\gamma} b}_{x\text{-deriv}} \right|$$

$$\leq C \sum_{\gamma} (1+|\xi|)^{m-|\gamma|} (1+|\xi|)^{l-|\beta|+|\gamma|}$$

$$\lesssim (1+|\xi|)^{m+l-|\beta|}$$

3) $\left(1+|\xi|^2\right)^{m/2} \in S^m \quad \forall m \in \mathbb{Z}$

"our usual representative from S^m".

$\partial_{\xi_i} \rightsquigarrow$ gives a factor $\dfrac{\xi_i}{1+|\xi|^2}$

Hence $\left| \partial_\xi^\beta \left(1+|\xi|^2\right)^{\frac{m}{2}} \right| \lesssim \left(1+|\xi|\right)^{m-|\beta|}$

(local operator only if $m \geq 0$ is even).

Hence

$$\boxed{a \in S^m \iff \left(1+|\xi|^2\right)^{-m/2} a \in S^0}$$

4) If $a \in S^m$ then

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi) \in S^{m-|\beta|}$$

5) Suppose $a \in S^m$ and $\gamma \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with $\gamma(0,0)=1$, then

$$a_\varepsilon(x,\xi) = a(x,\xi)\,\gamma(\varepsilon x, \varepsilon\xi)$$

is a compactly-supported symbol, and it is in S^m with estimates uniform in ε.

$$\lim_{\varepsilon\to 0^+} a_\varepsilon(x,\xi) = a(x,\xi) \quad \text{pointwise}$$

and $\forall \varphi \in \mathcal{S}$, $x \in \mathbb{R}^n$,

$$T_{a_\varepsilon}\varphi(x) \xrightarrow{\varepsilon\to 0} T_a\varphi(x)$$

(proved last time).

. Note $(1+|\xi|^2)^k \varphi \in \mathcal{S}$ $\forall^{k\in\mathbb{Z}}_{\varphi\in\mathcal{S}}$.

Thm: If $a \in S^m$ then

$$Op[a] : H^k \longrightarrow H^{k-m} \quad \text{is bdd}$$

$\forall k \geq \max\{m, 0\}$.

· Important for $m < 0$.

If $a \in S^{-([\frac{n}{2}]+k+1)}$ then

$$\mathrm{Op}[a] : \underset{\underset{H^0}{\shortparallel}}{L^2} \longrightarrow H^{[\frac{n}{2}]+k+1} \subseteq C^{k,}$$

So if $a \in S^{-\infty}$ then

$$\mathrm{Op}[a](L^2) \subseteq C^\infty.$$

"infinitely smoothing"

<u>Proof:</u> For $m = 0$ – proved last time.

$$(1+|\xi|^2)^{-m/2} a(x,\xi) \in S^0$$

Hence the corresponding operator is bdd in H^{k-m}. I.e. $\forall \varphi \in S$

$$\left\| x \mapsto \int_{\mathbb{R}^n} a(x,\xi) e^{i\xi\cdot x} \underbrace{(1+|\xi|^2)^{-m/2} \hat{\varphi}(\xi)}_{S \ni \hat{g}(\xi)} d\xi \right\|^2_{H^{k-m}}$$

$$\lesssim \|\varphi\|^2_{H^{k-m}} = \int \underbrace{(1+|\xi|^2)^{k-m} |\hat{\varphi}(\xi)|^2}_{(1+|\xi|^2)^{k} |\hat{g}(\xi)|^2} d\xi$$

$$\| g \|^2_{H^k}$$

Hence we got $\forall g \in \mathcal{S}$,

$$\left\| x \mapsto \int_{\mathbb{R}^n} a(x,\xi) e^{i\xi\cdot x} \hat{g}(\xi)\, d\xi \right\|_{H^{k-m}} \lesssim \|g\|_{H^k}.$$

$\blacksquare$

- A ΨDO of order $-m$ "gains" m derivatives in L^2-sense

Hölder - next week.

ΨDO symbol calculus and oscillatory integrals

Composing ΨDOs and conjugation.

Thm 1: For $a \in S^m$, $b \in S^l$ $\quad m, l \in \mathbb{R}$

$$\mathrm{Op}[a] \circ \mathrm{Op}[b] = \mathrm{Op}[c]$$

for $c \in S^{m+l}$, denoted by

$c = a \# b$.

. What is the product rule?

Example: multipliers, $a(x,\eta) = a(\eta)$

just multiply the symbols

$$\mathrm{Op}[a] \circ \mathrm{Op}[b] = \mathrm{Op}[ab]$$

commutative, $\boxed{\text{if}}$ $a(x,\xi) = a(\xi)$

In general, for large ξ

$$a \# b\,(x,\xi) \approx a(x,\xi)\, b(x,\xi)$$

Thm 1': For $a \in S^m$, $b \in S^l$,

$$a \# b\,(x,\xi) = a(x,\xi)\, b(x,\xi) + E$$

$$\uparrow$$

$$S^{m+l-1}$$

E is one-degree smoother,

or decays one order faster in ξ.

In fact, there is <u>asymptotic expansion:</u>
$\forall N \geq 0$

$$a \# b(x,\xi) = \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a \, \partial_x^\alpha b + \underset{\underset{S^{m+l-N}}{\uparrow}}{E}$$

· A formula for the composition is, for instance,

$$a \# b(x,\xi) = \lim_{\varepsilon \to 0} (2\pi)^{-n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\langle x-y, \eta-\xi\rangle} a_\varepsilon(x,\eta) \, b_\varepsilon(y,\xi) \, dy \, d\eta$$

where $a_\varepsilon(x,\xi) = a(x,\xi)\,\gamma(\varepsilon x, \varepsilon\xi)$
and $\gamma \in C_c^\infty(\mathbb{R}^n)$, $\gamma(0) = 1$. Any.

<u>Remark:</u> We will actually prove that $E \in S^{m+l-N}$ with estimates that depend only on the estimates of a and b (and on N, m, l, n).

$$a \# b = \sum_{|\alpha|<N} \frac{(-i)^\alpha}{\alpha!} \underbrace{\partial_\xi^\alpha a \, \partial_x^\alpha b}_{S^{m+l-|\alpha|}} + \underbrace{E}_{S^{m+l-N}}$$

<u>Thm 2</u>: Conjugation: $a \in S^m$,

$$\langle Op[a]^* \varphi, \psi \rangle := \langle \varphi, Op[a]\psi \rangle.$$

Then $Op[a]^* = Op[a^*]$ where $a^* \in S^m$, with

$$a^*(x,\xi) = \lim_{\varepsilon \to 0^+} (2\pi)^{-n} \int e^{i\langle x-y, \eta-\xi\rangle} \overline{a_\varepsilon(y,\eta)} \, dy \, d\eta$$

$\forall N$
$$= \sum_{|\alpha|<N} \frac{(-i)^\alpha}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a(x,\xi)} + \underbrace{E}_{S^{m-N}}$$

Brief remark about oscillatory integral and stationary phase:

$$\int e^{iN\phi(x)} \psi(x) \, dx$$

· Most contribution comes from around of point with $\nabla\phi(x) = 0$.

Because if $\nabla\phi(x) \neq 0$ can integrate by parts

$$\frac{\partial_j e^{in\phi(x)}}{\underbrace{in\,\partial_j\phi(x)}_{\neq 0}} = \underbrace{e^{in\phi(x)}}$$

<u>Remark:</u> 0^{th} order $a\# b(x,\xi) \approx a(x,\xi)\, b(x,\xi) + E$

<u>1^{st} order:</u>

$$\left[\, Op[a],\ Op[b] \,\right] = -i\, Op\left[\{a,b\}\right] + E$$

$$\{a,b\} = \sum_{j=1}^{n}\left[\frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j} - \frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j}\right]$$

$= w(X_a, X_b)$ when $w(X_a, \cdot) = da$

From now on we prove the composition rule and asymptotic expansion.

<u>Step 1:</u> Suppose $a \in S^m$, $b \in S^\ell$ are compactly-supported (in x and ξ).

Then $a(x,D)$ is an integral operator

$$(T_a \varphi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi) e^{ix\cdot\xi} \hat{\varphi}(\xi) d\xi$$

$$= \int_{\mathbb{R}^n} \underbrace{(2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi) e^{i\xi\cdot(x-y)} d\xi}_{K_a(x,y)} \varphi(y) dy$$

Therefore $T_a \circ T_b$ is again an integral operator with kernel

$$[K](x,y) = \int_{\mathbb{R}^n} K_a(x,z) K_b(z,y) dz$$

$$= (4\pi^2)^{-n} \int a(x,\eta)\, e^{i\eta\cdot(x-t)}\, b(t,\zeta)\, e^{i\zeta\cdot(t-y)}\, d\zeta\, d\eta\, dt$$

$$= (2\pi)^{-n} \int \underset{\substack{\uparrow \\ \text{we want}}}{c(x,\zeta)}\; e^{i\zeta\cdot(x-y)}\, d\zeta$$

This would imply that $T_a \circ T_b = T_c$

where

$$c(x,\zeta) = (2\pi)^{-n} \int a(x,\eta)\, b(t,\zeta)\, e^{A}\, d\eta\, dt$$

$$\begin{aligned} A &= i\eta\cdot(x-t) + i\zeta\cdot(t-y) - i\zeta\cdot(x-y) \\ &= i\eta\cdot(x-t) + i\zeta\cdot(t-x) \\ &= i(\eta-\zeta)\cdot(x-t) \end{aligned}$$

So

$$c(x,\zeta) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot(\eta-\zeta)}\, a(x,\eta)\, b(y,\zeta)\, dy\, d\eta$$

is smooth and compactly-supported. Today,

$$\hat{b}(\eta,\zeta) = \int_{\mathbb{R}^n} e^{-i\eta\cdot y}\, b(y,\zeta)\, dy$$

Fourier in the first variable. So,

$$c(x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot(\eta-\xi)} a(x,\eta)\, \hat{b}(\eta-\xi,\xi)\, d\eta$$

Next, what to do if a, b are not compactly supported?

Fix $\gamma \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with $\gamma(0)=1$

$$a_\varepsilon(x,\xi) = a(x,\xi)\,\gamma(\varepsilon x, \varepsilon\eta) \in S^m$$
$$b_\varepsilon(x,\xi) = b(x,\xi)\,\gamma(\varepsilon x, \varepsilon\eta) \in S^l$$

with uniform estimates in ε.

Define

$$c_\varepsilon = a_\varepsilon \# b_\varepsilon(x,\xi)$$
$$= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot(\eta-\xi)} a_\varepsilon(x,\eta)\, b_\varepsilon(y,\xi)\, d\eta\, dy$$

We will prove that $c_\varepsilon \in S^{m+l}$, with the asymptotic expansion with estimates uniform in ε.

<u>Justification of the limit $\varepsilon \to 0$:</u> Guided exercise: c_ε converges pointwise (up to subsequence) to some symbol $c \in S^{m+l}$, with the same asymptotic expansion, and

$$Op[c] = Op[a] \circ Op[b]$$

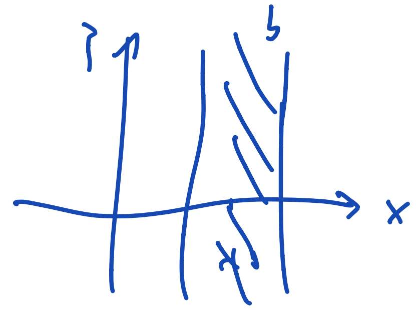
as operators on $\mathcal{S}$.

- Now we will prove the asymptotic expansion, uniformly in ε.

<u>Step 2:</u> Assume $a \in S^m$, $b \in S^l$, with

$$b(x,\xi) = 0 \quad \text{if} \quad |x - x_0| > 2$$

for some fixed $x_0 \in \mathbb{R}^n$



Thm $\forall \alpha, \beta$

$$\left| x^\alpha \partial_x^\beta \, b_\varepsilon(x,\xi) \right| \lesssim (1+|\xi|)^\ell$$

with implicit constants depending on α, β, a, b.

In other words, in terms of the norms $|\cdot|_N$ on $\mathcal{S}'$ (in the x-variable!),

$\forall N$ $\quad \left| b_\varepsilon(\cdot,\xi) \right|_N \lesssim (1+|\xi|)^\ell$

Recall $|\hat{\varphi}|_N \lesssim |\varphi|_{N+2n}$. Hence

$\forall N$,

$$\left| \hat{b}_\varepsilon(\cdot,\xi) \right|_N \lesssim (1+|\xi|)^\ell$$

i.e. $\forall \bar{N}$

$$\boxed{\left| \hat{b}_\varepsilon(\eta,\xi) \right| \leq \frac{C_{\bar{N}}}{(1+|\eta|)^{\bar{N}}} \cdot (1+|\xi|)^\ell}$$

where $\bar{C}_N$ depends only on a and b, in addition to $\bar{N}$.

· I stop writing subscript ε, for laziness.

$$c_\varepsilon(x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} a_\varepsilon(x, \xi+\eta)\, e^{ix\cdot\eta}\, \hat{b}_\varepsilon(\eta,\xi)\, d\eta$$

$\downarrow$ Taylor series in η

$$c_\varepsilon(x,\xi) = (2\pi)^{-n} \int \left[\sum_{|\alpha|<N} \frac{\partial_\xi^\alpha a(x,\xi)}{\alpha!} \eta^\alpha + R_N \right] \cdot e^{ix\cdot\eta}\, \hat{b}(\eta,\xi)\, d\eta$$

where $R_N = R_N(x,\xi,\eta)$, remainder in Taylor.

Each summand in α gives us a term in the asymptotic expansion:

$$(2\pi)^{-n} \int \frac{\partial_\xi^\alpha a(x,\xi)}{\alpha!} \eta^\alpha e^{ix\cdot\eta}\, \hat{b}(\eta,\xi)\, d\eta$$

$$= \frac{\partial_\xi^\alpha a(x,\xi)}{\alpha!} \cdot (2\pi)^{-n} \int e^{ix\cdot\eta} \widehat{D_x^\alpha b}(\eta,\xi)\,d\eta$$

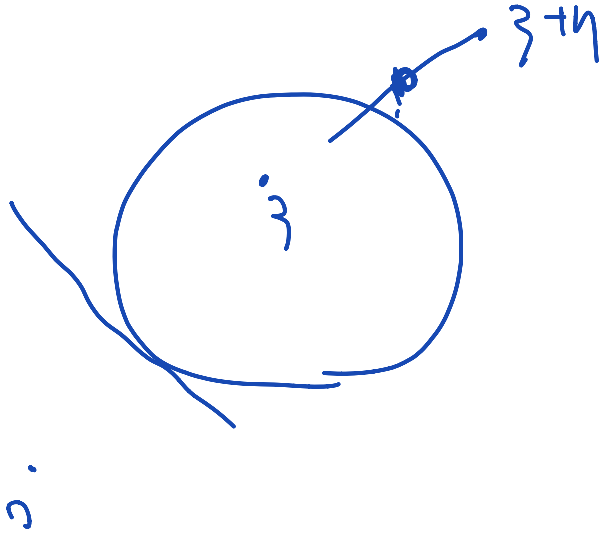
$$= (-i)^{|\alpha|} \cdot \frac{\partial_\xi^\alpha a(x,\xi)}{\alpha!} \underbrace{\partial_x^\alpha b(x,\xi)}_{D_x^\alpha}.$$

We need to bound the remainder in Taylor's formula. Since $\left|\partial_\xi^\alpha a(x,\xi)\right| \le (1+|\xi|)^{m-|\alpha|}$

then

$$\left|R_N(x,\xi,\eta)\right| \lesssim \begin{cases} |\eta|^N \cdot (1+|\xi|)^{m-N} & |\eta| < \frac{|\xi|}{2} \\ |\eta|^N & \eta \in \mathbb{R}^n \end{cases}$$

$N \ge m$



Enough to prove the asymptotic expansion for $N > m$.

Assume

In order to complete the proof of asymp. expansion (in Step 2), we need to look at

$$E(x,\xi) = c(x,\xi) - \sum_{|\alpha|<N} \frac{(-i)^{\alpha}}{\alpha!} \partial_\xi^\alpha a \, \partial_x^\alpha b$$

and prove that $E \in S^{l+m-N}$ with uniform estimates

Recall: $\forall \tilde{N}$

$$\left| \hat{b}(\eta,\xi) \right| \leq \frac{C_{\tilde{N}}}{(1+|\eta|)^{\tilde{N}}} \cdot (1+|\xi|)^{l}$$

and

$$\left| E(x,\xi) \right| = (2\pi)^{-n} \left| \int_{\mathbb{R}^n} R_N(x,\xi,\eta) e^{i\eta\cdot x} \hat{b}(\eta,\xi) \, d\eta \right|$$

$$\lesssim \int_{|\eta|<\frac{|\xi|}{2}} + \int_{|\eta|\geq\frac{|\xi|}{2}}$$

$\forall \tilde{N}$

$$\lesssim \int_{|\eta| < \frac{|\xi|}{2}} |\eta|^N \cdot (1+|\xi|)^{m-N+l} \frac{1}{(1+|\eta|)^{\bar{N}}} \, d\eta \qquad \bar{N} = N + 2n$$

$$+ \int_{|\eta| \geq \frac{|\xi|}{2}} |\eta|^N (1+|\xi|)^{l} \cdot \frac{1}{(1+|\eta|)^{\bar{N}}} \, d\eta \qquad \bar{N} \geq N + 2n$$

$$\lesssim (1+|\xi|)^{m+l-N} \cdot \int_{\mathbb{R}^n} \frac{d\eta}{(1+|\eta|)^{\underbrace{\bar{N}-N}_{\geq n+1}}}$$

$$+ (1+|\xi|)^{l} \cdot \left(\frac{1}{1+|\xi|}\right)^{\bar{N} - (N+2n)}$$

$$\lesssim (1+|\xi|)^{m+l-N}$$

This proves the $\alpha = \beta = 0$ inequality from the desired " $E \in S^{m+l-N}$ "

· Next, pick multiindices α, β, and need

$$|\partial_x^\alpha \partial_\xi^\beta E(x,\xi)| \lesssim (1+|\xi|)^{m+l-N-|\beta|}$$

Reduce matters to the case $\alpha=\beta=0$:

- We know: $\partial_\xi^\beta b \in S^{l-|\beta|}$, still supported in $|x-x_0|<2$, hence $\forall \tilde{N}$

$$\left|\widehat{\partial_\xi^\beta b}(\eta,\xi)\right| \lesssim \frac{C_{\tilde{N}}}{(1+|\eta|)^{\tilde{N}}} \cdot (1+|\xi|)^{l-|\beta|}$$

$$\widehat{\partial_\xi^\beta b}(\eta,\xi) = \partial_\xi^\beta \hat{b}(\eta,\xi)$$

- $R_N(x,\xi,\eta)$ was the remainder in Taylor formula for $a(x,\xi+\eta)$

$$a(x,\xi+\eta) = \sum_{|\gamma|<N} \frac{\partial_\xi^\gamma a(x,\xi)}{\gamma!}\eta^\gamma + R_N(x,\xi,\eta)$$

Therefore

$$\partial_x^\alpha \partial_\xi^\beta R_N(x,\xi,\eta) \text{ is the remainder in Taylor formula for } \underbrace{\partial_x^\alpha \partial_\xi^\beta a(x,\xi+\eta)}_{\in S^{m-|\beta|}}$$

Hence, similar estimate

$$|\partial_x^\alpha \partial_\xi^\beta R_N(x,\xi,\eta)| \leq \begin{cases} |\eta|^N \cdot (1+|\xi|)^{m-N-|\beta|} & |\eta| \leq \frac{|\xi|}{2} \\ |\eta|^N & \eta \in \mathbb{R}^n \end{cases}$$

$N > m$

So if we differentiate

$$\partial_x^\alpha \partial_\xi^\beta E(x,\xi) = \partial_x^\alpha \partial_\xi^\beta (2\pi)^{-n} \int R_N(x,\xi,\eta)\, e^{i\eta\cdot x}\, \hat{b}(\eta,\xi)\, d\eta$$

we get a constant number of summands, each bdd by $(1+|\xi|)^{l+m-N-|\beta|}$.

The main contribution will be

$$\int_{|\eta|<\frac{|\xi|}{2}} \frac{(1+|\xi|)^{m-N-|\gamma|}}{|\eta|^N} \cdot \frac{(1+|\xi|)^{l-|\beta-\gamma|}}{(1+|\eta|)^{\tilde{N}}}\, d\eta$$

from R_N

$$\lesssim (1+|\xi|)^{l+m-N-|\beta|}.$$

To summarize: We proved that $a\#b \in S^{m+l}$, with the asymptotic expansion, in the case $b(x,\xi)=0$ for $|x-x_0|>2$.

<u>Step 3:</u> For general $a\in S^m$, $b\in S^l$. Fix $x_0\in\mathbb{R}^n$.

decompose

$$b = \underbrace{\eta b}_{=b_1} + \underbrace{(1-\eta) b}_{=b_2}$$

where $\eta(x) = \begin{cases} 1 & \text{in } |x-x_0|<1 \\ 0 & \text{in } |x-x_0|>2 \end{cases}$

$\eta:\mathbb{R}^n\to[0,1]$.

Now, $b_1\in S^l$, satisfies the assumption of step 2. Therefore

$$T_a\circ T_{b_1} = T_{c_1} \qquad c_1\in S^{m+l}$$

asymptotic expansion.

What remains is to look at

$$c' = a \# b_2$$

Need to show that $\forall N$

$$\left| \partial_x^\alpha \partial_\xi^\beta c'(x_0, \xi) \right| \lesssim \frac{C_N}{(1+|\xi|)^N}$$

with estimates uniform in x_0 and ε.

$b_2(y,\xi) = 0$ for $|y - x_0| < 1$.

$$c'(x_0,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x_0-y)\cdot(\eta-\xi)} a(x_0,\eta)\, b_2(y,\xi)\, dy\, d\eta$$

$$\Delta_\eta e^{i(x_0-y)\cdot(\xi-\eta)} = -|x_0-y|^2 e^{i(x_0-y)\cdot(\xi-\eta)}$$

Integrate by parts in η

Hence, $\forall N$

$$c'(x,\xi) = \pm(2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\Delta_\eta^N e^{i(x_0-y)\cdot(\xi-\eta)}}{|x_0-y|^N} a(x_0,\eta)\, b_2(y,\xi)\, dy\, d\eta$$

$$= \pm(2\pi)^{N} \int_{\mathbb{R}^n} \frac{\Delta_\eta^N a(x_0,\eta)}{|x_0-y|^N} b_2(x,\xi)\, e^{i(x_0-y_0)\cdot(\xi-\eta)}\, d\eta\, dy$$

$$(I-\Delta_y)\, e^{i(x_0-y)\cdot(\xi-\eta)} =$$

$$e^{i(x_0-y)(\xi-\eta)}\left(1+|\xi-\eta|^2\right)$$

$$c'(x,\xi) = \pm(2\pi)^n \int_{\mathbb{R}^n} \frac{\Delta_\eta^{N_1} a(x_0,\eta)(1-\Delta_y)^{N_2} b_\gamma(y,\xi)}{|x_0-y|^{N_1}\cdot\left[(1+|\xi-\eta|)\right]^{N_2}} \cdot e^{i(x_0-y_0)\cdot(\xi-\eta)}\, d\eta\, dy$$

gain from diff.

no gain

$$|c'(x,\xi)| \lesssim \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(1+|\eta|)^{m-N_1}\,(1+|\xi|)^{\ell}}{|x_0-y|^{N_1}\left[(1+|\xi-\eta|)\right]^{N_2}}\, d\eta\, dy$$

$\mathbb{1}_{|y-x_0|\geq 1}$

$N_1 > 2n$

$|y-x_0|\geq 1$

Integrate w.r.t. y

$$\lesssim (1+|\xi|)^{\ell}\int_{\mathbb{R}^n} \frac{1}{(1+|\eta|)^{N_1-m}\,(1+|\xi-\eta|)^{N_2}}\, d\eta$$

$$(1+|\eta|)(1+|\xi-\eta|) \geq 1+|\xi|$$

$$\boxed{N_1 = N_2 = N+\ell+m+2n}$$

$$\lesssim \frac{(1+|\xi|)^{l}}{(1+|\xi|)^{N+l}} \int_{\mathbb{R}^n} \frac{1}{(1+|\eta|)^{rn}}$$

$$\lesssim \frac{1}{(1+|\xi|)^{N}}$$

This finishes the case $\alpha=\beta=0$ in the estimate for $c'(x_0,\xi)$.

This the same, as $\partial_x^\alpha \partial_\xi^\beta c'(x,\xi)\big|_{x=x_0}$ is again a sum of finitely many

$$\int e^{i(x-y)\cdot(\eta-\xi)} \left[\begin{array}{l}\text{derivative and}\\ \text{products by}\\ \text{monomials of}\\ a(x,\eta)\, b_2(y,\xi)\end{array}\right] dy\, d\xi$$

which is still supported on $|x_0-y|\geq 1$.

Same analysis applies and shows that

$$\forall r \ \left| \partial_x^\alpha \partial_\xi^\beta c^r(x,\xi)\big|_{x=x_0} \right| \leq \frac{C_{N,\alpha,\beta}}{(1+|\xi|)^N}.$$

$\blacksquare$

$a = a_\varepsilon$, $b = b_\varepsilon$ fulfilling.