4/5/2020

Last week $f \in L^1(\mathbb{R}^n)$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} f(x)\,dx$$

- $\|\hat{f}\|_\infty \le \|f\|_1 < \infty$
- $\hat{f}$ continuous, $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0.$
- Define $\mathcal{S}$, the class of Schwartz functions, i.e., for a smooth f,

$$f \in \mathcal{S} \iff \forall \alpha, \beta \quad x^\beta \partial^\alpha f \text{ is bounded.}$$

- $f \in \mathcal{S}$ decay at ∞ with their derivatives.
- $C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S} \subseteq L^1(\mathbb{R}^n)$
- If $f \in \mathcal{S}$, $x^\alpha f$, $\partial^\alpha f \in \mathcal{S}$

FT on $\mathcal{S}$:

$$\widehat{x^\alpha f}(\xi) = i^{|\alpha|}\left(\partial^\alpha \hat{f}\right)(\xi)$$

$$\widehat{\partial^\alpha f}(\xi) = i^{|\alpha|}\, \xi^\alpha \hat{f}(\xi)$$

Hence $\forall f \in \mathcal{S}$

$\forall \alpha, \beta$

$$\xi^\beta \partial^\alpha \hat{f} = c\, \widehat{\partial^\beta (x^\alpha f)}$$

hence bounded $\qquad\qquad \mathcal{S} \in L^1$

- Thus $\widehat{\mathcal{S}} \subseteq \mathcal{S}$.
- To show that $\widehat{\mathcal{S}} = \mathcal{S}$ we begin to prove the <u>Fourier inversion formula</u>:

$\forall f \in \mathcal{S}$

$$\hat{\hat{f}} = (2\pi)^n \cdot \check{f}$$

where $\check{f}(x) = f(-x)$.

Why is this formula correct, roughly?

$$\hat{\hat{f}}(-x) = \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} \hat{f}(\xi)\, d\xi$$

$$= \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} \left(\int_{\mathbb{R}^n} f(y) e^{-i\langle \xi, y\rangle} dy \right) d\xi$$

$$\underset{\text{problematic Fubini}}{``="} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{i\langle \xi, x-y\rangle} d\xi \right) f(y)\, dy$$

We hope that in som sense

$$``\int_{\mathbb{R}^n} e^{i\langle \xi, x-y\rangle} d\xi = (2\pi)^n \delta_{x=y}"$$

Two characteristic of a rigorous proof:

1) <u>Fubini</u>: When $\int |f(x,y)|\, d\mu(x)\, d\nu(y) < \infty$,

$$\int \left(\int f(x,y)\, d\mu(x) \right) d\nu(x)$$

$$= \int \left(\int f(x,y)\, d\nu(y) \right) d\mu(x).$$

2) Dominated convergence: If $f_m \to f$ μ-a.e. and $\forall n$ $|f_m| \le \psi$, $\int \psi\, d\mu$, then

$$\int f_m\, d\mu \xrightarrow{m\to\infty} \int f\, d\mu$$

<u>Proof of Fourier inversion in S</u>: Let $f \in S$, $x \in \mathbb{R}^n$. Then

$$\hat{\hat{f}}(-x) = \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} \hat{f}(\xi)\, d\xi$$

$$\boxed{\begin{array}{l} \cdot\ f, \hat{f} \in L^1 \\ \cdot\ f \text{ is continuous} \end{array}}$$

$$= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} \hat{f}(\xi) e^{-\frac{\varepsilon|\xi|^2}{2}}\, d\xi$$

$$= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \Big(\underbrace{\int_{\mathbb{R}^n} e^{i\langle x-y, \xi\rangle} e^{-\frac{\varepsilon|\xi|^2}{2}}\, d\xi}_{\left(\frac{1}{\sqrt{2\pi/\varepsilon}}\right)^n} \Big) f(y)\, dy$$

$$= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} \frac{(2\pi)^{n/2}}{\varepsilon^{n/2}}\, e^{-\frac{|x-y|^2}{2\varepsilon}} f(y)\,dy$$

$$=(2\pi)^n \cdot \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \underbrace{\frac{1}{(2\pi\varepsilon)^{n/2}}\, e^{-\frac{|x-y|^2}{2\varepsilon}}}_{\text{Gaussian density}} f(y)\,dy$$

$$\boxed{=(2\pi)^n \lim_{\varepsilon\to 0^+} \mathbb{E}\, f(x+\sqrt{\varepsilon}\, Z)} = f(x)$$

where Z is a standard Gaussian in $\mathbb{R}^n$,

density $(2\pi)^{-n/2} e^{-|x|^2/2}$

$Z=(Z_1,\dots,Z_n)$ i.i.d 1D Gaussians

$\mathbb{E} Z = 0$, $\mathbb{E}|Z|^2$

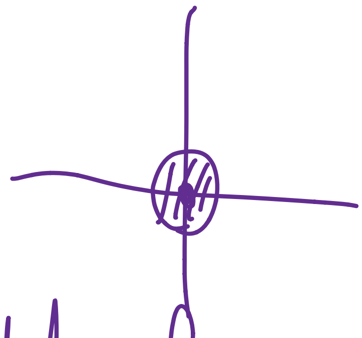
$$= \sum_{j=1}^n \mathbb{E} Z_j^2 = n.$$

$= f(x)$, thanks to the following

Proposition ("Approximate Unity")

Suppose that Z_δ is a random vector in $\mathbb{R}^n$ $\forall \delta > 0$, such that

$$\lim_{\delta \to 0} \mathbb{E}|Z_\delta|^2 = 0$$



Then for any continuous, bounded f,

$$f(0) = \lim_{\delta \to 0} \mathbb{E} f(Z_\delta).$$

Proof: May assume $\|f\|_\infty = 1$. Fix $\varepsilon_0 > 0$, and show for a sufficiently small $\delta > 0$

$$(*) \quad \mathbb{E}\left|f(Z_\delta) - f(0)\right| < \varepsilon_0$$

Since f is cont. at 0, $\exists \delta_0 > 0$ s.t. $\forall |x| < \delta_0$, $|f(x) - f(0)| < \frac{\varepsilon_0}{2}$

Hence,

$$\mathbb{E}\left|f(z_\delta) - f(0)\right| =$$

$$= \mathbb{E}\left|f(z_\delta) - f(0)\right| \mathbb{1}_{|z_\delta| < \delta_0}$$

$$+ \mathbb{E}\left|f(z_\delta) - f(0)\right| \mathbb{1}_{|z_\delta| > \delta_0}$$

$$\leq \frac{\varepsilon_0}{2} + 2\,\mathbb{P}\left(|z_\delta| \geq \delta_0\right)$$

$$\leq \frac{\varepsilon_0}{2} + 2\,\frac{\mathbb{E}\,|z_\delta|^2}{\delta_0^2} \xrightarrow{\delta \to 0} \frac{\varepsilon_0}{2}$$

So for a sufficiently small δ, we obtain (*). $\blacksquare$

<u>Remark:</u> The works if $f, \hat{f} \in L^1$, and continuous.

<u>Exercise:</u> If $f, \hat{f} \in L^1$ then f can be modified on a set of measure zero

to become continuous.

<u>Example:</u> $\widehat{\mathbb{1}_{[-1,1]}}(\xi) = 2\dfrac{\sin \xi}{\xi}$

in 1D, $\displaystyle\int_{-\infty}^{\infty} \left|\frac{\sin \xi}{\xi}\right| d\xi = +\infty$

<u>Solution/Hint:</u> Lebesgue's density thm:

for almost any $x \in \mathbb{R}^n$, $f \in L^1(\mathbb{R}^n)$,

$$\frac{1}{\mathcal{H}^n(B(x,\varepsilon))}\int_{B(x,\varepsilon)} |f(y) - f(x)|\, dy \xrightarrow{\varepsilon \to 0^+} 0$$

From this, $\hat{\hat{f}} = (2\pi)^n \check{f}$ a.e.

automatically continuous (and decays at ∞).

Later: F.T. in $L^1(\mathbb{R}^n)$ in sense of distribution. $\hat{f} = (2\pi)^n \check{f}$

<u>Corollary:</u> if $f \in L^1$ and $\hat{f} \equiv 0$, then $f \equiv 0$.

<u>Corollary (important)</u>

(Plancherel / Parseval)

$\forall f, g \in \mathcal{S}$,

$$\langle \hat{f}, \hat{g} \rangle = (2\pi)^n \langle f, g \rangle$$

$$\|\hat{f}\|_2^2 = \left((2\pi)^{n/2}\right)^2 \|f\|_2^2$$

<u>Proof:</u> First, we claim

$$\langle \hat{f}, g \rangle = \langle \check{f}, \hat{g} \rangle$$

$$\int \hat{f}(\xi)\, \overline{g(\xi)}\, d\xi$$

$$\parallel$$

$$\int e^{-i\langle \xi, x\rangle} f(x)\, \overline{g(\xi)}\, d\xi\, dx$$

$$\int f(-x)\, \overline{\hat{g}(x)}\, dx$$

$$= \int f(-x)\, \overline{e^{-i\langle \xi, x\rangle} g(\xi)}\, dx\, d\xi$$

$$\parallel$$

$$= \int f(-x)\, e^{i\langle \xi, x\rangle}\, \overline{g(\xi)}\, dx\, d\xi$$

$$x \to -x$$

$$= \int f(x)\, e^{-i\langle \xi, x\rangle}\, \overline{g(\xi)}\, dx\, d\xi$$

We know: $\forall f, g \in \mathcal{S}$

$$\langle \hat{f}, g \rangle = \langle f, \check{g} \rangle$$

Hence

$$\langle \hat{f}, \hat{g} \rangle = \langle \check{\hat{f}}, \hat{g} \rangle$$

$$= \langle \check{f}, (2\pi)^n \check{g} \rangle = (2\pi)^n \langle \check{f}, \check{g} \rangle$$

$$= (2\pi)^n \langle f, g \rangle$$ ∎

- Hence, up to 2π-factor, the Fourier transform is an L^2-isometry.

__Remark:__ Plancherel applies, same proof, if $f, \hat{f}, g, \hat{g} \in L^1 \cap L^2$.

__Exercise:__ If $f \in L^1$ is C^{n+1}-smooth, and $\partial^\alpha f \in L^1$ $\forall |\alpha| \le n+1$, then

$f \in L^1 \cap L^2$.

- Why did work with $\mathcal{S}'$, not $C_c^\infty(\mathbb{R}^n)$?

<u>Claim:</u> If $f, \hat{f} \in C_c^\infty(\mathbb{R}^n)$, then $f \equiv 0$. ("uncertainty principle")

<u>Proof:</u> Since $f \in \mathcal{S}'$, $\forall x \in \mathbb{R}^n$

$$(*) \quad f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} \underbrace{\hat{f}(\xi)}_{\text{compactly-supported}} \, d\xi$$

- Can we plug-in $z \in \mathbb{C}^n$ in place of $x \in \mathbb{R}^n$ in $(*)$?

Example: $\quad 2\dfrac{\sin x}{x} = \displaystyle\int_{-1}^{1} e^{-i\xi\cdot x}\, d\xi \quad x\in\mathbb{R}$

$$2\frac{\sin z}{z} = \int_{-1}^{1} e^{i\xi\cdot z}\, d\xi \quad z\in\mathbb{C}$$

entire function

$z\in\mathbb{C}^n$

$$\int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}\hat{f}(\xi)\, d\xi$$

$z = x + iy$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$

$$e^{i\langle z,\xi\rangle} = \underbrace{e^{i\langle x,\xi\rangle}}_{\text{bdd in } \xi}\ \underbrace{e^{-\langle y,\xi\rangle}}_{\text{not bounded}}$$

can grow exponentially

Since $\hat{f}$ is compactly-supported,

$$f(z) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle}\hat{f}(\xi)\, d\xi$$

is finite and well-defined $\forall z \in \mathbb{C}^n$.

Moreover, it is a holomorphic function in $\mathbb{C}^n$, which coincides with f on the real line. analytic continuation

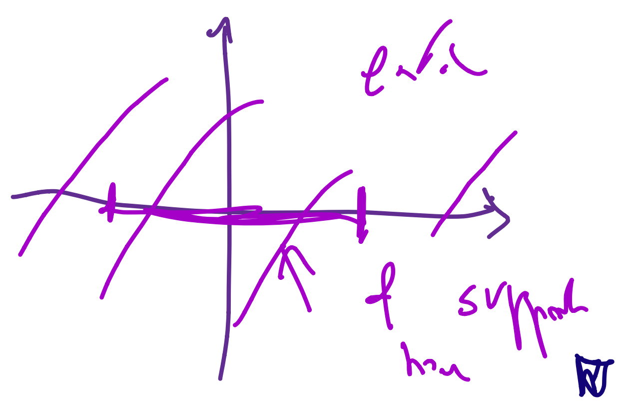
- f is cont. by bounded convergence

$f(z_1, .., z_n)$ is <u>holomorphic</u> if whenever we fix $n-1$ variables, it's holomorphic in the last variable.

(e.g., f is a uniform limit of a sequence holomorphic limit)

$$f(z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle \xi, z \rangle} \hat{f}(\xi)\, d\xi$$

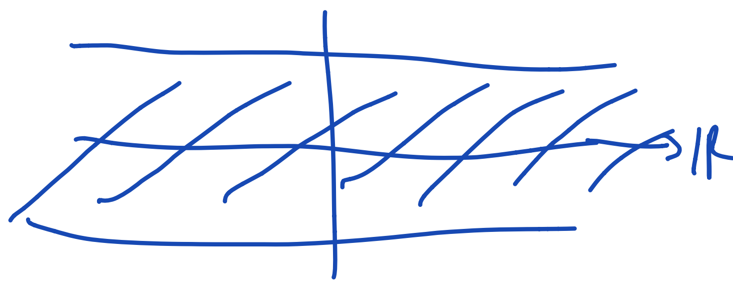
However, a holomorphic function in $\mathbb{C}$ that vanishes on a ray is zero. $\Longrightarrow f \equiv 0$.



Corollary: If $\hat{f}$ is compactly supported in $\mathbb{R}^n$, then f admits a holomorphic extension to $\mathbb{C}^n$.

Paley-Wiener: $|\hat{f}(\xi)| \le e^{-a|\xi|}$

$\xi \in \mathbb{R}$ always



- Back to uncertainty principle.

For $f \in L^2(\mathbb{R}^n)$, $\|f\|_2 = 1$, there are two probability distributions associated with f:

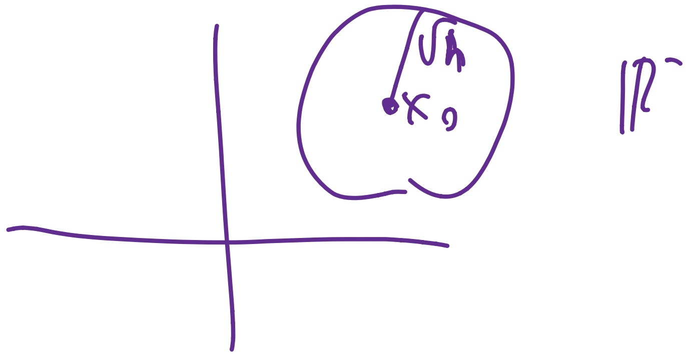
$$|f(x)|^2 \, dx \qquad \text{"position"}$$

$$(2\pi)^{-n} |\hat{f}(\xi)|^2 \, d\xi \qquad \text{"momentum"}$$

$$f(x) = e^{i\langle \xi, x\rangle} e^{-\frac{|x-x_0|^2}{2}}, \quad \xi \in \mathbb{R}^n$$

$$|\xi| = O(1)$$

$$i\frac{\partial f}{\partial t} = \Delta f$$



Heisenberg's uncertainty principle in 1D

For any $f \in \mathcal{S}$, $x_0, \xi_0 \in \mathbb{R}$,

$$\int_{\mathbb{R}} (x-x_0)^2 |f(x)|^2 dx \cdot \int (\xi-\xi_0)^2 |\hat{f}(\xi)|^2 d\xi \geq \frac{\pi}{2} \|f\|_2^4.$$

with equality for Gaussians.

Proof: $\widehat{f'}(\xi) = i\xi \hat{f}(\xi)$

$$\int_{-\infty}^{\infty} |f'|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widehat{f'}(\xi)|^2 d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi^2 \, |\hat{f}(\xi)|^2 .$$

Let's normalize $x_0 = 0$, $\xi_0 = 0$, $\|f\|_2 = 1$.

$$1 = \int_{-\infty}^{\infty} \underbrace{|f|^2}_{= f\bar{f}} = \int_{-\infty}^{\infty} \underbrace{x'}_{=1} \cdot f\bar{f} \, dx$$

$$= -\int_{-\infty}^{\infty} x \left(f'\bar{f} + f\bar{f'} \right) dx$$

$$\left(e^{-x^2/2}\right)' = -x e^{-x^2/2}$$

equality for f

$f(x) = e^{-|x|^2/2}$

$$\leq 2 \int_{-\infty}^{\infty} |x| \cdot |f| \cdot |f'|$$

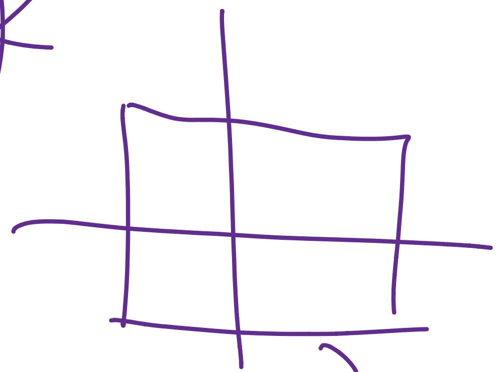
$$\leq 2 \sqrt{\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx} \sqrt{\int_{-\infty}^{\infty} |f'(\xi)|^2 d\xi}$$

$$= 2\sqrt{\int_{-\infty}^{\infty} x^2 |f(x)|^2} \cdot \sqrt{\frac{1}{2\pi}} \sqrt{\int_{-\infty}^{\infty} |\xi|^2 |\hat{f}(\xi)| d\xi}$$

$\triangle$

Open problem in $\mathbb{R}^n$

$K \subseteq \mathbb{R}^n$ convex, $K = -K$



The polar body is

$$K^\circ = \{ x \in \mathbb{R}^n ; \ \forall y \in K \ \ \langle x, y \rangle \le 1 \}$$

$K = B(\ell_p^n)$ $\qquad K^\circ = B(\ell_q^n)$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Q: (O)erskii - Ulanovski, Tao)

(last 10 years)

Given $K \subseteq \mathbb{R}^n$ convex, $K = -K$

does there exist $f \in L^2(\mathbb{R}^n)$, s.t.

$$\int_K |f|^2 \ge \frac{1}{2} \cdot \|f\|_2^2$$

- $\displaystyle\int_{20nK^\circ} |\hat{f}|^2 \geq \frac{1}{2} \cdot \|\hat{f}\|_2^2$

$20nK^\circ$ ↓ or another universal constant

- It then would imply: $\forall K \subseteq \mathbb{R}^n$ convex $K=-K$,

$$K \cap \mathbb{Z}^n = \{0\} \implies \mathbb{Z}^n + CnK^\circ = \mathbb{R}^n$$

not known.

Convolution

Def: For $f, g \in L^1(\mathbb{R}^n)$, their convolution is

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

whenever the integral converges.

<u>Claim:</u> $f * g \in L^1$ if $f, g \in L^1$

<u>Proof:</u> Fubini: $f(x) g(y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$

changing variables $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x-y \end{pmatrix}$

$(x, y) \mapsto f(x-y) g(y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$

Hence, for any almost $x \in \mathbb{R}^n$,

$$\int f(x-y) g(y) \, dy \text{ exists and finite,}$$

and

$$\|f * g\|_1 = \int |(f * g)(x)| \, dx$$

$$\underset{\text{Fubini}}{=} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y) g(x-y) \, dy \right| dx$$

$$\leq \int_{\mathbb{R}^n} |f(y)| \left(\int |g(x-y)| \, dx \right) dy$$

$$= \int_{\mathbb{R}^n} |f(y)|\, dy \cdot \|g\|_1 = \|f\|_1 \cdot \|g\|_1$$

$\square$

<u>Claim:</u> $\widehat{f * g} = \hat{f} \cdot \hat{g}$

<u>Proof:</u> $f, g \in L^1$

$$\widehat{f*g}(\xi) = \int_{\mathbb{R}^n} \Big(\underbrace{\int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy}_{(f*g)(x)} \Big) e^{-i\langle \xi, x\rangle} dx$$

Fubini

$$= \int_{\mathbb{R}^n} f(y)\, e^{-i\langle \xi, y\rangle} \Big(\underbrace{\int_{\mathbb{R}^n} g(x-y)\, e^{-i\langle \xi, x-y\rangle} dx}_{= \int g(x)\, e^{-i\langle \xi, x\rangle} dx} \Big)\, dy$$

$$= \int_{\mathbb{R}^n} f(y)\, e^{-i\langle \xi, y\rangle}\, \hat{g}(\xi)\, dy$$

$$= \hat{f}(\xi)\, \hat{g}(\xi).$$

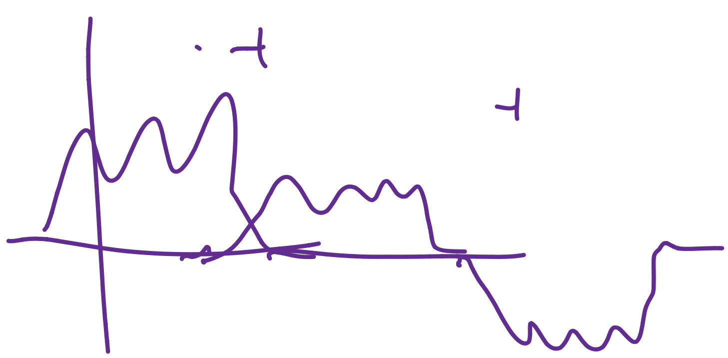
$\square$

- $f * g = g * f$
- Sometimes useful to think about $f*g$ as a weighted average of translates of f (with g the weight)

$$f*g(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

Set $f_y(x) = f(x-y)$, a translate of f

$$f*g = \int_{-\infty}^{\infty} \underbrace{f_y}_{\text{an element of a Banach space}} \ \underbrace{g(y)\,dy}_{\text{weight}}$$



- Convolutions takes the best of two functions, i.e.,

<u>Claim:</u> If $g \in L^1(\mathbb{R}^n)$,

i) $f \in L^p \implies f * g \in L^p$

ii) $f \in \Lambda(\alpha) \implies f * g \in \Lambda(\alpha)$

$\Lambda(\alpha)$: α- Hölder function

$0 < \alpha \leq 1$

$$\|f\|_{\Lambda(\alpha)} = \sup_{x \in \mathbb{R}^n} |f(x)| + \sup_{\substack{x, y \in \mathbb{R}^n \\ 0 < |x-y| \leq 1}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

iii) $f \in C^{m,\alpha} \implies f * g \in C^{m,\alpha}$

iv) $f \in \mathcal{S} \cancel{\implies} f * g \in \mathcal{S}$

$\boxed{\text{OOPS! That's not true!}}$

$$\|f\|_{C^{m,\alpha}} = \sup_{x, |\alpha| \leq m} |\partial^\alpha f(x)| + \max_{|\alpha| = m} \|\partial^\alpha f\|_{\Lambda(\alpha)}$$

Proof: i, ii, iii : A Banach space X of functions on $\mathbb{R}^n$, where the norm is translation invariant:

$$\| f_y \| = \|f\| \quad \forall y \in \mathbb{R}^n$$

Even if $f \notin L^1$, but in such a space, we may define

$$X \ni f * g = \int_{\mathbb{R}^n} \underset{\substack{\uparrow \\ X}}{f_y}\, g(y)\, dy$$

Banach-space valued integral.

Therefore, by the triangle inequality

$$\|f*g\|_X = \Big\| \int_{\mathbb{R}^n} f_y \underset{\substack{\downarrow \\ \text{scalar} \\ \text{function}}}{g(y)}\, dy \Big\|$$

$$\leq \int_{\mathbb{R}^n} \|f_y\|_X \cdot |g(y)|\, dy$$

$$= \|f\|_X \int_{\mathbb{R}^n} |g(y)|\,dy = \|f\|_X \cdot \|g\|_1.$$

Hence the convolution operator

$$f \longmapsto f * g$$

is a bounded operator on X of norm

$$\leq \|g\|_1$$

$\blacksquare$

<u>Exercise:</u> Poisson summation formula.

$\forall f \in \mathcal{S}$,

$2\pi m$

$$\sum_{m \in \mathbb{Z}^n} f(m) = \cancel{(2\pi)^{n/2}} \sum_{m \in \mathbb{Z}^n} \hat{f}(m)$$

Oops

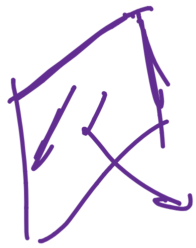
(Actually, $\sum f(x+m)$ and compute its FT)

<u>Slices of measurable sets and the Radon transform</u>

For $\theta \in S^{n-1} = \{x \in \mathbb{R}^n ; |x| = 1\}$
$t \in \mathbb{R}$ set

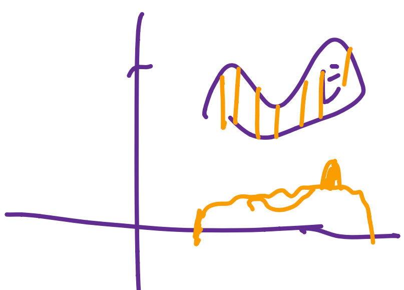
$P_{\theta,t} = \{x \in \mathbb{R}^n ; \langle x, \theta \rangle = t\}$

a hyperplane in $\mathbb{R}^n$
orthogonal to θ
distance $|t|$ from 0.



- Suppose $E \subseteq \mathbb{R}^n$ measurable set (bounded).
- Fix $\theta \in S^{n-1}$, and consider

$$(*) \quad t \longmapsto \mathcal{H}_{n-1}(E \cap P_{\theta,t}) \qquad t \in \mathbb{R}$$



By Fubini, the function $(*)$ is measurable in t, and its integral in t equals $\mathcal{H}_n(E)$.

Thm (I saw it in Falconer '80)

Assume $n \geq 3$, $E \subseteq \mathbb{R}^n$ is bounded, Borel measurable.
Then for almost any $\theta \in S^{n-1}$, the function

$$(*) \qquad t \longmapsto \mathcal{H}_{n-1}(E \cap P_{\theta,t}) \qquad (t \in \mathbb{R})$$

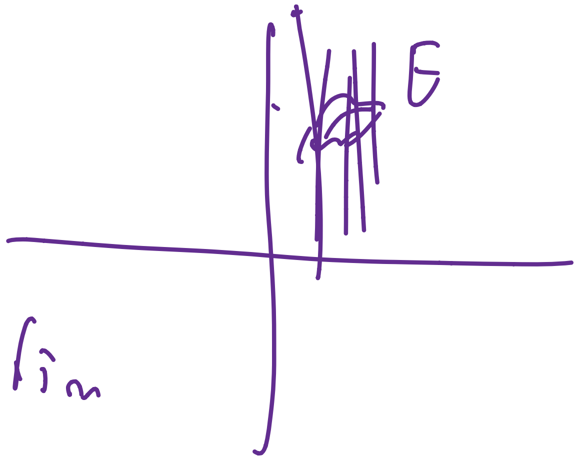
is continuous in $\mathbb{R}$, and in fact α-Hölder for any $0 < \alpha < \frac{1}{2}$.

- FALSE in 2D.

If E = Besicovitch set
Kakeya
compact

$\mathcal{H}^2(E) = 0$, but $\in E$
in any direction

Then for any $\theta \in S^{n-1}$ the function $(*)$ has integral zero by Fubini, but its maximum is at least 1.



· A non-negative function of integral zero and positive maximum is not continuous.

Radon transform

(kind of x-ray)

For $f \in L^1(\mathbb{R}^n)$ define

$$Rf(\theta, t) = \int_{P_{\theta,t}} f \, d\mathcal{H}_{n-1}$$

$\theta \in S^{n-1}$, $t \in \mathbb{R}$

$$\left(Rf : S^{n-1} \times \mathbb{R} \to \mathbb{C} \right)$$

whenever the integral is defined.

Properties: 1) By Fubini, $\forall \theta \in S^{n-1}$, well-defined for almost any t,

and

$$\int_{-\infty}^{\infty} Rf(\theta, t)\,dt = \int_{\mathbb{R}^n} f$$

2) If $f \in C_c(\mathbb{R}^n)$

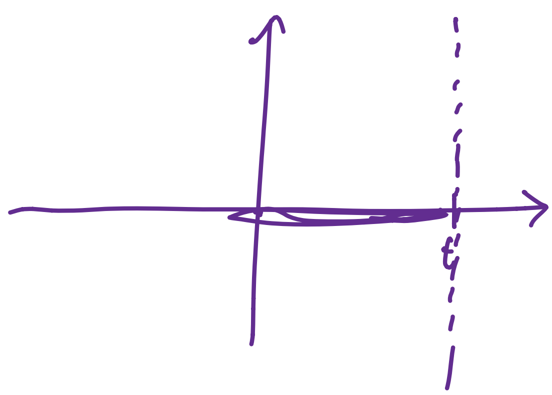
then Rf is continuous

and compactly-supported in $S^{n-1} \times \mathbb{R}$.

(follows from bounded convergence).

__Example:__ $f(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$

What is $Rf(\theta, t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$



$$f(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

__Convenient notation:__ $\hat{R}$

$\hat{R}f$ is FT of Rf in t-variable

$$\hat{R}f(\theta, \lambda) = \int_{-\infty}^{\infty} e^{-it\lambda} Rf(\theta, t)\, dt$$

<u>Lemma</u>: (n-dim FT is 1-dim FT of RT)

For any $f \in L^1(\mathbb{R}^n)$,

$$\hat{R}f(\theta, \lambda) = \hat{f}(\lambda\theta)$$