8/6/20

Last time:

$0 < t < n$ $\quad \widehat{\frac{1}{|x|^t}} = c_{n,t} \frac{1}{|x|^{n-t}}$

For a compactly-supported Borel measure μ

$$I_t(\mu) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1}{|x-y|^t}\, d\mu(y)\, d\mu(x)$$

$$= \left\langle \mu * \frac{1}{|x|^t}, \mu \right\rangle$$

$$\left(\text{Recall } \langle F, \varphi \rangle := F(\bar{\varphi}) \right)$$

$F \in \mathcal{S}^*$, $\varphi \in \mathcal{S}$

$\boxed{\text{exercise}}$

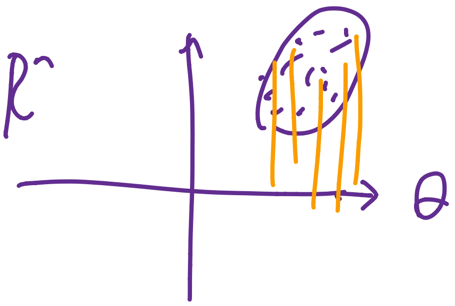
$$= (2\pi)^{-n} \left\langle \hat{\mu} \cdot \widehat{\frac{1}{|x|^t}}, \hat{\mu} \right\rangle$$

$$\widehat{\frac{1}{|x|^t}} = \frac{c}{|x|^{n-t}}$$

$$= c \int_{\mathbb{R}^n} \frac{1}{|x|^{n-t}} |\hat{\mu}(x)|^2 \, dx$$

Projections of fractals

$K \subseteq \mathbb{R}^n$



For $\theta \in S^{n-1}$

$$P_\theta(x) = \langle x, \theta \rangle$$

Interested in $P_\theta(K)$ for a fractal $K \subseteq \mathbb{R}^n$.

Thm (Marstrand, Kaufmann) Let $K \subseteq \mathbb{R}^n$ be a compact with $\dim(K) > 1$ (Hausdorff dimension). Then for almost any $\theta \in S^{n-1}$,

$$\mathcal{H}^1(P_\theta(K)) > 0.$$

Example: $K = \text{Cantor} \times \text{Cantor} \subseteq \mathbb{R}^2$

$$\dim(K) = \log_3 2 + \log_3 2 > 1$$

$$P_\theta(C \times C) = \alpha C + \beta C$$

Proof: By Frostman lemma $\exists \mu \neq 0$ measure supported on K with

$$I_1(\mu) < \infty.$$

$$\parallel$$

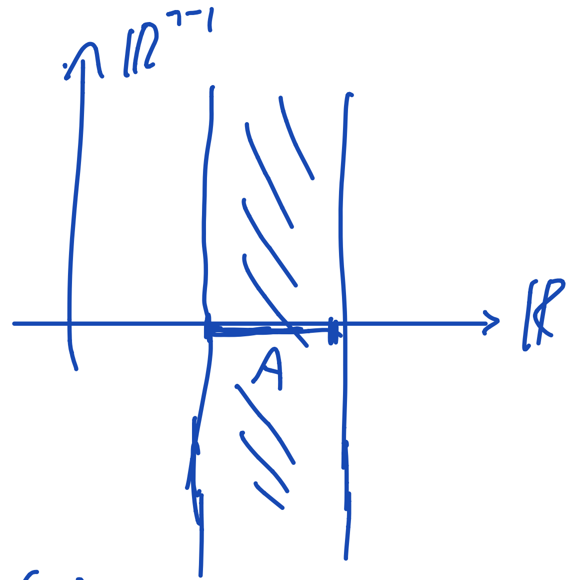
$$\int_{\mathbb{R}^n} \frac{1}{|x|^{n-1}} |\hat{\mu}(x)|^2 \, dx < \infty$$

· For $\theta \in S^{n-1}$ denote

$$(P_\theta)_* \mu = \mu_\theta$$

I.e., that for $A \subseteq \mathbb{R}$ Borel

$$\mu_\theta(A) = \mu(P_\theta^{-1}(A))$$



$\forall \varphi : \mathbb{R} \to \mathbb{R}$, $C_c(\mathbb{R})$,

$$\int_{\mathbb{R}} \varphi \, d\mu_\theta = \int_{\mathbb{R}^n} \varphi(\underbrace{\langle x, \theta \rangle}_{\overset{\parallel}{P_\theta(x)}}) \, d\mu(x)$$

We will show that $\mu_\theta \in L^2(\mathbb{R})$ for almost any $\theta \in S^{n-1}$.

absolutely cont. with density in $L^2(\mathbb{R})$

Since $\mu_\theta \not\equiv 0$ and $\mu_\theta \in L^2(\mathbb{R})$, its support has positive Lebesgue measure,

but

$$\mathrm{Supp}(\mu) \subseteq K$$

$$\Downarrow$$

$$\mathrm{Supp}(\mu_\theta) \subseteq P_\theta K$$

which consequently has positive Lebesgue measure.

(A point in $\mathbb{R}^n$ is NOT in $\mathrm{Supp}(\mu)$ if it has a nbhd of zero μ-measure.

. $\mathrm{Supp}(\mu)$ is the minimal closed set of full μ-measure.)

$$\int_{\mathbb{R}} \varphi \, d\mu_\theta = \int_{\mathbb{R}^n} \varphi(\underset{P_\theta(x)}{\underset{\shortparallel}{\langle x, \theta \rangle}}) \, d\mu(x)$$

$$\Downarrow$$

$$\widehat{\mu_\theta}(t) = \hat{\mu}(t\theta)$$

$$\int_{-\infty}^{\infty} e^{-its}\, d\mu_\theta(s) \quad = \quad \int e^{-it\langle\theta, x\rangle}\, d\mu(x)$$

$$\varphi(s) = e^{-its}$$

Same as: $\widehat{Rf}(\theta, t) = \hat{f}(\theta t)$

· We need to show that for almost any θ, $\|\mu_\theta\|_2 < \infty$.

Apply Plancherel:

$$\int_{S^{n-1}} \|\mu_\theta\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi} \int_{S^{n-1}} \|\widehat{\mu_\theta}\|^2_{L^2(\mathbb{R})} =$$

$$= \frac{1}{2\pi} \int_{S^{n-1}} \int_{-\infty}^{\infty} |\widehat{\mu_\theta}(t)|^2\, dt\, d\theta$$

$$= \frac{1}{\pi} \int_{S^{n-1}} \int_0^{\infty} \frac{|\hat{\mu}(r\theta)|^2}{|r\theta|^{n-1}} r^{n-1}\, dr\, d\theta$$

$\overset{\text{polar coordinates}}{=}$ $x = r\theta$

$$\frac{1}{\pi} \int_{\mathbb{R}^n} \frac{|\hat{\mu}(x)|^2}{|x|^{n-1}} dx = c\, I_1(\mu) < \infty.$$

[Formal justification from right to left).

Then $\mu_\theta \in L^2(\mathbb{R})$ for almost any θ. $\blacksquare$

__Thm__ (Falconer '99, Peres - Schlag)

$K \subseteq \mathbb{R}^n$, $\dim(K) > 2$.
compact

Then for almost any $\theta \in S^{n-1}$, $P_\theta(K)$ has non-empty interior.
(contains an interval)

__Example:__ $K \subseteq \mathbb{R}^2$ Kakeya, $\mathbb{Q}^2 = \{q_n\}_{n \geq 1}$.

$$\overline{B(0, \tfrac{1}{2})} \setminus \bigcup_{n=1}^{\infty} (K_{\delta_n} + q_n)$$

$\uparrow$ open

has dim $= 2$, but for any $\theta \in S^1$,
$P_\theta(K)$ is nowhere dense

<u>Proof:</u> By Frostman $\exists \mu$ supported on K
with $I_s(\mu) < \infty$ for some $s > 2$.

$$\mu_\theta = (P_\theta)_* \mu$$

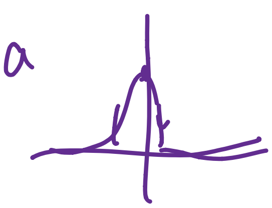
We will show that for almost any
$\theta \in S^{n-1}$, $\widehat{\mu_\theta} \in L^1(\mathbb{R})$

Hence $\hat{\hat{\mu}}_\theta \in C(\mathbb{R})$

as a distribution $\check{\mu}_\theta = \frac{1}{2\pi} \hat{\hat{\mu}}_\theta$

Hence, if $\widehat{\mu_\theta} \in L^1$, then

$0 \not\equiv \mu_\theta$ is a measure on $\mathbb{R}$ with a
continuous density.



This density is positive on some interval I_θ

(by continuity), hence

$$\mathrm{Supp}(\mu_\theta) \supseteq I_\theta$$

interval in $\mathbb{R}$ of positive length

However

$$P_\theta(K) \supseteq \mathrm{Supp}(\mu_\theta)$$

Hence $P_\theta(K)$ contains an interval.

- Enough to show that for almost any $\theta \in S^{n-1}$,

$$\int_{-\infty}^{\infty} |\hat{\mu}_\theta(t)|\, dt < \infty$$

- Enough to prove that

$$\int_{|t|>1} |\hat{\mu}_\theta(t)|\, dt < \infty$$

(because $|\hat{\mu}_\theta(t)| \leq \mu_\theta(\mathbb{R}) \leq \mu(\mathbb{R}^n) < \infty$)

So $\int_{-1}^{1} |\hat{\mu}_\theta(t)|\, dt < \infty$

Now, we compute:

$$\int_{S^{n-1}} \int_{|t|\geq 1} |\hat{\mu}_\theta(t)|\, dt =$$

$$\leq \sqrt{\int_{S^{n-1}} \int_{|t|\geq 1} |\hat{\mu}_\theta(t)|^2\, t^{s-1}\, dt} \cdot \sqrt{\int_{S^{n-1}} \int_{|t|\geq 1} \frac{dt}{t^{s-1}}}$$

$\wedge$ ∞ as $s > 2$

$$\leq 2C_s \sqrt{\int_{S^{n-1}} \int_0^\infty \frac{|\hat{\mu}(r\theta)|^2}{|r\theta|^{(n-1)+(1-s)}}\, r^{n-1}\, dr\, d\theta}$$

polar coordinates $x = r\theta$

$$= 2C_s \sqrt{\int_{\mathbb{R}^n} \frac{|\hat{\mu}(x)|^2}{|x|^{n-s}}\, dx}$$

$$= c_{n,s} \sqrt{I_s(\mu)} < \infty.$$

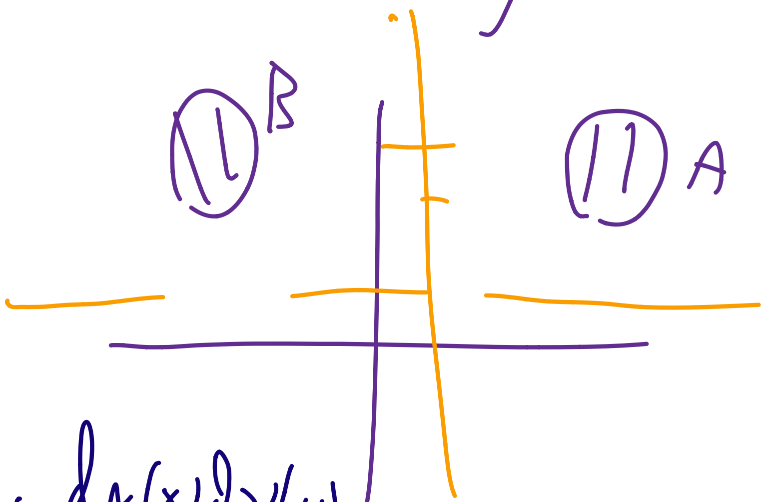
$\square$

Exercise $A, B \subseteq \mathbb{R}^2$, $\dim(A) > 1$, $\dim(B) > 1$

Then for a positive prob. of choosing $\theta \in S^1$,

$$\mathcal{H}^1\left(P_\theta(A) \cap P_\theta(B) \right) > 0$$

Hint:



$$I_t(\mu, \nu) := \iint \frac{1}{|x-y|^t} \, d\mu(x) \, d\nu(y)$$

by prop $\;=\; \int_{\mathbb{R}^2} \frac{1}{|x|^t} \hat{\mu}(x) \overline{\hat{\nu}(x)} \, dx.$

Mattila–Orponen

$\langle \mu_\theta, \nu_\theta \rangle > 0$ for positive measure of θ

$\int \langle \mu_\theta, \nu_\theta \rangle > 0$

S^{n-1}

$\overset{?}{=} \langle \hat{\mu}_\theta, \hat{\nu}_\theta \rangle$

Littlewood-Paley decompositions

- For $f \in L^1(\mathbb{R}^n)$, "the smoother f is, the fast its F.T. decays at ∞".

E.g.,

- If $\hat{f} = O\left(\frac{1}{|x|^{n+1}}\right)$ then $\hat{f} \in L^1$ and f is cont.

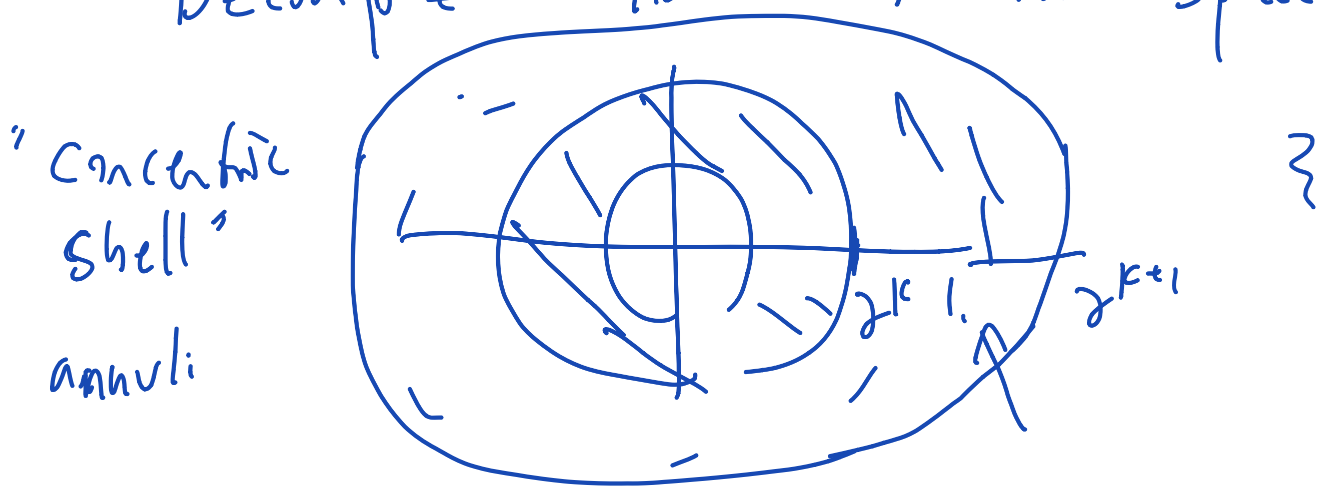
- If $\hat{f} = O\left(\frac{1}{|x|^{n+k}}\right)$ then $\forall |\alpha| = k$ $\widehat{\partial^\alpha f} \in L^1$ hence f is C^k-smooth.

- If $\partial^\alpha f \in L^1$ $\forall |\alpha| \le k$ $\Rightarrow \hat{f}(x) = o\left(\frac{1}{|x|^k}\right)$

· Not that precise.

A more precise way to understand smoothness of f from decay of $\hat{f}$:

Decompose $\mathbb{R}^n$ Fourier space

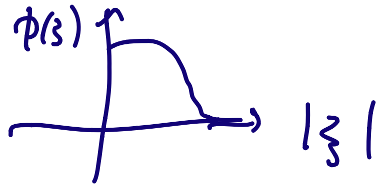


$\sin(\xi \cdot x)$ does $\sim 2^k$ oscillations per unit length if $2^{k-1} < |\xi| < 2^{k+1}$

· We divide the Fourier space into concentric shells smoothly via $\boxed{\text{partition of unity.}}$

Throughout this lecture (one hour) we fix

$\phi \in C_c^\infty(\mathbb{R}^n)$ with



- $\begin{cases} \phi(x) = 1 & |x| \le 1 \\ \phi(x) = 0 & |x| > \frac{3}{2} \end{cases}$

- $0 \le \phi \le 1$

  ϕ is radially non-increasing

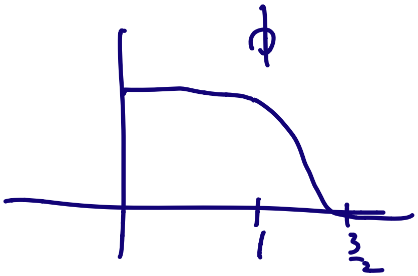
- $|\partial^\alpha \phi(x)| \le C_\alpha \quad \forall x \in \mathbb{R}^n$

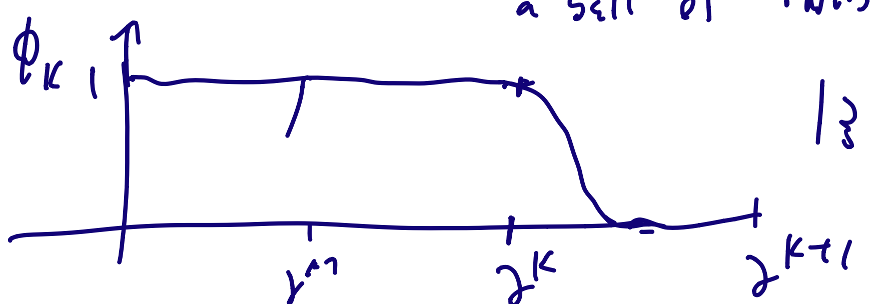
Construct a P.O.U from ϕ as follows:

For $k \ge 0$

$$\phi_k(\xi) = \phi(2^{-k}\xi)$$

roughly indicator of a ball of radius 2^k.

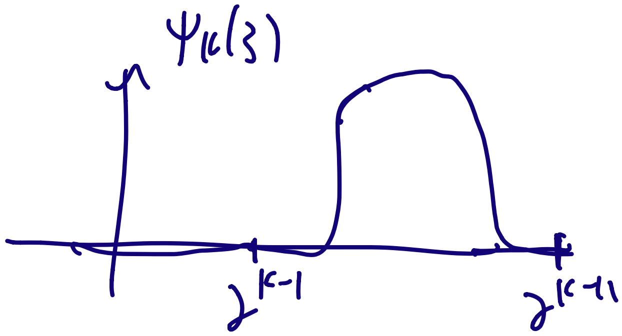




Define

$k \geq 1 \quad \psi_k(\xi) = \phi_k(\xi) - \phi_{k-1}(\xi)$

$k = 0 \quad \psi_0(\xi) = \phi_0(\xi) = \phi(\xi)$

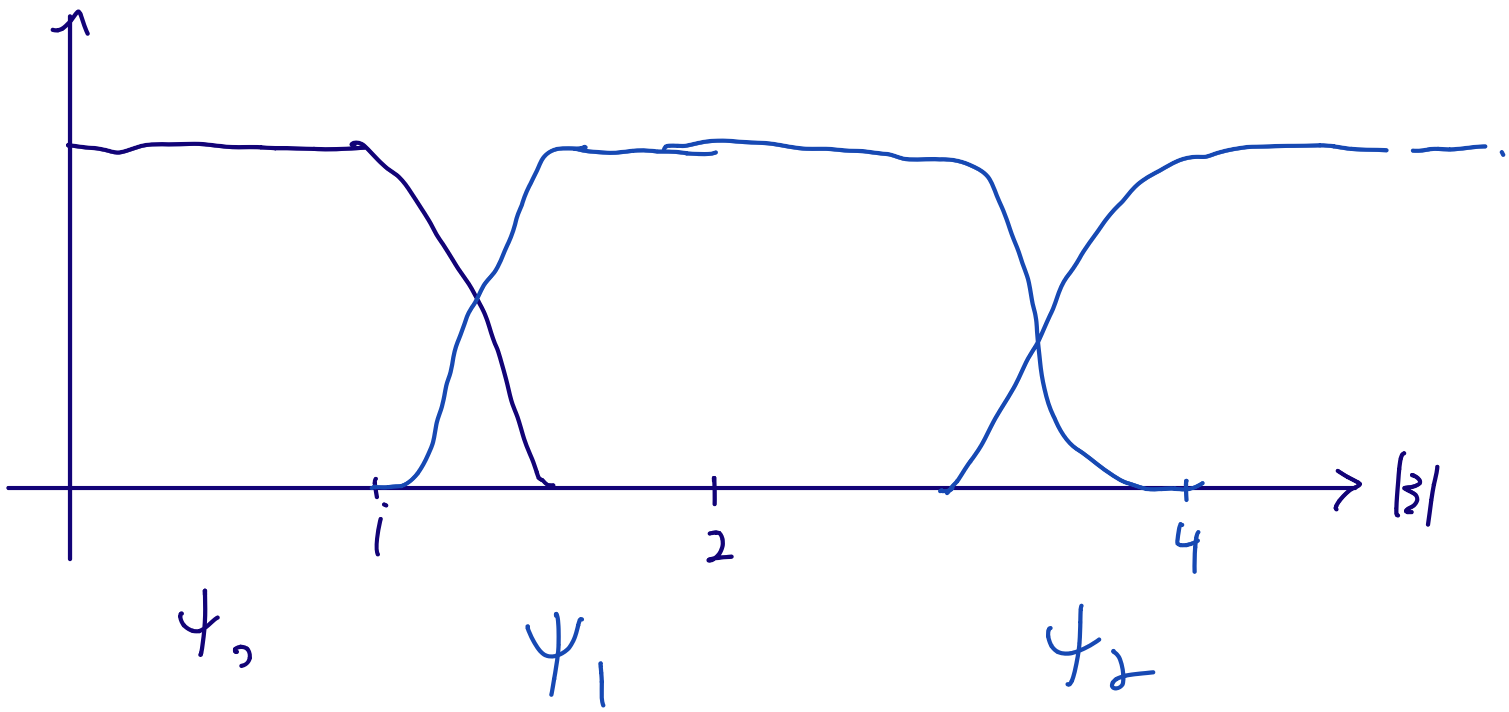


$\psi_k(\xi) = \psi(2^{-k}\xi)$

where

$\psi(\xi) = \phi(\xi) - \phi(2\xi) \geq 0$

Draw all of the ψ_k's.



ψ_k is roughly the indicator of a

shell $2^{k-1} \le |\xi| \le 2^{k+1}$

$\mathrm{supp}(\psi_k) \subseteq \{ \quad -\text{"}- \quad \}$.

<u>Lemma:</u> $\psi_k \in C_c^\infty(\mathbb{R}^n)$,

- $\forall \xi \in \mathbb{R}^n$, $\quad \displaystyle\sum_{k=0}^{\infty} \psi_k(\xi) \equiv 1$

the sum consists of just 1 or 2 non-zero summands.

In fact,

- $\displaystyle\sum_{k=0}^{N} \psi_k(\xi) = \phi_N(\xi)$

- $\left|\partial^\alpha \psi_k(\xi)\right| \le C_\alpha 2^{-k|\alpha|} \quad \forall \xi \in \mathbb{R}^n$.

(as $\psi_k(\xi) = \psi(2^{-k}\xi)$)

<u>Proof:</u> Obviously.

Operator Partition of Unity

Defn: For $f \in \mathcal{S}'^*$ and $K \geq 0$ set

$$\widehat{P_K f} = \underset{\substack{\downarrow \\ \mathcal{S}}}{\psi_K} \cdot \underset{\substack{\uparrow \\ \mathcal{S}'^*}}{\hat{f}}$$

(actually $C_c^\infty(\mathbb{R}^n)$)

In other words,

$$P_K f = \tilde{\psi}_K * f$$

where $\tilde{\psi}_K := (2\pi)^{-n} \check{\hat{\psi}}_K$

i.e. $\widehat{\tilde{\psi}_K} = \psi_K$.

So $P_K f = \underset{\substack{\uparrow \\ \mathcal{S}}}{\tilde{\psi}_K} * f$.

ψ_K is compactly-supported, but not $\tilde{\psi}_K$.

Remarks:

•) P_k is a Fourier multiplier, in Fourier space, just multiply.

•) A common notation:

$$D^\alpha := (-i)^{|\alpha|} \partial^\alpha$$

$\forall f \in \mathcal{S}$, $\widehat{D^\alpha f}(\xi) = \xi^\alpha \hat{f}(\xi)$

Some people denote

$$P_k = \psi_k(D)$$

•) $$P_k f = \tilde{\psi}_k * f$$

a convolution operator with a Schwartz function,

$\forall f \in \mathcal{S}^*$, $P_k f \in C^\infty$ of moderate growth $P_k f = \underset{\mathcal{S}\ni}{\tilde{\psi}_k} * \underset{\mathcal{S}^*}{f}$

- $P_k : \mathcal{S} \to \mathcal{S}$ cont. in Schwartz topology
(2 Fouriers & one mult.)

- Since $-1 \leq \psi_k \leq 1$
(could make $0 \leq \dots \leq 1$).

$$\| P_k f \|_2 \leq \| f \|_2$$

Why is $P_k : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ a contraction?

$$\| P_k f \|_2 = (2\pi)^{-n/2} \| \widehat{P_k f} \|_2 = (2\pi)^{-n/2} \| \psi_k \hat{f} \|_{L^2}$$

$$\leq (2\pi)^{-n/2} \| \hat{f} \| = \| f \|_2 . \qquad |\psi_k| \leq 1$$

Also $P_k : \mathcal{S}^* \to \mathcal{S}^*$ weakly cont.

<u>Lemma:</u> For $f \in \mathcal{S} \,/\, \mathcal{S}^* \,/\, L^2(\mathbb{R}^n)$

$$\sum_{k=0}^{\infty} P_k \underset{\substack{\uparrow \\ C^\infty}}{f} = f$$

when sum converges in Schwartz topology / weakly in $\mathcal{S}'$ / in $L^2(\mathbb{R}^n)$.

$$\operatorname{supp}\left(\widehat{P_k f}\right) \subseteq \left\{ 2^{k-1} \le |\xi| \le 2^{k+1} \right\}$$

· $P_k f$ is the component of f with frequencies $\sim 2^k$

oscillates in lengthscale around 2^{-k}.

<u>Proof:</u> By continuity of F.T., enough to prove

$$\sum_{k=0}^{\infty} \widehat{P_k f} = \hat{f}$$

''

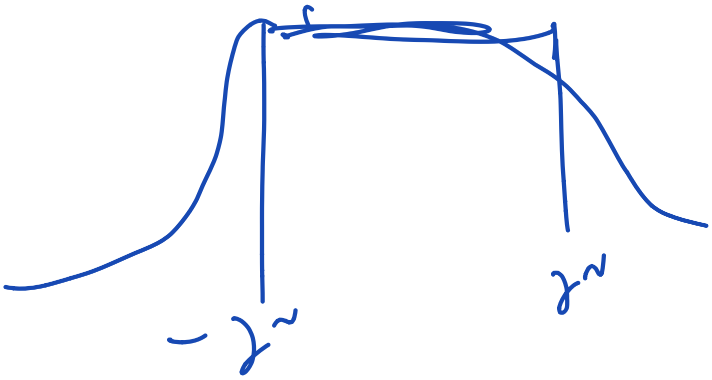
$$\sum_{k=0}^{\infty} \psi_k(\xi)\, \hat{f}(\xi) = \hat{f}$$

I.e, need to prove

$$\left(\sum_{k=0}^{N} \psi_k(\xi)\right) \hat{f} \xrightarrow{N\to\infty} \hat{f}$$

$g = \hat{f}$

$\forall g \qquad \Phi_N g \longrightarrow g$



In L^2 - ✓

$\mathcal{S}$ - ✓

$\mathcal{S}^*$ ✓

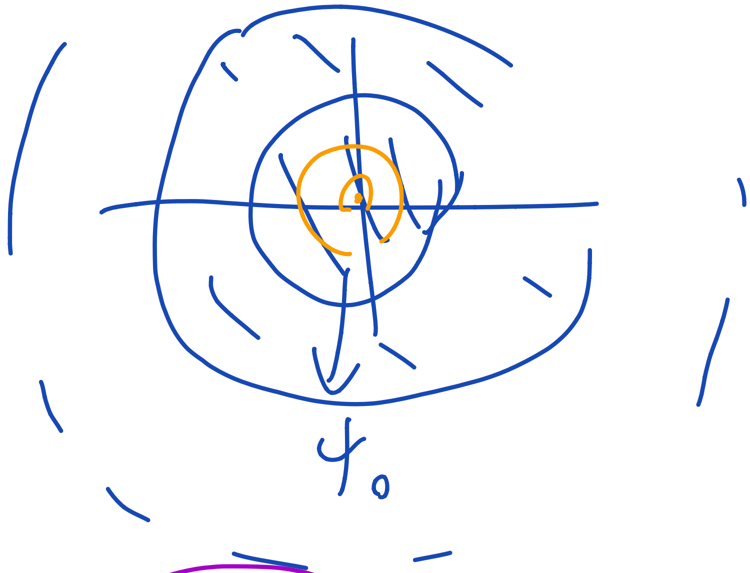
$$\sum_{k=0}^{\infty} P_k f = f$$

Convergence in the space where f belongs

$\mathcal{S}$ / L^2 / $\mathcal{S}^*$

Remarks:

Could take $k \in \mathbb{Z}$ in place of $k \geq 0$ and decompose



$$f = \sum_{k \in \mathbb{Z}} Q_k f + \text{harmonic polynomial}$$

where $\operatorname{supp} \widehat{Q_k f} \subseteq \{ 2^k \leq |\xi| \leq 2^{k+1} \}$ also for $k \leq 0$.

its F.T. is supported at $\{0\}$.

Hölder norm: $0 < s \leq 1$

$$\| f \|_{C^s} = \| f \|_\infty + \sup_{\substack{x, y \in \mathbb{R}^n \\ |x-y|<1}} \frac{|f(x) - f(y)|}{|x-y|^s}$$

We can put or not put $|x-y|<1$, the result would be equivalent.

$$\|f\|_{C^{m,s}} = \max_{|\alpha|\le m} \|\partial^\alpha f\|_{C^s}$$

$m \ge 0$

$$C^s \subseteq L^\infty \subseteq \mathcal{S}'$$

Sobolev norms (easier, L^2 variants)

For $k \ge 0$ and $f \in L^2(\mathbb{R}^n)$,

$$\|f\|_{H^k} \sim \sum_{|\alpha|\le k} \|\partial^\alpha f\|_2$$

where the implied constants depend on k & n.

$$\|f\|_{H^s}^2 := \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s |\hat{f}(\xi)|^2 d\xi$$

$s > 0$ $H^s \subseteq L^2 \subseteq \mathcal{S}'$

Lemma: The two def. of Sobolev coincide

(for integer $s = k$)

Enough to prove for $f \in \mathcal{S}$,

Proof: 1) $\forall |\alpha| \le k$

$$\|\partial^\alpha f\|_2^2 = c \int |\xi^\alpha|^2 |\hat{f}(\xi)|^2 d\xi$$

$$\le c \int \left(1+|\xi|^2\right)^K \left|\hat{f}(\xi)\right|^2 d\xi$$

2) $$\left(1+|\xi|^2\right)^k \le \left(1+\sum \xi_i^{2k}\right) \cdot C_{n,k}$$

$$\int \left(1+|\xi|^2\right)^k \left|\hat{f}(\xi)\right|^2 d\xi$$

$$\le C_{n,k} \sum_{i=1}^{n} \int_{\mathbb{R}^n} \left(1+\xi_i^{2k}\right) \left|\hat{f}(\xi)\right|^2 d\xi$$

$$= C\|f\|_2^2 + C' \sum_{i=1}^{n} \int_{\mathbb{R}^n} \left|\xi_i^k \hat{f}(\xi)\right|^2 d\xi$$

$$= C\|f\|_2^2 + C' \sum_{i=1}^{n} \left\| \frac{\partial^k f}{\partial x_i^k} \right\|_2^2 . \qquad \blacksquare$$

Remark: $\underset{\subseteq L^2}{H^s}$ and $\underset{\subseteq L^\infty}{C^s}$ are the closure of $\mathcal{S}$ under the corresponding norm.

Think:

0) define C^s norm on $\mathcal{S}$

1) Consider the completion as an abstract Banach space, in

which $\mathcal{S}$ is dense.

2) Identity between the completion and a subspace of L^∞.

3) Prove that $f \in L^\infty$ for which the norm is finite is a limit of Schwartz functions

Expressing Hölder / Sobolev norms in terms of Littlewood - Paley decomp.

Lemma: $\forall f \in H^s$,

$$\boxed{\|f\|_{H^s}^2 \sim \sum_{k=0}^{\infty} 2^{2ks} \|P_k f\|_2^2}$$

with implied constants depending on n and s.

Proof: Enough to prove for $f \in \mathcal{S}$.

$$\infty > \|f\|_{H^s}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\hat{f}(\xi)|^2 \, d\xi$$

$$= \int_{\mathbb{R}^n} (1+|\xi|^2)^s \underbrace{\sum_{k=0}^{\infty} \psi_k(\xi)}_{\text{"1"}} |\hat{f}(\xi)|^2 d\xi$$

Now, $\underset{\geq 0}{a} + \underset{\geq 0}{b} = 1 \Rightarrow \frac{1}{2} \leq a^2 + b^2 \leq 1$

Hence, up to factor 2,

$$\|f\|_{H^s}^2 \sim \sum_{k=0}^{\infty} \int (1+|\xi|^2)^s \underbrace{\psi_k^2(\xi) |\hat{f}(\xi)|^2}_{|\widehat{P_K f}(\xi)|^2} d\xi$$

$$= \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \underbrace{(1+|\xi|^2)^s}_{\substack{\sim \\ 2^{2k} \\ \text{in support}}} \underbrace{|\widehat{P_K f}(\xi)|^2}_{\substack{\text{supported} \\ \text{in} \\ \{2^{k-1} < |\xi| < 2^{k+1}\}}} d\xi$$

$$\sim \sum_{k=0}^{\infty} 2^{2ks} \int |\widehat{P_k f}(\xi)|^2 d\xi$$

$$\sim \sum_{k=0}^{\infty} 2^{2ks} \|P_k f\|_2^2 .$$

$\square$

Thm: For $0 < s < 1$, $f \in C^s$,

$$\|f\|_{C^s} \sim \sup_{k \geq 0} 2^{ks} \|P_k f\|_\infty$$

Exercise: Think about $s = 1$.

implied constants depend on n and s.

$|\xi| \sim 2^k$

$\psi_k \hat{f} \sim \frac{1}{2^{ks}}$

Roughly, $f \in C^s$ is like $\hat{f} = O\left(\frac{1}{|\xi|^s}\right)$

but the thm is the precise way.

Bernstein's lemma: Suppose $f \in L^\infty$

$\mathrm{Supp}(\hat{f}) \subseteq B(0, R)$.

Then,

$$\|\nabla f\|_\infty \leq C R \|f\|_\infty$$

Exercise: $\|\nabla f\|_\infty \sim R \cdot \|f\|_\infty$.

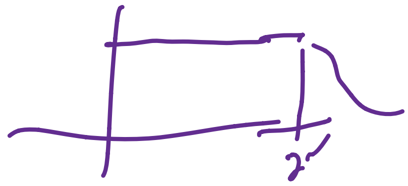
if $\mathrm{supp}(\hat{f}) \subseteq \{\frac{R}{2} < |\xi| < 2R\}$

May assume $f \in \mathcal{S}$,

<u>Proof:</u> Pick k with

$$R < 2^k \le 2R$$

$$\left(\begin{array}{l} \widehat{\tilde{\Psi}_k} = \psi_k, \quad P_k f = \tilde{\Psi}_k * f \\ \sum_{k=0}^{n} \psi_k = \phi_n \\ \widehat{\tilde{\Phi}_n} = \phi_n \end{array} \right)$$



$$\hat{f} \cdot \mathbb{1}_{B(0,R)} = \hat{f}$$

$$\hat{f} \cdot \phi_k = \hat{f}$$

(because $\phi_k(x) = 1$ on $|x| \le 2^k$)

i.e. $$\boxed{f * \tilde{\Phi}_k = f}$$

when $$\tilde{\Phi}_k(\xi) = (2\pi)^{-n} \hat{\Phi}_k(-\xi)$$

$$f(x) = \int_{\mathbb{R}^n} \tilde{\Phi}_k(x-y) f(y)\, dy$$

Φ_k is a scaling of ϕ by 2^k

$\tilde{\Phi}_k$ is the F.T. of Φ_k

(up to 2π-factor and $\xi \longleftrightarrow -\xi$)

$$\Phi_k(x) = \phi(2^{-k}x)$$

$$\widehat{\Phi_k}(\xi) = (2^k)^n \hat{\phi}(2^k \xi)$$

Hence

$$\tilde{\Phi}_k(\xi) = (2^k)^n \tilde{\phi}(2^k \xi)$$

$$f(x) = \int_{\mathbb{R}^n} \tilde{\Phi}_k(x-y)\, f(y)\, dy$$

$$= 2^{kn} \int_{\mathbb{R}^n} \tilde{\phi}(2^k(x-y))\, f(y)\, dy$$

$$\partial_{x_i} f(x) = 2^{kn} \int_{\mathbb{R}^n} 2^k (\partial_{x_i} \tilde{\phi})(2^k(x-y))\, f(y)\, dy$$

$$\left|\frac{\partial f}{\partial x_i}(x)\right| \le 2^{(n+1)k} \int_{\mathbb{R}^n} |\nabla\tilde{\phi}|\left(2^k(x-y)\right)|f(y)|\,dy$$

$$\le 2^{(n+1)k} \int_{\mathbb{R}^n} |\nabla\tilde{\phi}|\left(2^k(x-y)\right)dy \cdot \|f\|_\infty$$

$$= 2^k \cdot \underbrace{2^{kn}}\int_{\mathbb{R}^n} |\nabla\tilde{\phi}|\left(2^k y\right)\underbrace{dy} \cdot \|f\|_\infty$$

$$\overset{z=2^k y}{=} 2^k \cdot \|f\|_\infty \cdot \int_{\mathbb{R}^n} |\nabla\tilde{\phi}(z)|\,dz$$

$\dot{\phi}$ was fin, $\nabla\tilde{\phi} \in S$ is fin. thus

$$\left|\frac{\partial f}{\partial x_i}(x)\right| \le C\,2^k\,\|f\|_\infty$$

$$\le C'\,R\,\|f\|_\infty. \qquad \square$$

<u>Corollary:</u> $\forall f \in S^*$

$$\|\nabla P_k f\|_\infty \sim 2^k \,\|P_k f\|_\infty$$

because $\operatorname{supp} \widehat{P_k f} \subseteq \{2^{k-1} < |\xi| < 2^{k+1}\}$.

The thm tells us: $f \in C^s \iff$ when taking $\hat{f}$, multiplying by an approximate of a shell of radius R, and applying inverse FT, we get a function of L^∞-norm

$$\leq \frac{C}{R^s}$$