22/6/20

Last time:

Smoothness and Fourier Transform

1) $0 < s < 1$

$$\|f\|_{C^s} \sim \sup_{k \geq 0} 2^{ks} \|\underbrace{P_k f}_{\psi_k(D) f}\|_\infty$$

2) $$\|f\|^2_{H^s} = \sum_{k=0}^{\infty} \left(2^{ks} \|P_k f\|_2\right)^2$$

3) $$H^{s+\frac{n}{2}} \subseteq C^s$$ Sobolev embedding

$$H^{s+k+\frac{n}{2}} \subseteq C^{k,s}$$ Morrey

- This was used to prove elliptic regularity of homogeneous, Fourier-multiplier operators:

1) The Laplacian Δ is homogeneous of degree 2

$$f_r(x) = f(rx)$$

$$\Delta f_r = r^2 (\Delta f)_r$$

Fourier multiplier by $-|\xi|^2$

$\forall f \in \mathcal{S}^*$ $\quad \widehat{\Delta f}(\xi) = -|\xi|^2 \hat{f}(\xi)$

Careful: homogeneous distribution does not have the same degree of homogeneity

(Clarify for yourself:

$pv\left(\frac{1}{x}\right)$ is (-1)-homogeneous as tempered distribution

but

$\varphi \longmapsto \varphi * pv\left(\frac{1}{x}\right)$ in $\mathcal{S}$

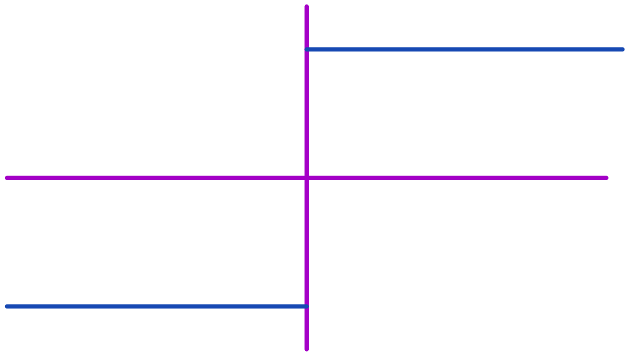
is 0-homogeneous

2) The Hilbert transform

$$Hf = \frac{1}{\pi} \cdot f * pv\left(\frac{1}{x}\right)$$

$(Hf)_r = H(f_r)$, 0-homogeneous

$f \in \mathcal{S}$ $\qquad \widehat{Hf}(\xi) = -i \underbrace{\operatorname{sgn}(\xi)}_{\text{multiplier}} \hat{f}(\xi)$



<u>Def:</u> A Fourier multiplier operator

$$\widehat{Tf}(\xi) = \underbrace{m(\xi)}_{\text{multiplier}} \hat{f}(\xi)$$

is k-homogeneous (or s-homogeneous, non-integer) and <u>elliptic</u> if

- $m(\lambda \xi) = \lambda^k m(\xi) \qquad \lambda > 0$
- $m(\xi) \neq 0$ for $\xi \neq 0$.

<u>Thm</u> (elliptic regularity for homogeneous Fourier multipliers)

Let T be a k-homogeneous, elliptic, Fourier multiplier. Then

1) <u>Sobolev case:</u>

$\forall u$

$$Tu \in H^s \implies u \in H^{k+s}$$

2) <u>Hölder case:</u> $\forall u$

$$Tu \in C^s \implies u \in C^{k,s}_{loc}$$

$$\boxed{\begin{array}{l} 3) \quad Tf = \dfrac{\partial f}{\partial z} \quad \text{or} \quad Tf = \dfrac{\partial f}{\partial \bar{z}} \\ \text{then the multiplier is} \\ \qquad m(\underbrace{\xi_1, \xi_2}_{\mathbb{R}^2}) = c \cdot (\xi_1 \pm i \xi_2) \\ \text{elliptic.} \end{array}}$$

· This is a generalization (exercise!) of the thm for Laplacian and Hilbert transform using:

<u>Basic lemma:</u> $\beta \in \mathcal{S}$, $\beta(0) = 0$.

$\beta_k(\xi) = \beta(2^{-k}\xi)$. Then

$$\| \beta_k(D) f \|_\infty \lesssim 2^{-ks} \|f\|_{C^s}$$

implied cons[t]s depend on β.

Used it for $\beta(\xi) = \dfrac{\xi^\alpha}{m(\xi)} \psi(\xi)$

$|\alpha| = k$, m is k-homogeneous.

Hilbert Transform

- Why ever convolve with $pv\left(\frac{1}{x}\right)$?

$$Hf = \frac{1}{\pi}\, pv\left(\frac{1}{x}\right) * f$$

$$\widehat{Hf}(\xi) = -i \cdot sgn(\xi)\, \hat{f}(\xi)$$

Last time we proved:

1) $\forall f \in \mathcal{S}$,

$$\|Hf\|_{C^s} \lesssim \|f\|_{C^s} + \|f\|_{L^1}$$

Then H uniquely extends to a continuous linear operator $H: C^s \cap L^1 \longrightarrow C^s$

2) A formula: $\forall f \in C^s \cap L^1,\ x \in \mathbb{R}$

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|y| \geq \varepsilon} \frac{f(x-y)}{y}\, dy$$

$$\int_{|y-x| \geq \varepsilon} \frac{f(y)}{x-y}$$

3) $\|Hf\|_2 = \|f\|_2$ by Plancherel.

$$H^2 = -Id$$

Remark: For $1 < p < \infty$, $H: L^p \to L^p$ is well-defined and continuous.

$p = \infty$?
$$H(L^\infty) + L^\infty = BMO$$

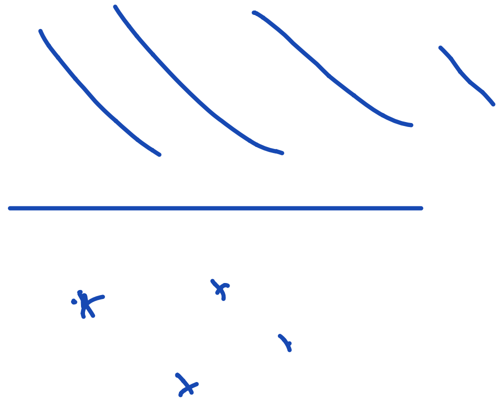
$$\sup_{I \subseteq \mathbb{R}} \iint_{I \times I} |f(x) - f(y)| \frac{dx}{|I|} \cdot \frac{dy}{|I|}$$

· Relation of Hilbert transform to the

Cauchy integral

- Suppose f holo. in $\{z \in \mathbb{C} ;\ \mathrm{Im}(z) \geq 0\}$ and $f(z) = O\left(\frac{1}{1+|z|}\right)$ as $z \to \infty$

E.g.



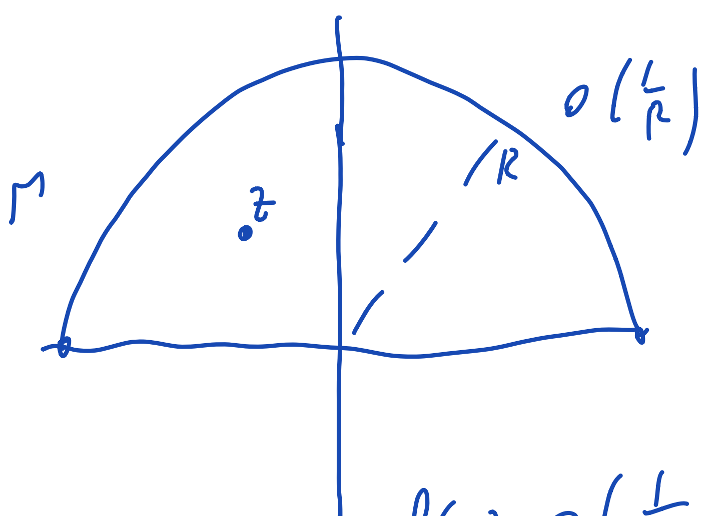
$\sum \frac{c_j}{(z-z_j)^{n_j}} e^{iz}$

Cauchy integral formula: Any such f satisfies, for $\mathrm{Im}\, z > 0$

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt$$

absolute convergence

Proof:



By Cauchy

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w-z}\, dw$$

$f(w) = O\left(\frac{1}{1+|w|}\right)$

$$= \frac{1}{2\pi i} \int_{-R}^{R} \frac{f(t)}{t-z}\, dt + O\left(\pi R \cdot \frac{1}{R^2}\right)$$

$$\xrightarrow{R\to\infty} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \underbrace{\frac{f(t)}{t-z}}_{O\left(\frac{1}{t^2}\right) \text{ at } \infty} dt \, .$$

absolute converge. ∎

<u>Def:</u> For $f \in L^1$ (or $f(t) = O\left(\frac{1}{1+|t|}\right)$ at ∞)

define its <u>Cauchy integrals</u> as

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt$$

for $z \in \text{UHP} = \{ z \in \mathbb{C} ; \operatorname{Im} z > 0 \}$

- It recovers f if f has a holo. extension to $\{\operatorname{Im} z \geq 0\}$ with decay $O\left(\frac{1}{1+|z|}\right)$ at ∞.
- It is <u>always</u> holomorphic in UHP.

Proposition (Cauchy integral through FT)

For $f \in L^1$ (or $f(t) = O\left(\frac{1}{1+|t|}\right)$),

then its Cauchy integral is

$$F(z) = \frac{1}{2\pi} \int_0^\infty e^{iz\cdot\xi} \hat{f}(\xi)\, d\xi$$

I.e., $F_y(x) = F(x+iy)$

$$F_y(x) = \frac{1}{2\pi} \int_0^\infty e^{ix\cdot\xi} \underbrace{e^{-y\cdot\xi} \hat{f}(\xi)}\, d\xi$$

i.e. F_y is the inverse Fourier transform of $\mathbb{1}_{[0,+\infty)}\, e^{-y\cdot\xi}\, \hat{f}(\xi)$.

Proof:

$$\int_0^\infty e^{-\alpha s}\, ds = \frac{1}{\alpha} \qquad \mathrm{Re}(\alpha) > 0$$

$z \in \mathbb{H}$, $t \in \mathbb{R}$

$$\int_0^\infty e^{i(z-t)\xi}\, d\xi = \frac{1}{-i(z-t)} = \frac{1}{i(t-z)}$$

$$\mathrm{Im}(z-t) = \mathrm{Im}(z) > 0$$

$$\operatorname{Re}\big(i(z-t)\big) < 0 \quad \checkmark$$

Therefore, the Cauchy integral is

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(t)}{i(t-z)}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{0}^{\infty} e^{i(z-t)\xi}\, d\xi \right) f(t)\, dt$$

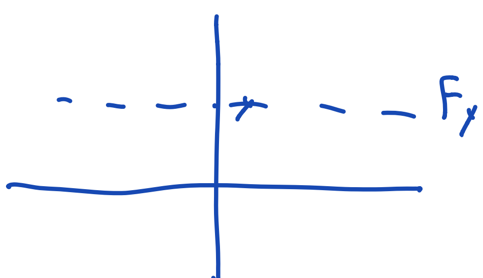
$$\overset{\text{Fubini}}{=} \frac{1}{2\pi} \int_{0}^{\infty} e^{iz\cdot\xi} \left(\int_{-\infty}^{\infty} e^{-it\xi} f(t)\, dt \right) d\xi$$

majorant

$$\left(e^{-\operatorname{Im}(t)|\xi|} f(t) \right)$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} e^{iz\cdot\xi}\, \hat{f}(\xi)\, d\xi. \qquad \blacksquare$$

Corollary: 1) $\widehat{F_y}(\xi) = \mathbb{1}_{[0,+\infty)}(\xi)\, e^{-y\cdot\xi}\, \hat{f}(\xi)$



$$= \widehat{\frac{f + iHf}{2}}(\xi)\, e^{-y\cdot|\xi|}$$

$$\mathbb{1}_{[0,\infty)}(\xi) = \frac{1 + i(-i \cdot \operatorname{sgn}(\xi))}{2}$$

$$\widehat{F_y}(\xi) = \widehat{\frac{f + iHf}{2}} \cdot \underbrace{e^{-y|\xi|}}_{\substack{\text{FT of the} \\ \text{Cauchy distribution} \\ x \mapsto \frac{1}{\pi}\frac{y}{x^2+y^2}}}$$

Hence

2) $$F_y = \frac{f + iHf}{2} * \underbrace{\left(\frac{1}{\pi}\frac{y}{x^2+y^2}\right)}_{\substack{\text{Poisson integral} \\ \text{Harmonic} \\ \text{extension.}}}$$

3) For $f \in L^1 \cap C^s$, the Cauchy integral is the Poisson integral (harmonic extension) of $\frac{f + iHf}{2}$ to UHP.

(hence $F_y \longrightarrow \frac{f+iHf}{2}$ pointwise in $\mathbb{R}$, as $y \to 0$).

<u>Corollary:</u> If f is holo. in $\{\operatorname{Im} z \geq 0\}$ and $f(z) = O\left(\frac{1}{1+|z|}\right)$ at ∞ then

$$H(\operatorname{Re} f) = \operatorname{Im} f$$

$$H(\operatorname{Im} f) = -\operatorname{Re} f$$

<u>Proof:</u> $f = F$ when F is the Cauchy integral of f. Therefore

$$\begin{tikzcd}
 & F_y(x) = F(x+iy) \arrow[dl] \arrow[dr, "y \to 0"] & \\
\frac{f + iHf}{2} & & f
\end{tikzcd}$$

say, convergence in L^1_{loc}

Hence $\boxed{f = \frac{f+iHf}{2}}$ on $\mathbb{R}$

$$\Updownarrow$$

$$iHf = f$$

$$\Downarrow$$

$$H(\operatorname{Re} f) = \operatorname{Im} f$$
$$H(\operatorname{Im} f) = -\operatorname{Re} f. \qquad \boxtimes$$

- Theory of boundary values of holomorphic f-ns.

- <u>End with a question:</u> Find a hidden $SL_2(\mathbb{R})$-symmetry of the Hilbert transform.

  <u>Hint:</u> Möbius map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ acts on UHP.

<u>Part III:</u> Introduction to ΨDO's,

<u>Pseudodifferential operators</u>

- We've dealt with Fourier multiplier, mostly:

- Laplacian, $\frac{\partial}{\partial x_i}$ — differential operators, local.
- Hilbert transform: non-local
- Smoothing properties — related.
- Large class of operators, including differential operator, approx. closed under composition, inversion, conjugation.

References:
- Stein: harmonic analysis real variable methods
- Muscalu-Schlag: Classical and multilinear harmonic analysis.

- A general differential operator of order k, is

$$\forall \varphi \in \mathcal{S}, \quad L\varphi(x) = \sum_{|\alpha| \leq k} a_\alpha(x) \, \overbrace{\partial^\alpha}^{-i\partial^\alpha = \xi^\alpha} \varphi(x)$$

- How does it act in frequency space?

$$L\varphi(x) \overset{(2\pi)^{-n}}{=} \sum_{|\alpha| \le k} a_\alpha(x) \int_{\mathbb{R}^n} \xi^\alpha i^{|\alpha|} e^{i\xi\cdot x} \hat{\varphi}(\xi)\, d\xi$$

$$L\varphi(x) \overset{(2\pi)^{-n}}{=} \int_{\mathbb{R}^n} \underbrace{\left(\sum_{|\alpha| \le k} a_\alpha(x) i^{|\alpha|} \xi^\alpha \right)}_{a(x,\xi)} \hat{\varphi}(\xi) e^{i\xi\cdot x}\, d\xi$$

- The function $a(x, \xi)$

"$a : T^*M \to \mathbb{C}$"

$x \in \mathbb{R}^n$: position, time

$\xi \in (\mathbb{R}^n)^*$: frequency, momentum, Fourier variable

- is called "the symbol of the differential operator",

$$a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$$

(lives in the cotangent bundle)

$$a(x, \xi) = \sum_{|\alpha| \le k} a_\alpha(x)\, \xi^\alpha (-i)^{|\alpha|}$$

for a diff. operator of order k, the symbol is polynomial of degree k in ξ.

<u>Def:</u> Let $a(x,\xi)$ be a smooth function of $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, of moderate growth in $\mathbb{R}^{2n}$, called a **symbol**. For $\varphi \in \mathcal{S}$ define $a(x,D)\varphi : \mathbb{R}^n \to \mathbb{C}$ as follows:

$$\left(a(x,D)\varphi\right)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot x}\, \hat{\varphi}(\xi)\, d\xi$$

- since $a(x,\xi)$ is of moderate growth, the integral converges, as $\hat{\varphi} \in \mathcal{S}$.
- Differentiating under the integral sign, we see $a(x,D)\varphi$ is a smooth

function.

(Later: $a(x,D)\,\varphi \in \mathcal{S}'$).

- Any diff. operator with smooth of moderate growth is a ΨDO.

<u>Remark:</u> · If $a(x,\xi)$ grows like $|\xi|^k$ in the ξ-variable, then $a(x,D)$ is a bit like a diff. operator of order k.

<u>Notation:</u> $a(x,D) = Op[a] =$ standard quantization of the symbol a.

$$a(x,D)\varphi = \boxed{(2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot x}\, \hat{f}(\xi)\, d\xi}$$

<u>Vague, intuitive remarks</u>

1) Littlewood - Paley

$a(\xi)$, cut into pieces $|\xi| \sim 2^k$.

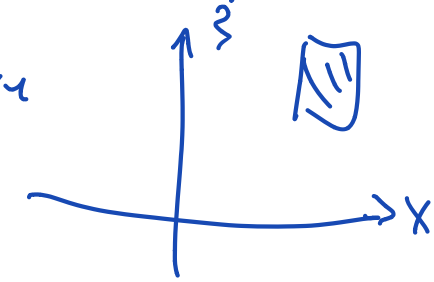
ΨDO:

$$a(x,\xi) = \sum_k a_k(x,\xi)$$

where $\mathrm{Supp}\; a_k(x,\xi) \subseteq B(x_0, 1) \times B(\xi_0, 1)$

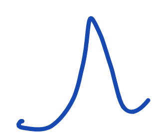
time - frequency space

phase spaces



2) The effect of $a(x,D)\varphi$?

Roughly

- Multiply φ by cutoff f-n near x
- Compute Fourier transform 
- Multiply by $\xi \mapsto a(x,\xi)$
- Apply inverse Fourier transform.

3) Ways to take $f(x)$ position-space

and transform it into $\tilde{f}(x,\xi)$

$\boxed{\text{wavelets}}$ localized F.T. $\uparrow_{\mathbb{R}^n}$ $\uparrow_{\mathbb{R}^n}$

wave-packet transform (Gabor transform)

$$\tilde{f}(x_0,\xi_0) = \int_{\mathbb{R}^n} f(x) e^{-i\xi_0\cdot x} e^{-\frac{(x-x_0)^2}{2}} dx$$

For some operators (elliptic), it's true

$$\widetilde{a(x,D)f} \approx a(x,\xi)\tilde{f}(x,\xi)$$

Def: (Order of a symbol)

We say that a symbol $a(x,\xi)$ belongs to S^m (the class of symbols of order $\leq m$) if $\boxed{m \in \mathbb{Z}}$

- $a(x,\xi) \lesssim (1+|\xi|^m)$

in the following precise sense:

$\forall \alpha,\beta \quad \exists A_{\alpha,\beta} \in (0,+\infty)$ s.t. $\forall x,\xi$

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leq A_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

- There are many symbol classes

$\left(\begin{array}{ccc} F & * & \varphi \\ \uparrow & & \uparrow \\ \xi^* & & \xi \end{array} \right.$ tempered function, with derivatives growing this way $\Big)$.

- Plain vanilla symbol class, people allow some growth in x, singularity at $\xi = 0$ (for us the Hilbert transform is excluded).

<u>Examples</u>:

- The symbol of a reasonable diff. operator of order m is in S^m.

- $\mathrm{sgn}(\xi)$ wants to be a symbol of order 0, and there are variants of S^0 which allow it.

Thm (Calderon - Vaillancourt)

A ΨDO of order 0 (i.e., symbol in S^0) is a bounded operator in L^2, i.e. $\exists M > 0$, $\forall \varphi \in \mathcal{S}$

$$\| a(x,D)\varphi \|_2 \le M \cdot \|\varphi\|_2.$$

Remarks: A bit like Plancherel for multipliers:

If $a(x,\xi) = a(\xi)$

then $a(x,D)$ is a Fourier multiplier

$a \in S^0 \Rightarrow \|a\|_\infty < \infty$

so then by Plancherel:

$$\| \underset{a(D)}{a(x,D)} \varphi \|_2 = (2\pi)^{-n/2} \| a(\xi)\hat{\varphi}(\xi)\|_2$$

$$\le \sup|a| \cdot \|\varphi\|_2$$

Different frequencies are orthogonal by Plancherel.

$$\| a(D)\varphi \| = (2\pi)^{-n/2} \| a(\xi)\hat{\varphi}(\xi) \|_2$$

$\forall \varphi_1, \varphi_2 \in \mathcal{S}$

- $\mathrm{Supp}(\hat{\varphi}_1) \cap \mathrm{Supp}(\hat{\varphi}_2) = \emptyset$

$$\Downarrow$$

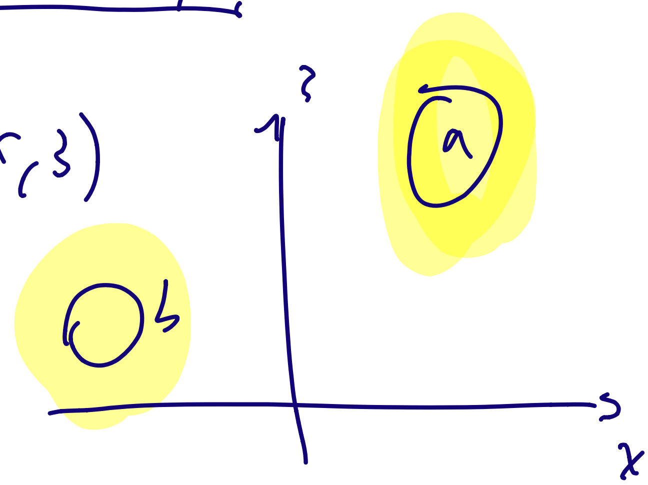
$$\varphi_1 \perp \varphi_2$$

orthogonality of diff. frequencies

- In proving Calderon-Vaillancourt we will use cut-off functions, and:

Almost orthogonality

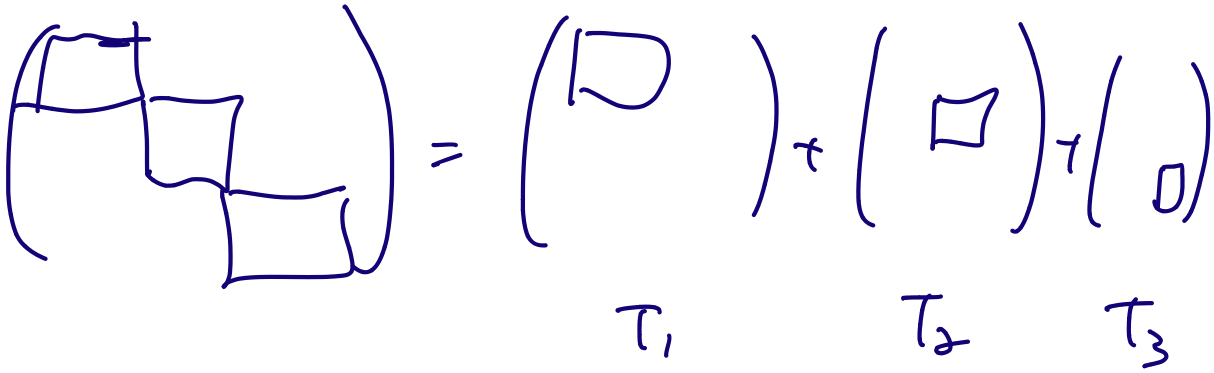
$a(x,\xi)$ and $b(x,\xi)$ disjoint support in phase space



$a(x,D)\varphi$ and $b(x,D)\varphi$ are

somewhat orthogonal.

<u>Example</u> Linear algebra, matrices

$T =$ 

- $\mathrm{Im}(T_i) \perp \mathrm{Im}(T_j)$ for $i \neq j$

$$\Updownarrow$$

$$\mathrm{Im}(T_i) \subseteq \mathrm{Im}(T_j)^{\perp} = \ker(T_j^*)$$

$$\Updownarrow$$

$$T_j^* T_i = 0$$

- In this case

$$\|T\|_{2\to 2} = \max_j \|T_j\|_{2\to 2}$$

<u>Lemma</u> (Cotlar - Stein)

Let $T_1, \dots T_N$ be bdd linear operators on a Hilbert space with $\forall j, k$

$$\| T_j^* T_k \| \leq \gamma^2 (j-k)$$

$$\| T_j T_k^* \| \leq \gamma^2 (j-k)$$

for some $\gamma(j)$, $j \in \mathbb{Z}$. Then

$$\left\| \sum_{j=1}^{N} T_j \right\| \leq \sum_{j \in \mathbb{Z}} \gamma(j)$$

Reminder of spectral theory: Any self-adjoint operator is unitary-equivalent to a multiplication operator:

$$Sf(x) = m(x) \cdot f(x) \quad \text{in } L^2(\mu)$$

a bdd, real-valued multipliers.

Hence $\cdot \ \|S\| = \sup |m|$

$\cdot \ \|S^n\| = \|S\|^n$.

- $\|S\| = \sup_{\|x\|\le 1} \|Sx\| = \sup_{\|x\|\le 1} \langle Sx, x\rangle$

for any operator $\qquad$ self-adjoint.

- In particular, T^*T is always self adjoint, so

- $\|(T^*T)^n\| = \|T^*T\|^n$

- $\|T^*T\| = \sup_{\|x\|\le 1} \langle T^*Tx, x\rangle = \sup_{\|x\|\le 1} \|Tx\|^2$

$= \|T\|^2.$

<u>Proof of Cotlar-Stein</u>: Take large n.

write
$$T = \sum_{i=1}^{N} T_i$$

$$(T^*T)^n = T^*T\,T^*T \cdots T^*T$$

$$= \left(\sum T_i^\dagger\right)\left(\sum T_j\right)\cdots$$

$$= \sum_{\substack{j_1,\dots,j_n=1\\ k_1,\dots,k_n=1}}^{N} \underbrace{(T_{j_1}^\dagger (T_{k_1})(T_{j_2}^\dagger)(T_{k_2})\cdots) T_{j_n}^\dagger T_{k_n}}_{\text{two bounds for each summand}}$$

(I): $\leq \|T_{j_1}^\dagger T_{k_1}\| \cdot \|T_{j_2}^\dagger T_{k_2}\| - \|T_{j_n}^\dagger T_{k_n}\|$

$\underset{\gamma(j_1-k_1)}{\leq}$ $\quad$ $\underset{\gamma(j_2-k_2)}{\leq}$

(II) $\leq \|T_{j_1}^\dagger\| \cdot \|T_{k_1}^\dagger T_{j_2}\| \cdot \|T_{k_2}^\dagger T_{j_3}\| \cdots$

$\leq \gamma(k_1 - j_1)$

$$\sqrt{\|T_{j_1}^\dagger T_{j_1}\|} \cdots \cdot \|T_{k_{n-1}}^\dagger T_{j_n}\| \cdot \|T_{k_n}\|$$

$\underset{\gamma(0)}{\leq}$ $\quad$ $\underset{\gamma(0)}{\leq}$

geometric average of bounds

Hence,

$$\|(T^\dagger T)^n\| \leq \sum_{\substack{j_1,\dots,j_n=1\\ k_1,\dots,k_n=1}}^{N} \gamma(j_1-k_1)\gamma(j_2-k_2)\cdots\gamma(j_n-k_n) \cdot$$

$$\cdot \gamma(0) \cdot \gamma(k_1-j_2)\cdots \gamma(k_2-j_3)\cdots\gamma(k_{n-1}-j_n)$$

Rewrite: So far, we proved

$$\|(T^*T)^n\| \leq$$

$$\sum_{\substack{j_1,\dots,j_n=1 \\ k_1,\dots,k_n=1}}^{N} \gamma(0)\,\gamma(\hat{j}_1-k_1)\,\gamma(k_1-\hat{j}_2)\,\gamma(\hat{j}_2-k_2)\,\gamma(k_2-\hat{j}_3)\cdots\gamma(k_{n-1}-\hat{j}_n).$$

Next week - to be continued