29/6/20

Last time: Given a smooth $a(x,\xi)$ ($x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$) of moderate growth, define a ΨDO (= pseudo-diff. operator)

$$a(x,D) = Op[a] = T_a$$

For $\varphi \in \mathcal{S}$

$$(T_a \varphi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi) e^{i\xi \cdot x} \hat{\varphi}(\xi) d\xi$$

- $T_a \varphi$ is always C^∞-smooth. (Actually a Schwartz function)

Example: 1) $a(x,\xi) = a(\xi)$. Here $a(x,D) = a(D)$ is just a Fourier-multiplier:

$$a(x,D)\varphi = a(D) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(\xi) e^{i\xi \cdot x} \hat{\varphi}(\xi) d\xi$$

$$\widehat{a(D)\varphi}(\xi) = a(\xi)\,\hat{\varphi}(\xi)$$

· A ΨDO is roughly "a multiplier depending also on ξ".

2) Differential operator: $a(x,\xi)$ is a poly. in ξ.

Def: (Symbol class S^m): For $m \in \mathbb{Z}$

$a(x,\xi) \in S^m$ if

$$\forall_{\alpha,\beta} \quad \left| \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right| \leq A_{\alpha,\beta} \left(1+|\xi|\right)^{m-|\beta|}$$

when $A_{\alpha,\beta} > 0$ depending only on α, β.

Example: If $L = \sum_{|\alpha| \leq m} a_\alpha(x)\, \partial^\alpha$

and a_α are "nice enough" functions,

we get a differential operator of order m that belongs to S^m.

· Today: Only the case $m=0$.
ΨDO of order 0.

I.e., $a(x,\xi)$ is bdd.

· We don't "excite" high freq., so we expect to preserve smoothness. We expect analytic properties like the Hilbert transform

(today) · $a(x,D)$ should be bdd in $L^2(\mathbb{R}^n)$

($\Downarrow$ ..) · — " — — " — in $H^k(\mathbb{R}^n)$, Sobolev

(next week or HW) · bdd from $L^\infty \cap C^s$ to C^s [Hölder norms) for $0 < s < 1$.

(not in this class) · bdd in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$

Thm (Calderon–Vaillancourt)

If $a \in S^0$ then $a(x,D)$ is extended extended to a bdd linear operator in

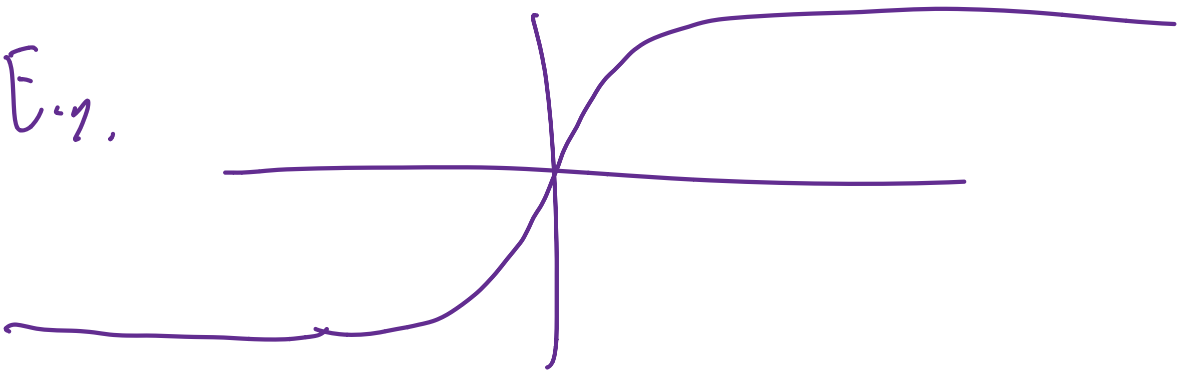
$$L^2(\mathbb{R}^n)$$

- In fact, rather than $a \in S^0$ enough to assume that

$$\partial_x^\alpha a(x,\xi), \quad \partial_\xi^\alpha a(x,\xi)$$

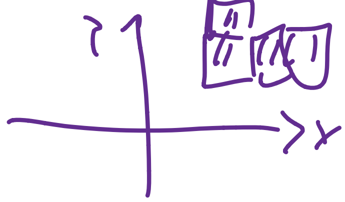
bdd for any $|\alpha| \le 2n+1$.

E.g.



$\in S^0$

- Proof strategy: cut a ΨDO into "local pieces" in phase space



Lemma 1 (Cotlar-Stein, "almost orthogonality")

Let $I \subseteq \mathbb{Z}^d$ be a finite set of indices, $(T_j)_{j \in I}$ are bdd, linear operators on a Hilbert space. Assume that $\forall j,k \in \mathbb{Z}^d$

$$\| T_j^* T_k \| \le \gamma^2(j-k)$$

$$\| T_j T_k^* \| \le \gamma^2(j-k)$$

for some $\gamma : \mathbb{Z}^d \to (0,\infty)$. Then

$$\Big\| \sum_{j \in I} T_j \Big\| \le \sum_{j \in \mathbb{Z}^d} \gamma(j).$$

Proof: Set $T = \sum_{j \in I} T_j$.

$$\|T\|^2 = \|\underbrace{T^* T}_{\text{self-adjoint}}\|, \quad \|(T^*T)^n\| = \|T^*T\|^n.$$

We stopped at: Fix $n \ge 1$.

$$(T^*T)^n = \sum_{\substack{j_1 \ldots j_n \in I \\ k_1 \ldots k_n \in I}} (T_{j_1}^* T_{k_1})(T_{j_2}^* T_{k_2} \ldots) T_{j_n}^* T_{k_n}$$

$$\|(T^*T)^n\| \le \sum_{\substack{j_1, \ldots, j_n \in I \\ k_1 \ldots k_n \in I}} \gamma(0)\, \gamma(j_1 - k_1)\gamma(k_1 - j_2)\, \gamma(j_2 - k_2) \ldots \gamma(j_n - k_n)$$

$$\le \gamma(0) \sum \qquad \approx \; \sum_{j_n \in \mathbb{Z}^d} \gamma(k_{n-1} - j_n) \sum_{k_n \in \mathbb{Z}^d} \gamma(j_n - k_n)$$

$$\le \gamma(0) \cdot A^{2n-1}$$

where $A = \sum_{j \in \mathbb{Z}^d} \gamma(j)$

Hence

$$\|T\| = \|(T^*T)^n\|^{1/2n} \le$$

$$\le \left(\gamma(0) \cdot A^{2n-1}\right)^{\frac{1}{2n}} \xrightarrow{n \to \infty} A.$$

∎

· The next lemma is about <u>integral operators</u>, i.e, operators of the form

$$Tf(x) = \int \underbrace{K(x,y)}_{\text{the "kernel" of the integral operator}} f(y)\, d\mu(y)$$

where we assume

$$A := \sup_y \int |K(x,y)|\, d\mu(x) < \infty$$

$$B := \sup_x \int |K(x,y)|\, d\mu(y) < \infty$$

<u>Lemma 2</u> ("Schur's test")

The integral operator is well-defined on L^p, and $\forall\ 1 \le p \le \infty$

$$\| T \|_{p \to p} \le A^{\frac{1}{p}}\, B^{1-\frac{1}{p}} \le \max\{A, B\}$$

<u>Proof:</u> Recall Hölder's inequality

$\forall\, 0<\lambda<1 \quad \int f^{\lambda} g^{1-\lambda} \le \left(\int |f|\right)^{\lambda} \left(\int |g|\right)^{1-\lambda}$

Now, for any p, for any x

$$|Tf(x)| = \left|\int K(x,y) f(y)\, d\mu(y)\right|$$

$$\le \int |K(x,y)|^{1-\frac{1}{p}} |K(x,y)|^{\frac{1}{p}} |f(y)|\, d\mu(y)$$

$$\le \left(\int |K(x,y)|\, d\mu(y)\right)^{1-\frac{1}{p}} \left(\int |K(x,y)|\, |f(y)|^{p}\right)^{\frac{1}{p}}$$

$$\le B^{1-\frac{1}{p}} \left(\int |K(x,y)| \cdot |f(y)|^{p}\, d\mu(y)\right)^{1/p}$$

Hence,

$$\int |Tf(x)|^{p}\, d\mu(x)$$

$$\le \left(B^{1-\frac{1}{p}}\right)^{p} \cdot \int\int |K(x,y)|\, |f(y)|^{p}\, d\mu(x)\, d\mu(y)$$

$$\le \left(B^{1-\frac{1}{p}}\right)^{p} \cdot A \cdot \int |f(y)|^{p}\, d\mu(y)$$

Then $\|Tf\|_p \le B^{1-\frac{1}{p}} \cdot A^{\frac{1}{p}} \|f\|_p$.

$\square$

- Integral operators are like matrix multiplication

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy$$

like, for $A \in \mathbb{C}^{n \times n}$,

$$(Av)_i = \sum A_{ij} V_j$$

How do we compose integral operators?

$$Tf(x) = \int A(x,y) f(y)\, dy$$

$$Sf(x) = \int B(x,y) f(y)\, dy$$

then $T \circ S$ has kernel

$$C(x,y) = \int A(x,z) B(z,y)\, dz$$

because

$$T(Sf)(x) = \int A(x,z)\, Sf(z)\, dz$$

$$= \int A(x,z) \int B(z,y)\, f(y)\, dy\, dz$$

$$= \int \underbrace{\left(\int A(x,z)\, B(z,y)\, dz \right)}_{\text{the kernel of the composition}} f(y)\, dy$$

<u>Conjugation:</u>

$$Tf(x) = \int K(x,y)\, f(y)\, dy$$

$$T^* f(x) = \int \underbrace{\overline{K(y,x)}}_{\text{the new kernel}} f(y)\, dy$$

$$\langle Tf, g \rangle = \langle f, T^* g \rangle$$

- These are compact operators, trace class usually.

Remark: A ΨDO is order 0 is a singular integral operator, or Calderon-Zygmund operator, i.e.

$$Tf(x) = \int K(x,y) f(y)\, dy$$

is valid only when $x \notin \overline{\mathrm{Supp}(f)}$

$$|K(x,y)| \sim \frac{1}{|x-y|^n}$$

a bit like p.v. integral, or like Hilbert operator.

· Our approach here: Suppose $a(x,\xi) \in S^m$ is compactly-supported in ξ and x.

Then $a(x,D)$ is a honest integral operator:

$\forall f \in S$

$$a(x,D)\varphi = (2\pi)^{-n}\int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot x}\,\hat{\varphi}(\xi)\,d\xi$$

$$= (2\pi)^{-n}\int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot x}\left(\int_{\mathbb{R}^n} e^{-i\xi\cdot y}\,\varphi(y)\,dy\right)d\xi$$

$\int |a(x,\xi)|\cdot|\varphi(y)|\,d\xi\,dy < \infty$

$$= \int_{\mathbb{R}^n} \underbrace{(2\pi)^{-n}\int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot(x-y)}\,d\xi}_{\substack{\shortparallel \\ K_a(x,y)}}\,\varphi(y)\,dy$$

Again,

$$a(x,D)\varphi(x) = \int K_a(x,y)\,\varphi(y)\,dy$$

when

$$K_a(x,y) = (2\pi)^{-n}\int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot(x-y)}\,d\xi$$

$$= (2\pi)^{-n}\,\hat{a}(x,y-x)$$

when $\hat{a}(x,\cdot)$ is the Fourier transform in ξ, the second variable.

. In this case, $a(x,\cdot) \in S'$

$$\Downarrow$$

$$\hat{a}(x,\cdot) \in S'$$

so $$\left|K_a(x,y)\right| \leq C_N \cdot \frac{1}{|x-y|^N} \quad \forall N$$

<u>Lemma 3:</u> ("small interaction between faraway regions in phase space")

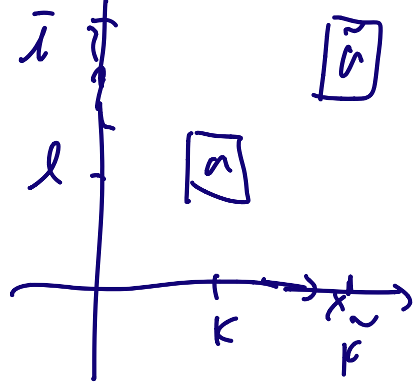
Let $a(x,\xi)$, $\tilde{a}(x,\xi)$ be two smooth compactly-supported functions ("symbols"). Say that for some $k, l, \tilde{k}, \tilde{l} \in \mathbb{Z}^n$ s.t.

$$\operatorname{Supp} a(x,\xi) \subseteq (k,l) + [-5,5]^n$$

$$\operatorname{Supp} \tilde{a}(x,\xi) \subseteq (\tilde{k},\tilde{l}) + [-5,5]^n$$

Consider the operators on S'

$$Tf(x) = \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot x} f(\xi)\, d\xi$$



$$\tilde T f(x) = \int_{\mathbb{R}^n} \tilde a(x,\xi)\, e^{i\xi\cdot x} f(\xi)\, d\xi$$

(no hat!) . Then $\forall N$

$$\|\tilde T^* T\|_{2\to 2} \le C_N \frac{\|a\|_{C^N}\cdot\|\tilde a\|_{C^N}}{(1+|k-\tilde k|)^N\,(1+|l-\tilde l|)^N}$$

$$\|\tilde T\, T^*\|_{2\to 2} \le \quad \text{—''—}$$

could have take $\chi_{|k-\tilde k|\le 10}$

where $\|a\|_{C^N} = \sup_{\substack{x\in\mathbb{R}^{2n}\\ |\alpha|\le N}} |\partial^\alpha a(x)|$.

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4. $a(x,\xi) = \delta_{x_0,\xi_0}$

$\hat a(x, x-y) = e^{i\xi_0\cdot(x-y)}\cdot\delta_{x_0}$

$\int \hat a(x, x-t)\, \hat b(t, t-y)\, dt = e^{i\xi_0\cdot(x-t)}\, e^{i\eta_0\cdot(t-y)}\, dt$

$K_a(x,t)$ $\quad K_b(t,y)$ $\quad t = x_0$ $\quad \eta_0 = \xi_0$

Proof: The kernel of T is

$$a(x,y)\, e^{iy\cdot x}$$

The kernel of $\tilde{T}$ is $\tilde{a}(x,y)\, e^{iy\cdot x}$

Hence the kernel of $\tilde{T}^* T$ is

$$[K](x,y) = \int_{\mathbb{R}^n} \overline{\tilde{a}(z,x)}\, e^{-iz\cdot x}\, a(z,y)\, e^{iy\cdot z}\, dz$$

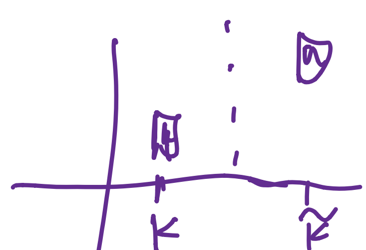
$$= \int_{\mathbb{R}^n} e^{iz\cdot(x-y)}\, \overline{\tilde{a}(z,x)}\, a(z,y)\, dz$$

- Recall: $\tilde{a}$ lives near $(\tilde{k}, \tilde{l})$
  a " " (k, l)

- Hence if $\|k - \tilde{k}\|_\infty \geq 20$ then

$$[K](x,y) \equiv 0 \quad \forall x,y, \text{ as}$$

$$\overline{\tilde{a}(z,x)}\, a(z,x) \equiv 0 \quad \forall x,x,z.$$



In all cases, the integral defining $[K](x,y)$ is only non-zero in some cube of sidelength 10,

$$(k,\ell) + [-5,5]^2$$

- Hence, by Schur's test,

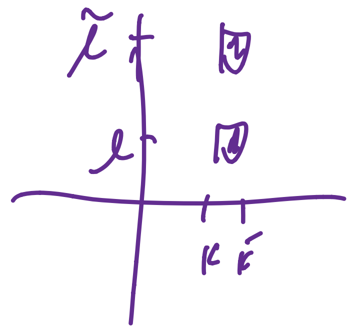
$$\|\tilde{T}^* T\|_{2\to 2} \le C \cdot \sup_{z,x,y} \left| \overline{\tilde{a}(z,x)}\, a(z,y) \right|$$

$$\le C \cdot \|a\|_{C^0} \cdot \|\tilde{a}\|_{C^0}$$

- Recall: we assume that

$$\|k - \tilde{k}\|_\infty \le 10$$

- Note we already proved the lemma if $\|\ell - \tilde{\ell}\|_\infty \le 10$



- It remains to consider the case $\|\ell - \tilde{\ell}\|_\infty \ge 10$.

$$[K](x,y) = \int_{\mathbb{R}^n} e^{i z\cdot(x-y)} \overline{\check{\sigma}(t,x)}\, a(t,y)\, dz$$

The integral is non-zero only when

$$\|x-\tilde{l}\|_\infty < 5, \qquad \|y-l\|_\infty \le 5$$

Hence

$$|x-y| \sim |l-\tilde{l}|$$

for $(x,y) \in \operatorname{supp}[K]$.

Our goal, by Schur's test: $\forall N$

$$\left|[K](x,y)\right| \le C_N \frac{\|a\|_{C^N} \cdot \|\check{a}\|_{C^N}}{|x-y|^N}$$

$$[K](x,y) = \int_{\mathbb{R}^n} e^{i z\cdot(x-y)} \overline{\check{\sigma}(t,x)}\, a(t,y)\, dz$$

- Decay of FT under strong smoothness

  How to integrate by parts

$$\pm i \frac{x-y}{|x-y|^2} \cdot \nabla_z e^{iz\cdot(x-y)} = e^{iz\cdot(x-y)}$$

$$[K](x,y) = \pm i \int \sum_{j=1}^{n} \frac{x_j - y_j}{|x-y|^2} \frac{\partial}{\partial z_j} e^{iz\cdot(x-y)} \cdot \left(\overline{\tilde{a}(z,x)}\, a(z,y) \right) dz$$

$$= \sum_{j=1}^{n} \frac{x_j - y_j}{|x-y|^2} (\mp i) \cdot \int_{\mathbb{R}^n} e^{iz\cdot(x-y)} \frac{\partial}{\partial z_j} \overline{\tilde{a}(z,x)}\, a(z,y)\, dz$$

Denote $c(z,x,y) = \overline{\tilde{a}(z,x)}\, a(z,y)$

$$= \dots \sum_{j_1 \dots j_N = 1}^{n} \frac{\prod_{k=1}^{N} (x_{j_k} - y_{j_k})}{|x-y|^{2N}} (\pm i) \cdot \int_{\mathbb{R}^n} e^{iz(x-y)} \underbrace{\frac{\partial^N}{\partial z_1 \dots \partial z_N} c(z,x,y)}\, dz$$

$$\leq \frac{1}{|x-y|^N}$$

· its z-support in in a cube of sidelength 2λ (λ?)

· its L^∞-norm is at most

$C_N \|a\|_{C^N} \cdot \|\tilde{a}\|_{C^N}$

Then

$$|[K](x,y)| \leq C_N \frac{1}{|x-y|^N} \|a\|_{C^N} \cdot \|\tilde{a}\|_{C^N}$$

if $\|x-\tilde{\ell}\|_\infty < 5$, $\|y-\ell\|_\infty < 5$.

By Schur's test

$$\int |[K](x,y)|\,dy \quad \text{and} \quad \int |[K](x,y)|\,dx$$

are all by

$$C_N \frac{1}{|x-y|^N} \|a\|_{C^N} \cdot \|\tilde{a}\|_{C^N}$$

So $$\|\tilde{T}^* T\|_{2\to 2} \leq C_N \frac{\|a\|_{C^N} \cdot \|\tilde{a}\|_{C^N}}{(1+|\ell-\tilde{\ell}|^N)}$$

and it vanishes if $\|k-\tilde{k}\| \geq 10$. $\square$

. What we actually proved:

<u>Claim:</u> If $K(x,y) = \int_{T \subset \mathbb{R}^n} e^{it(x-y)} c(x,y,t)\,dt$

then $\forall n$
$$|K(x,y)| \leq C_n \cdot |T| \cdot \frac{\|c\|_{C^n}}{|x-y|^n}.$$

<u>Proof of Calderon-Zygmund:</u> We need to show that $\forall \varphi \in \mathcal{S}'$,

$$\|a(x,D)\varphi\|_2 \leq C \|\varphi\|_2$$

. Since $a \in S^0$, we may assume that

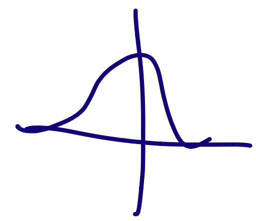
$$|\partial_x^\alpha a(x,\xi)| \leq 1 \qquad \forall |\alpha| \leq n+1$$
$$|\partial_\xi^\alpha a(x,\xi)| \leq 1$$

$$a(x,D)\varphi = (2\pi)^{-n} \int a(x,\xi)\, e^{i\xi \cdot x}\, \underset{\mathcal{S}' \ni}{\hat{\varphi}}(\xi)\, d\xi$$

- By Plancherel thm, $\varphi \mapsto (2\pi)^{-n/2}\hat{\varphi}$ is an L^2-isometry, so enough to prove L^2-boundness of

$$Tf(x) = \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot x} f(\xi)\, d\xi .$$

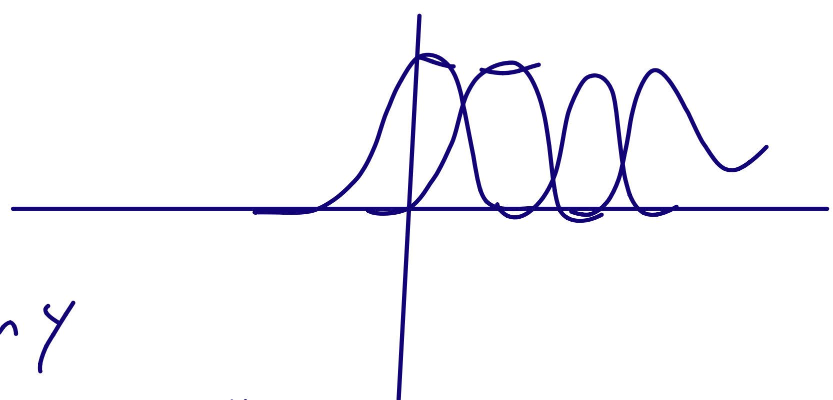
__Claim:__ $\exists\, \chi \in C_c^\infty(\mathbb{R}^n)$,



$\mathrm{Supp}(\chi) \subseteq [-2,2]^n$, with $0 \le \chi \le 1$

and

$$\sum_{k\in\mathbb{Z}^n} \chi(x-k) \equiv 1$$



Why? Take any bump function η with

$$\eta(x) = \begin{cases} 0 & x \notin [-2,2]^n \\ 1 & x \in [-1,1]^n \end{cases}$$

and

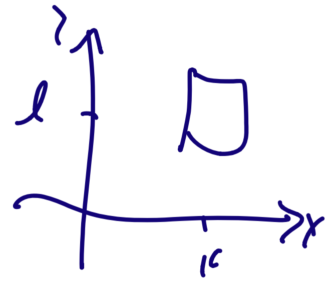
$$\chi = \frac{\eta(x)}{\underbrace{\sum_{k \in \mathbb{Z}} \eta(x-k)}_{\geq 1}} \in C_c^\infty(\mathbb{R}).$$

satisfies the requirement of the claim.

<u>Decompose the symbol:</u> For $k, l \in \mathbb{Z}^n$

$$\chi_{k,l}(x,\xi) = \chi(x-k)\,\chi(\xi - l)$$

$$\forall x \quad \sum_{k,l \in \mathbb{Z}^n} \chi_{k,l}(x) \equiv 1$$



$$a_{k,l}(x,\xi) = a(x,\xi)\,\chi_{k,l}(x,\xi)$$

Now, this is a compactly-supported, smooth function

- $\operatorname{supp} a_{k,l} \subseteq (k,l) + [-2,2]^n$
- $\| a_{k,l} \|_{C^{2n+1}} \leq C$

Define

$$T_{k,l} f(x) = \int_{\mathbb{R}^n} a_{k,l}(x,\xi) e^{i\xi\cdot x} f(\xi)\, d\xi$$

By Lemma 3, with $N = 2n+1$,

$\forall\, k, l, \tilde{k}, \tilde{l} \in \mathbb{Z}^n$

$$\| T^*_{\tilde{k},\tilde{l}} T_{k,l} \|_{2\to 2} \leq \frac{C}{(1+|k-\tilde{k}|)^{2n+1} \cdot (1+|l-\tilde{l}|)^{2n+1}}$$

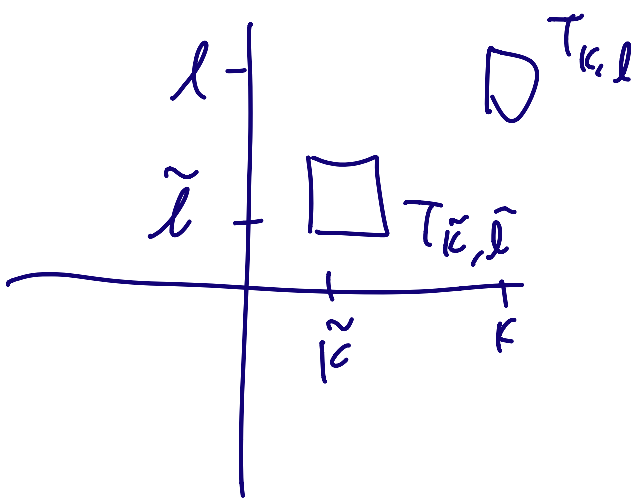
$$\| T_{\tilde{k},\tilde{l}} T^*_{k,l} \|_{2\to 2} \leq \quad - \text{''} -$$

Use Lemma 1, Cotlar-Stein,

with

$$\gamma(k,l) = \frac{1}{(1+|k|)^{n+\frac{1}{2}} (1+|l|)^{n+\frac{1}{2}}}$$

Then $\forall N$,



$$\underbrace{\Big\| \sum_{\substack{|k|<N \\ |l|<N}} T_{k,l} f \Big\|_2}_{T_N f} \le \underbrace{\sum_{k,l \in \mathbb{Z}^n} \frac{1}{(1+|k|)^{n+\frac12}(1+|l|)^{n+\frac12}}}_{= A \;<\; \infty} \|f\|_2$$

• Need to take $N \to \infty$. Fix $x \in \mathbb{R}^n$

$$T_N f(x) = \int_{\mathbb{R}^n} \Big(\sum_{\substack{|k|<N \\ |l|<N}} a_{k,l}(x,\xi) \Big) e^{i\xi\cdot x} f(\xi)\, d\xi$$

bdd in L^∞

$$\Big\downarrow \; N \to \infty$$

$|f|$ is the majorant

$$Tf(x) = \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot x} f(\xi)\, d\xi$$

Hence by Fatou's lemma

$$\int_{\mathbb{R}^n} |Tf|^2 \le \liminf_{N\to\infty} \int_{\mathbb{R}^n} |T_N f|^2 \le A \int |f|^2 .$$

$\blacksquare$

· L^2 boundness should imply Sobolev boundness

<u>Corollary</u> ("Sobolev regularity")

$a \in S^0 \implies \forall\, k \geq 0$

$a(x,D) : H^k \to H^k$ is bdd.

(no need $a \in S^0$, enough $\forall\, |\alpha| \leq 2n + k + 1$ $\partial_x^\alpha a$ and $\partial_\xi^\alpha a$ are bdd).

<u>Proof:</u> · $\|f\|_{H^k} \sim \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_2$

for $f \in S$.

$$T_a f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot x}\, \underset{\substack{\uparrow \\ S}}{\hat{f}(\xi)}\, d\xi$$

$\forall\, |\alpha| \leq k$,

$\partial_x^\alpha T_a f(x) =$

$$(2\pi)^{-n} \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} \int_{\mathbb{R}^n} \partial_x^\gamma a(x,\xi)\, (i\xi)^{\alpha-\gamma} e^{i\xi\cdot x} \cdot \hat{f}(\xi)\, d\xi$$

$$= (2\pi)^{-n} \sum_{\gamma} \cdot \pm i \underbrace{\int_{\mathbb{R}^n} \underset{\substack{\uparrow \\ S^0}}{\partial_x^\gamma a(x,\xi)}\, e^{i\xi\cdot x} \underbrace{\xi^{\alpha-\gamma} \hat{f}(\xi)}_{\widehat{\partial^{\alpha-\gamma} f}}\, d\xi}_{\text{This has } L^2\text{-norm at most}}$$

$$C \left\| \partial^{\alpha-\gamma} f \right\|_2 \le C \|f\|_{H^k}$$

by Calderon-Vaillancourt, as

$$|\alpha - \gamma| \le |\alpha| \le k$$

thus $\| \partial^\alpha (T_a f) \|_2 \le C \|f\|_{H^k}$. $\forall \alpha$, $|\alpha| \le k$

So. $$\|T_a f\|_{H^k} \lesssim \|f\|_{H^k}.$$

■

<u>Lemma:</u> For any $m \in \mathbb{Z}$, $a \in S^m$

$$a(x,D)\varphi : \mathcal{S} \longrightarrow \mathcal{S}'$$

is a continuous, linear operator.

<u>Proof:</u> We know $a \in S^m$, i.e.

$$\left|\partial_x^\alpha \partial_y^\beta a(x,y)\right| \lesssim (1+|y|)^{m-|\beta|}.$$

Since $f \mapsto \hat{f}$ is cont. in Schwartz topology, so enough to prove that

$$Tf(x) = \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot x} f(\xi)\, d\xi$$

is continuous in $\mathcal{S}$.

Why is $\|Tf\|_\infty \lesssim |f|_{2n+m+1}$?

$$|Tf(x)| \le \int_{\mathbb{R}^n} \underbrace{\frac{|a(x,y)|}{1+|y|^m}}_{\text{bdd}} \cdot \left|(1+|y|^n) f(y)\right| dy$$

its L^1 norm
is bdd by
$$\left|(1+|y|^m)\, f(y)\right|_{2n}$$
$$\lesssim |f|_{2n+m}$$

$$\Rightarrow \|Tf\|_\infty \lesssim |f|_{2n+m}$$

- Why is $\partial_x^\beta Tf(x)$ is bdd in L^∞ by $|f|_{2n+m+|\beta|}$

$$\partial^\beta Tf(x) = \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} \int_{\mathbb{R}^n} \partial_x^\gamma a(x,\xi)\,(i\xi)^{\alpha-\gamma} e^{i\xi\cdot x} \cdot f(\xi)\,d\xi$$

Each summand is handled the same way, divide by $(1+|\xi|)^{m+|\beta|}$ and multiply.

- Why is $x^\alpha \partial_x^\beta Tf(x)$ is bdd in L^∞ by $C\cdot |f|_{2n+m+|\alpha|+|\beta|}$?

$x^\alpha \partial_x^\beta Tf(x)$ is a sum of $2^{|\beta|}$ terms, each of the form

$\xi = y$

$$\int_{\mathbb{R}^n} x^\alpha \partial_x^\gamma a(x,\xi)\, \xi^{\beta-\gamma} e^{i\xi\cdot x} f(\xi)\, d\xi$$

$$= \pm i \int_{\mathbb{R}^n} \partial_x^\gamma a(x,\xi) \left(\partial_\xi^\alpha e^{i\xi\cdot x}\right) \xi^{\beta-\gamma} f(\xi)\, d\xi$$

Integrate by parts (in ξ) - get sum of terms of the form

$$= \int_{\mathbb{R}^n} \underbrace{\partial_\xi^{\gamma_1} \partial_x^{\gamma_1} a(x,\xi)}_{\leq (1+|\xi|)^C} \cdot e^{i\xi\cdot x} \xi^{\gamma_3} \partial^{\gamma_4} f(\xi)\, d\xi$$

Again multiply and divide by $(1+|\xi|)^C$ and we're done. $\blacksquare$

$S^{m,n}$ $\quad |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq (1+|\xi|)^{m-|\beta|} (1+|x|)^{m_2-|\alpha|}$