13/7/20

Last time: $a \in S^m$, $b \in S^l$, $m, l \in \mathbb{R}$

$$Op[a] \circ Op[b] = Op[c]$$

$c \in S^{m+l}$

Recall: • $Op[a] = a(x,D) = T_a$ is

$$a(x,D)\varphi = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\xi\cdot x}\, \hat{\varphi}(\xi)\, d\xi$$

• $a(x,\xi) \in S^m$ roughly means

" $|a(x,\xi)| \lesssim (1+|\xi|)^m$ "

plus condition on derivatives

Composition rule: Asymptotic expansion: $\forall N>0$

$$c(x,\xi) = a \# b\, (x,\xi)$$

$$= \sum_{|\alpha| < N} \frac{(-i)^{|\alpha|}}{\alpha!} \underbrace{\partial_\xi^\alpha a\, \partial_x^\alpha b}_{S^{m+l-|\alpha|}} + \underbrace{E}_{S^{m+l-N}}$$

The approx. is local in phase space

Rule for the adjoint: $a \in S^m$, define

$Op^*[a]$ via $\forall \varphi, \psi \in \mathcal{S}$

$$\langle Op^*[a]\varphi, \psi \rangle := \langle \varphi, Op[a]\psi \rangle$$

Now, $Op^*[a] = Op[a^*]$ where

$a^* \in S^m$ with asymptotic expansion

$$\forall N \quad a^*(x,\xi) = \sum_{|\alpha| < N} \frac{(-1)^{|\alpha|}}{\alpha!} \underbrace{\partial_\xi^\alpha \partial_x^\alpha \overline{a(x,\xi)}}_{S^{m-|\alpha|}} + \underbrace{E}_{S^{m-N}}$$

Def: ("Action of ψDO on $\mathcal{S}^*$")

For $a \in S^\infty = \bigcup_{m \in \mathbb{R}} S^m$ and $F \in \mathcal{S}^*$

define $a(x,D)F \in \mathcal{S}^*$ via $\forall \varphi \in \mathcal{S}$

$$\langle Op[a]F, \varphi \rangle := \langle F, Op^*[a]\varphi \rangle$$

(and $\langle F, \varphi \rangle = F(\bar{\varphi})$ "$=$" $\int F\bar{\varphi}$ ")

well-defined by continuity of a ΨDO in $\mathcal{S}$.

<u>Lemma:</u> Suppose $a \in S^{-\infty} = \bigcap_{m \in \mathbb{R}} S^m$

("infinitely smoothing"), $F \in \mathcal{S}'$

Then $a(x,D)F \in C^\infty$.

Last week: $a \in S^{-\infty}$ then

$Op[a](L^2) \subseteq C^\infty$.

<u>Direct proof:</u> $\left| \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right| \lesssim \frac{C_{N,\alpha,\beta}}{(1+|\xi|)^N}$

$\forall \alpha, \beta, N$.

I.e., $a(x,\cdot) \in \mathcal{S}$ in ξ-variable

Recall that $Op[a]$ is an <u>integral</u>
<u>operator</u> with kernel

$$K_a(x,y) = (2\pi)^{-n} \hat{a}(x, y-x)$$

where the F.T. is in ξ, second

variable. Now,

$\forall x$ $$K_a(x,\cdot) \in \mathcal{S}$$
and
$$\partial_x^\alpha K_a(x,\cdot) \in \mathcal{S}$$
with uniform estimates in x and α (growth at most poly. in x).

In the homework (#4), you are asked to prove that an integral operator with such kernels maps $\mathcal{S}^*$ to C^∞.
(like the proof that $\underset{\mathcal{S}^*}{F} * \underset{\mathcal{S}}{\varphi} \in C^\infty$).

$\blacksquare$

<u>Exercise:</u> If $a \in S^{-\infty}$ then T_a is a compact operator in $L^2(\mathbb{R}^n)$.
In particular, $\forall\, 0 \neq \lambda \in \mathbb{C}$
$$\ker(T_a - \lambda I) \text{ is finite-dim.}$$

- Next: ΨDOs are "pseudo-local", they do not destroy C^∞-smoothness.

<u>Proposition</u>: Let $a \in S^\infty$, $F \in \mathcal{S}'$, $\Omega \subseteq \mathbb{R}^n$ open. Assume $F|_\Omega \in C^\infty$ (i.e., it coincides with C^∞-function when acting on $\mathcal{S}$-functions supported in Ω).

Then
$$a(x,D)\,F\,\Big|_\Omega \in C^\infty .$$

<u>Proof:</u> Fix $x_0 \in \Omega$. Fix $\eta \in C_c^\infty(\Omega)$ with $\eta \equiv 1$ near x_0. Now

$$a(x,D)F = a(x,D)\underbrace{\left(\eta F\right)}_{\substack{\uparrow \\ C_c^\infty(\mathbb{R}^n) \\ \uparrow \\ \mathcal{S}}} + a(x,D)\left((1-\eta)F\right)$$

$\in S^{m}$

Need to prove that

$$a(x,D)\,[\underbrace{(1-\eta)}_{\psi} F]$$

is smooth near x_0.

$\cdot$ ψ vanishes near x_0.

Hence $\exists\, \varphi \in C_c^\infty$ with $\varphi \equiv 1$ near x_0,

$$\operatorname{Supp}(\varphi) \subseteq \{\eta \equiv 1\}$$

$$\overline{\operatorname{Supp}(\varphi)} \cap \overline{\operatorname{Supp}(\psi)} = \emptyset$$

Need to prove

$$\varphi\, a(x,D)\,[\psi F] \in C^\infty$$

Let us show that

$$(*) \qquad \varphi\, a(x,D)\, \psi \in S^{-\infty}$$

(i.e., a ΨDO corresponding to a symbol in $S^{-\infty}$). By the Lemma

$$\varphi\, a(x,D)\, \psi\, F \in C^\infty$$

Now, why $\forall m \in \mathbb{Z}$

$$\varphi \# a \# \psi \in S^m$$

because $\overline{\mathrm{Supp}(\varphi)} \cap \overline{\mathrm{Supp}(\psi)} = \emptyset$.

By the composition law

$$\underbrace{\varphi \# a \# \psi}_{(x,\xi)} = \sum \partial^\alpha \varphi\, \partial^\beta a\, \partial^\gamma \psi + \underset{\substack{\uparrow \\ S^m}}{E}$$

at the same point (x,ξ)

$$\equiv 0$$

Hence $\varphi \# a \# \psi \in S^{-\infty}$. ∎

Def: ("singular support of $F \in S'$")

For $F \in S'$, $x \in \mathbb{R}^n$, we say that $x \notin \mathrm{Singular}(F)$

if F coincides with a C^∞-function in some nbhd of x.

Example: $\mathrm{Singular}\left(\mathrm{pv}\left(\frac{1}{x}\right)\right) = \{0\}$

Corollary: ("A ΨDO is pseudo-local")
$\forall a \in S^\infty$, $F \in \mathcal{S}'$

$$\mathrm{Singular}(T_a F) \subseteq \mathrm{Singular}(F).$$

Example:

$$F = \int_{S^{n-1}} a(x) \, (\partial_r)^{17}$$

Then

$a(x,D)F\big|_{\mathbb{R}^n \setminus S^{n-1}}$ is in $C^\infty(\mathbb{R}^n - S^{n-1})$.

Pseudo-inverses of elliptic operators

Def: $a \in S^m$, $m \in \mathbb{R}$, is called elliptic of order m if $\exists C > 0$ s.t.

$$|a(x,\xi)| \geq \frac{1}{C} |\xi|^m \quad \text{for } |\xi| > C.$$

· it grows at maximal speed at ∞, under constraint $a \in S^m$.

Example: Δ has symbol $-|\xi|^2$, it is elliptic of order 2.

Main property: $a \in S^m$ elliptic of order m

Then
$$\frac{\varphi(\xi)}{a(x,\xi)} \in S^{-m}$$

where $\varphi \in C^\infty(\mathbb{R}^n)$

$$\varphi(\xi) = \begin{cases} 1 & |\xi| > 2C \\ 0 & |\xi| \leq C \end{cases}$$

Why? need

$$\left| \partial_x^\alpha \partial_\xi^\beta \frac{\varphi(\xi)}{a(x,\xi)} \right| \lesssim \frac{1}{(1+|\xi|)^{m+|\beta|}}$$

- $\alpha = \beta = 0$ - ✓
- α, β arbitrary.
- $\frac{\partial}{\partial \xi_i}$ gives a factor $- \frac{\partial a / \partial \xi_i}{a}$ $\overset{\leq |\xi|^{m-1}}{\underset{\geq |\xi|^m}{}}$

  so $\left| \frac{\partial a / \partial \xi_i}{a} \right| \lesssim \frac{1}{|\xi|}$
- $\frac{\partial}{\partial x_i}$ gives $\frac{\partial_{x_i} a}{a}$ $\overset{\leq |\xi|^m}{\underset{\geq |\xi|^m}{}}$, bdd.

and it works, $\frac{\varphi(\xi)}{a(x,\xi)} \in S^{-m}$.

<u>Example:</u> Quantum harmonic oscillator is

$$Lu = -\Delta u + \frac{|x|^2}{4} u(x)$$

its symbol is $a(x,\xi) = |\xi|^2 + \frac{|x|^2}{4}$ elliptic, a discrete set of eigenvalues

$$\frac{n}{2},\ \frac{n}{2}+1,\ \frac{n}{2}+2, \ldots$$

$\left(-\Delta + \left(|\nabla\varphi|^2 - \Delta\varphi \right) \text{ has zero minimal eigenvalue, with eigenfunction } e^{-\varphi} \right)$

$-\Delta + \frac{|x|^2}{4} - E$ is not invertible

if $E \in \left\{ \frac{n}{2}, \frac{n}{2}+1, \ldots \right\}$

<u>Thm</u> (A "parametrix", pseudo-inverse for elliptic operator)

Let $a \in S^m$ elliptic of order m, $m \in \mathbb{Z}$. Then $\exists\, b \in S^{-m}$ s.t.

$$a \# b = 1 + \underset{S^{-\infty}}{\underset{\uparrow}{E}}$$

$$b \# a = 1 \mod S^{-\infty}.$$

· "b is almost an inverse". Will be used:

1) Kernel of a is finite-dimensional

2) to prove that u is m derivatives smoother than $T_a u$.

<u>Proof:</u> First we construct

$$b_j \in S^{-m-j} \qquad j=0,1,\dots$$

s.t.

$$a \# \sum_{j=0}^{N-1} b_j = 1 \mod S^{-N}$$

· What should be b_0? Try

$$b_0(x,\xi) = \frac{\varphi(\xi)}{a(x,\xi)}$$

where $\underset{C^\infty}{\varphi(\xi)} = \begin{cases} 0 & |\xi| \le C \\ 1 & |\xi| > 2C \end{cases}.$

<u>Know:</u> 1) $b_0 \in S^{-m}$ ✓

2) $\underset{S^m}{a} \# \underset{S^{-m}}{b_0}(x,\xi) = \underset{S^0}{a(x,\xi) b_0(x,\xi)} + \underset{S^{-1}}{E}$

i.e.,

$$a \# b_0 = a b_0 \mod S^{-1}$$

$$= \varphi(\xi) \mod S^{-1}$$

$$= 1 + \underbrace{(\varphi - 1)}_{\substack{\text{compactly-supp in } \xi, \\ \text{independent in } x \\ \varphi - 1 \in S^{-\infty} \subseteq S^{-1}}} \mod S^{-1}$$

Hence $a \# b_0 = 1 + \underset{S^{-1}}{E}$.

Next, by recursion, set

$$b_N = \frac{1 - \sum_{j=1}^{N-1} a \# b_j}{a} \varphi$$

$$= \underbrace{b_0}_{S^{-m}} \underbrace{\left(1 - \sum_{j=1}^{N-1} a \# b_j\right)}_{S^{-N}}$$

by induction hypothesis

Hence $b_N \in S^{-m-N}$,

and

$$a \# \sum_{j=0}^{N} b_j = a \# b_N + a \# \sum_{j=0}^{N-1} b_j$$

$$= a \# \sum_{j=0}^{N-1} b_j + a \cdot b_N \mod S^{-N-1}$$

$$= a \# \sum_{j=0}^{N-1} b_j + \left(1 - \sum_{j=1}^{N-1} a \# b_j\right)\varphi \mod S^{-N-1}$$

$$= a \# \sum_{j=0}^{N-1} b_j + \left(1 - \sum_{j=1}^{N-1} a \# b_j\right) \mod S^{-N-1}$$

(as $\varphi - 1$ is compactly-supp)

$$= 1 \mod S^{-N-1}.$$

- Next, we need $b \in S^{-m}$ s.t.

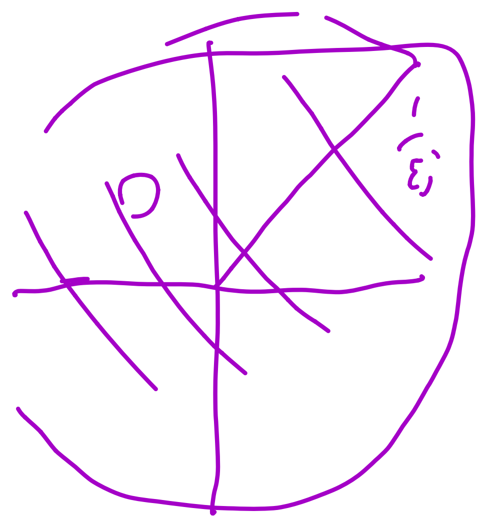
$$\forall N \quad b = \sum_{j=0}^{N-1} b_j \mod S^{-m-N}$$

(Usual notation: $b \sim \sum_{j=0}^{\infty} b_j$)
asymptotic series

- The idea: Find $\varepsilon_j \searrow 0$ fast enough such that

$$(*) \quad b(x,\xi) := \sum_{j=0}^{\infty} b_j(x,\xi)\, \varphi(\varepsilon_j \xi)$$

is the desired symbol.



$\varphi(\varepsilon_j \xi)$

How to determine ε_j?

Need: $\forall |\alpha|, |\beta| \le j$, $\varepsilon_j < \frac{1}{j}$.

$$\left| \partial_x^\alpha \partial_\xi^\beta \left(b_j(x,\xi)\,\varphi(\varepsilon_j \xi) \right) \right| \le \frac{2^{-j}}{(1+|\xi|)^{m+j-|\beta|-1}}$$

Already know: $b_j \in S^{-m-j}$

$$\left| \partial_x^\alpha \partial_\xi^\beta b_j(x,\xi) \right| \le \frac{C_{j,\alpha,\beta}}{(1+|\xi|)^{m+j-|\beta|}}$$

So it is possible to find these ε_j.

Claim: 1) $b \in S^{-m}$

2) $b - \sum_{j=0}^{k-1} b_j \in S^{-m-k}$

- The sum $(*)$ defining b is a finite sum near any (x,ξ), hence $b \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.
- Enough to prove 2).

Abbreviate $\varphi_j(\xi) = \varphi(\varepsilon_j \xi)$.

Fix $k \ge 1$. Then $\forall \alpha, \beta$

$$\left|\partial_x^\alpha \partial_\xi^\beta \sum_{j\ge k+1} b_j \varphi_j\right| \le \sum_{j=k+1}^{\infty} \frac{2^{-j}}{(1+|\xi|)^{m+j-|\beta|-1}}$$

$$\le \frac{1}{(1+|\xi|)^{m+k-|\beta|}}.$$

Hence $\sum_{j\ge k+1} b_j\varphi_j \in S^{-m-k}$.

We now decompose

$$b - \sum_{j=0}^{k-1} b_j = \underbrace{\sum_{j=0}^{k-1} (1-\varphi(\varepsilon_j;\xi))\, b_j}_{\in S^{-\infty}} + \underbrace{\overset{S^{-m-k}}{\overset{\cup\!\!\!|}{b_k}}\, \overset{S^{0}}{\overset{\cup\!\!\!|}{\varphi_k}}}_{\in S^{-m-k}} + \underbrace{\sum_{j\ge k+1} b_j\varphi_j}_{\in S^{-m-k}}$$

compact support in ξ.

Hence $b - \sum_{j=0}^{k-1} b_j \in S^{-m-k}$, proving 2).

My mistake was not to treat $b_k\varphi_k$ separately. It is now fixed.

We constructed $b \in S^{-m}$ with

$$a \# b = 1 \mod S^{-\infty}.$$

Why $b \# a = 1 \mod S^{-\infty}$?

May repeat the construction and construct $\tilde{b} \in S^{-m}$ with

$$\tilde{b} \# a = 1 \mod S^{-\infty}$$

Now,

$$\begin{aligned} \tilde{b} &= \tilde{b} \# (a \# b) && \mod S^{-\infty} \\ &= (\tilde{b} \# a) \# b && \text{— " —} \\ &= b && \mod S^{-\infty} \end{aligned}$$

Hence $b \# a = 1 \mod S^{-\infty}$. ∎

<u>Examples:</u> 1) $a \in S^m$ elliptic, $k \geq 0$.

Suppose that

$$\underbrace{a(x,D)\varphi}_{=} \in H^k.$$

Then necessary $f \in H^{k+m} + C^\infty$ (i.e. f differs from H^{k+m}-function by C^∞). Why? $b \in S^{-m}$ with

$$\mathrm{Op}[b] \circ \mathrm{Op}[a] = Id + \underset{\substack{\uparrow \\ \mathrm{Op}[S^{-\infty}]}}{E}$$

Then

$$\underbrace{\underset{\substack{\uparrow \\ S^{-m}}}{\mathrm{Op}[b]} \underset{\substack{\uparrow \\ H^k}}{u}}_{H^{k+m}} = u + \underset{\substack{\uparrow \\ C^\infty}}{Eu}, \quad E \in S^{-\infty}$$

- We see from this example that we gain m derivatives in L^2-sense when solving $Lu = f$ with $L \in S^m$ elliptic.
- What about Hölder regularity?

Thm: If $a \in S^0$, $0 < s < 1$, $f \in C^s(\mathbb{R}^n)$ then also

$$a(x,D) f \in C^s.$$

Remark: 1) This implies that for $a \in S^{-m}$

$$f \in C^s \Rightarrow a(x,D) f \in C^{m,s}$$

just look at $\underbrace{\partial_x^\alpha a(x,D)}_{S^0} f$

$|\alpha| = m$

2) For elliptic ΨDO of order m,

If $Lu = f$, $f \in C^s$

$$\Rightarrow u \in C^s + C^\infty$$

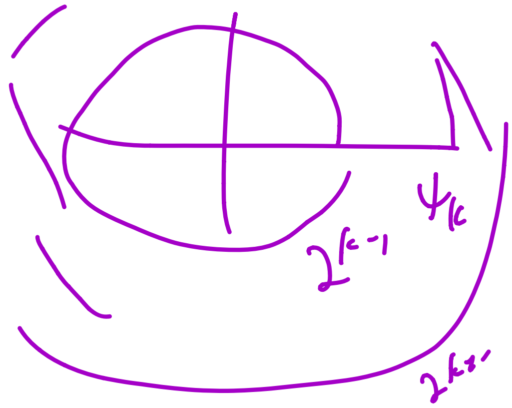
because of the parametrix, $L = Op[a]$

$b \# a = 1 + S^{-\infty}$, $b \in S^{-m}$

Littlewood-Paley

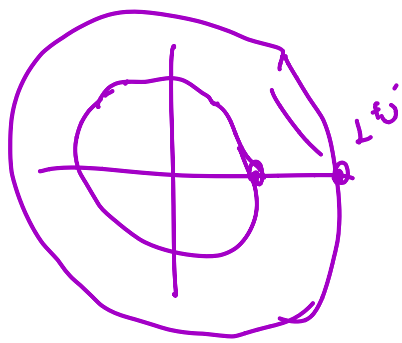
Recall: $\|f\|_{C^s} \sim \sup_{k \geq 0} \|P_k f\|_\infty \cdot 2^{ks}$

where $\widehat{P_k f} = \psi_k \hat{f}$



Lemma: If $a \in S^m$ define

$$a_\ell(x,\xi) = a(x,\xi)\,\psi_\ell(\xi)$$



Then for any $\ell \geq 0$,

$$\|T_{a_\ell}\|_{\infty \to \infty} \leq C\, 2^{\ell m}$$

where C depends only on a.

Proof: Express T_{a_j} as an integral operator and apply Schur's test. Here are the details: The operator $Op[a_\ell]$ is an integral operator with kernel

$$K_{a_\ell}(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} a_\ell(x,\xi)\, e^{i\xi\cdot(x-y)}\, d\xi$$

- The integral is only on $\{2^{\ell-1} \leq |\xi| \leq 2^{\ell+1}\}$ for $\ell \geq 1$, and on $\{|\xi| \leq 1\}$ for $\ell = 0$.

<u>Claim:</u> For any even number M,

$$|K_{a_\ell}(x,y)| \leq \frac{C_{M,a}}{|x-y|^M} \cdot 2^{\ell(n+m-M)}$$

Indeed, integrate by parts using

$$\Delta_\xi e^{i\xi\cdot(x-y)} = -|x-y|^2 e^{i\xi\cdot(x-y)}$$

$$K_{a_\ell}(x,y) = \frac{\pm(2\pi)^{-n}}{|x-y|^{2M}} \int_{|\xi| \leq 2^{\ell+1}} \Delta_\xi^M a_\ell(x,\xi)\, e^{i\xi\cdot(x-y)}\, d\xi$$

If $\ell = 0$, then $|\Delta_\xi^M a_0(x,\xi)| \leq C_M$ for $|\xi| \leq 1$, proving claim with $\ell = 0$.

If $\ell \geq 1$, use $\left|\Delta_\xi^M a_\ell(x,\xi)\right| \lesssim (1+|\xi|)^{m-2M}$

$\sim 2^{\ell(m-2M)}$

when $2^{\ell-1} \leq |\xi| \leq 2^{\ell+1}$. Hence

$$|K_{a_\ell}(x,y)| \lesssim \frac{1}{|x-y|^{2m}} 2^{\ell(m-2n)} \cdot \underbrace{2^{\ell n}}_{\substack{\text{volume}\\ \text{of}\\ \text{ring}\\ 2^{\ell-1} \leq |\xi| \leq 2^{\ell+1}}}$$

This proves the claim.

Now, by Schur's test,

$$\|T_{a_\ell}\|_{\infty \to \infty} \leq \sup_x \int_{\mathbb{R}^n} |K_{a_\ell}(x,y)|\, dy$$

For any x,

$$\int_{\mathbb{R}^n} |K_{a_\ell}(x,y)|\, dy \lesssim \overbrace{\int_{|x-y|<2^{-\ell}}}^{M=0} + \overbrace{\int_{|x-y|\geq 2^{-\ell}}}^{M=2n}$$

$$\lesssim (2^{-\ell})^n \cdot 2^{\ell(n+m)} + 2^{\ell(m-n)} \int_{|y|\geq 2^{-\ell}} \frac{1}{|y|^{2n}}$$

$$\lesssim 2^{\ell m} + 2^{\ell(m-n)} \cdot \int_{2^{-\ell}}^{\infty} r^{-2n} r^{n-1} dr$$

$$\lesssim 2^{lm} + 2^{l(m-n)} 2^{ln} \lesssim 2^{lm}. \quad \blacksquare$$

What is the symbol of $\partial_x^\alpha T_{a_j}$?

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \underbrace{a(x,\xi)\psi_j(\xi)e^{i\xi\cdot x}}\, \hat{\varphi}(\xi)\,d\xi$$

The symbol would be:

$$e^{-i\xi\cdot x}\partial_x^\alpha\left(a(x,\xi)e^{i\xi\cdot x}\right)\psi_j(\xi) \in S^{m+|\alpha|}$$

and

$$\left(\partial_x^\alpha T_a\right)P_j = \partial_x^\alpha\left(T_{a_j}\right)$$

Hence, by the lemma, $\forall\alpha$,

$$\|\partial_x^\alpha T_{a_j}\|_{\infty\to\infty} \lesssim 2^{j(m+|\alpha|)}.$$

with implied constant depending on a and α, m.

Proof of thm: $f \in C^s$, $a \in S^0$

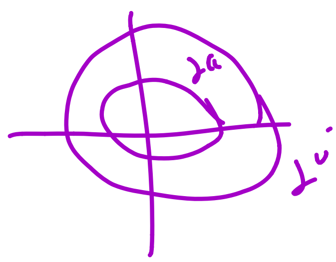
$$f = \sum_{j=0}^{\infty} \underbrace{P_j f}_{= f_j}$$

$$T_a f = \sum_{k=0}^{\infty} T_{a_k} f = \sum_{k,j=0}^{\infty} T_{a_k} f_j$$

If $|j-k| \geq 1$ then

$$T_{a_k} f_j = 0$$

$\hat{f}_j$



Hence

$$T_a f = \sum_{k=0}^{\infty} T_{a_k} f = \sum_{k=0}^{\infty} \underbrace{T_{a_k} \Big(\underbrace{f_{k-1} + f_k + f_{k+1}}_{f_k'} \Big)}_{F_k}$$

and $\|f_j'\|_\infty \lesssim 2^{-js}$

because f is s-Hölder.

The constants in this proof are independ

$\forall$ i and j.

Now, $\forall \alpha$

$$\|\partial_x^\alpha F_j\|_\infty = \|\partial_x^\alpha T_{a_j} f_j'\|_\infty \quad (*)$$

$$\le \|\partial_x^\alpha T_{a_j}\|_{\infty\to\infty} \cdot \|f_j'\|_\infty$$

$$\lesssim 2^{j(|\alpha|)} \cdot 2^{-js} \lesssim 2^{j(|\alpha|-s)}$$

. Use $(*)$ for $\alpha = 0$, $\sum \|F_j\|_\infty < \sum 2^{-js} < \infty$

Then $T_a f = \sum_{j=0}^\infty F_j \in L^\infty$.

Next we need to know $T_a f \in C^s$,

i.e.

$$\|P_j T_a f\|_\infty \lesssim 2^{-js}$$

or

$$\|P_j (\sum_i F_i)\|_\infty \lesssim 2^{-js}.$$

. We understand $T_a P_j = T_{a_j}$, but here we need to understand

$P_j T_{a_i}$.

We will use the rule of composition:

We pick $\alpha_0, \alpha_1, \dots, \alpha_n : \mathbb{R}^n \to \mathbb{R}$

s.t.

$$(A) \quad 1 = \alpha_0(\xi) + \sum_{j=1}^{n} \alpha_j(\xi)\, \xi_j$$

where

$$C^\infty \ni \alpha_j(\xi) = \begin{cases} \xi_j / |\xi|^2 & |\xi| \ge 2 \\ 0 & |\xi| \le 1 \end{cases}$$

$$\uparrow S^{-1}$$

$$\alpha_0 \in C_c^\infty(\mathbb{R}^n)$$

From (A)

$$\mathrm{Id} = \underset{\substack{\uparrow \\ S^{-\infty}}}{\overset{\substack{\mathrm{Op}[\alpha_0] \\ \|}}{S^{(0)}}} + \sum_{|\alpha|=1} \underset{\substack{\uparrow \\ S^{-1}}}{S^{(\alpha)}}\, \partial_x^\alpha$$

$$\boxed{Id = \sum_{|\alpha|\le l} S^{(\alpha)}\,\partial_x^{\alpha}}$$

Only for $l = 0, 1$, when $S^{(\alpha)}$ is a ΨDO in S^{-l}, which is in fact a Fourier multiplier.

· Fourier multipliers commute.

Therefore $S^{(\alpha)}$ commutes with P_j

with ∂_x^α

Consequently, $l = 0, 1$

$$P_j(F_i) = P_j\Big(\sum_{|\alpha|\le l} \underset{\substack{\vee\\ S^{-l}}}{S^{(\alpha)}}\,\partial_x^{\alpha} F_i\Big)$$

$$F_i = T_{a_i} f = T_{a_i} f_i'$$

$$= \sum_{|\alpha|\le l} \underbrace{\underset{\substack{\vee\\ S^{-l}}}{S^{(\alpha)}}\, P_j}\,\underbrace{\left(\partial_x^{\alpha} F_i\right)}$$

ℓ operator norm in L^∞ at most $2^{-j\ell}$

L^∞-norm at most $2^{i(|\alpha|-s)} \le 2^{i(\ell-s)}$

Hence $\forall i, j$, $\forall \ell = 0, 1$

$$(\square) \quad \| \underset{\substack{\| \\ T_a P_i f}}{P_j (F_i)} \|_\infty \lesssim 2^{-j\ell} \cdot 2^{i(\ell - s)}$$

$$\lesssim 2^{-js}$$

Now, $\forall j$

$$\| P_j \underset{\substack{\| \\ T_a f}}{\left(\sum_i F_i \right)} \|_\infty$$

$$\le \underset{\ell=1}{\sum_{i \le j} \| P_j (F_i) \|_\infty} + \underset{\ell=0}{\sum_{i \ge j} \| P_j (F_i) \|_\infty}$$

$$\lesssim \sum_{i=1}^{j} 2^{-j} 2^{i(1-s)} + \sum_{i=j}^{\infty} 2^{-is}$$

$$\lesssim 2^{-j} \cdot 2^{j(1-s)} + 2^{-js} \lesssim 2^{-js}.$$

$\blacksquare$

Chit-chat about further developments

If $a(x,\xi)$ is compactly-supported then $Op[a]$ is of trace-class in $L^2(\mathbb{R}^n)$ $\left(\sum |\lambda_k| < \infty \right.$ λ_k singular-values$\left.\right)$

and the trace is

$$\operatorname{Tr} Op[a] = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi)\, dx\, d\xi$$

[In general, if $Tf(x) = \int K(x,y) f(y)\, dy$ then $\operatorname{Tr}[T] = \int_{\mathbb{R}^n} K(x,x)\, dx$]

$$K_a(x,y) = (2\pi)^{-n} \hat{a}(x, y-x)$$

This may be used for counting eigenvalues of some operators like

$$-h^2 \Delta + V(x) \quad , \quad h > 0 \text{ parameter}$$

$V(x)$: some potential $V : \mathbb{R}^n \to \mathbb{R}$

<u>Weyl's law:</u> Assume

1) $|V(x)| \geq \frac{1}{C} |x|^k$ for $|x| > C$

2) $|\partial_x^\alpha V(x)| \leq C(1+|x|^k) \quad \forall \alpha$

Then $-h^2\Delta + V$ has a complete sequence of L^2-eigenfunctions, eigenvalues

$$E_0 < E_1 \leq E_2 \leq \dots \nearrow \infty$$

and $\forall\, a < b \in \mathbb{R}$

$$\#\{ j : E_j \in [a,b] \}$$

$$= \frac{1}{(2\pi h)^n} \Big[\big| \{ (x,\xi) : \underbrace{|\xi|^2 + V(x)}_{a(x,\xi)} \in [a,b] \} \big| + \underbrace{o(1)}_{\text{in } h} \Big]$$

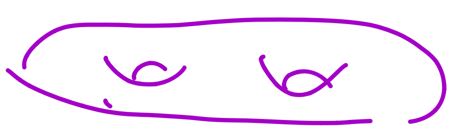
Shnirelman thm: Let (M, g) be a compact Riemannian mfld, $-\Delta_g$ is the Laplacian, with eigenvalues

$$\underset{0}{\overset{\lambda_0}{\shortparallel}} < \lambda_1 \le \lambda_2 \le \dots \nearrow \infty$$

eigenfunctions $\varphi_0, \varphi_1, \varphi_2, \dots$ with $\varphi_0 \equiv 1$

$\|\varphi_j\|_2 = 1$, orthonormal.

Assume: The geodesic flow on M is ergodic.



Then there exist a sequence of integers of density 1, such that along this sequence,

$$\underbrace{|\varphi_j|^2\,dx}_{\text{prob. measure on } M} \xrightarrow{\text{weakly}} \text{the uniform probability measure on } M$$

along a subsequence of density one.

<u>Open problem</u> (random wave conjecture):

For any $F \in C_c^\infty(\mathbb{R})$, even, along a subsequence of density one,

$$\frac{1}{\mathrm{vol}(M)}\int_M F(\varphi_j) \xrightarrow[\text{subseq.}]{j\to\infty} \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} F(x)\, e^{-\frac{x^2}{2}}\,dx$$

Gaussian value distribution.

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