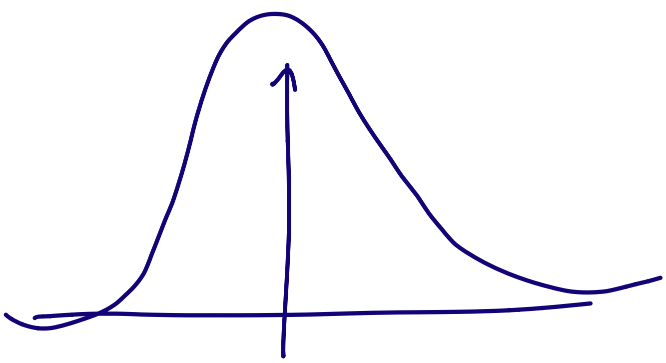
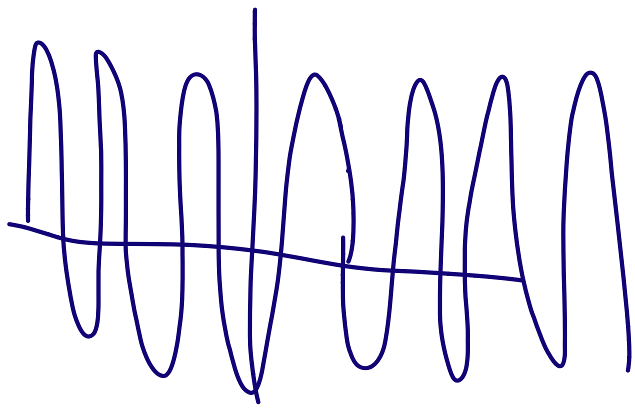
Harmonic Analysis

It has something to do with Fourier series / transform.

$$\int_{\mathbb{R}} f(x) \cos(\xi x)\, dx$$



f



$\cos(\xi x)$

oscillatory wave

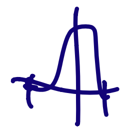
· Lots of delicate cancellation.

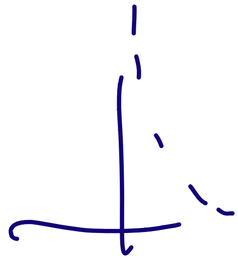
Today: No cancellation
measure theory

$$\int_{\mathbb{R}} \frac{1}{|x|^\alpha}\,\theta(x)\,dx = \begin{cases} +\infty & \alpha \geq 1 \\ < \infty & \alpha < 1 \end{cases}$$

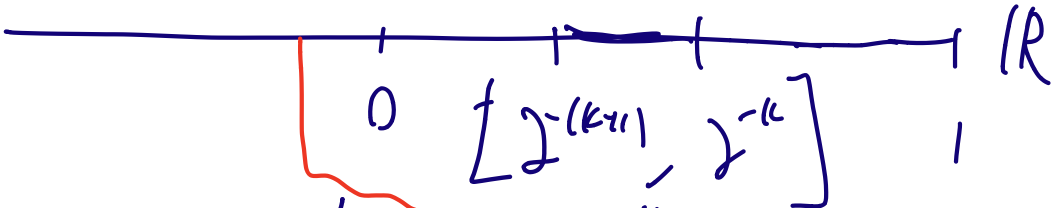
θ: some compactly-supported Cont. function

$\theta(0) = 1$





Why?



$$\int_{I_k} \frac{dx}{|x|^\alpha} \sim (2^{-k})^\alpha \cdot 2^{-k}$$

values of integrand

$$\sum_{k=0}^{\infty} \int_{I_k} \sim \sum_{k=0}^{\infty} (2^{-k})^\alpha\, 2^{-k}$$

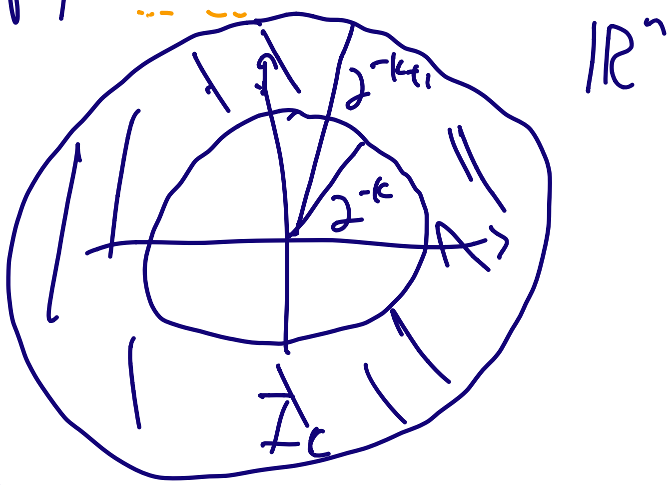
$$= \sum_{k=0}^{\infty} 2^{k(\alpha-1)} = \begin{cases} +\infty & \alpha > 1 \\ \frac{1}{1-2^{\alpha-1}} & \alpha < 1 \end{cases}$$

· Similarly in $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \frac{1}{|x|^\alpha}\,\theta(x)\,dx = \begin{cases} +\infty & \alpha \geq n \\ <\infty & \alpha < n \end{cases}$$

$$\begin{cases} \theta(0) = 1 & \text{cont.} \\ \theta \geq 0 \\ \operatorname{supp}(\theta) \subset\subset \mathbb{R}^n \end{cases}$$

Why?



$$I_k = \{2^{-k} < |x| \leq 2^{-k+1}\}$$

$$\int_{I_k} \frac{dx}{|x|^\alpha} \sim (2^{+k})^\alpha \cdot \underbrace{(2^{-k})^n}_{\text{volume}} = 2^{k(\alpha-n)}$$

$$\int_{\mathbb{R}^n} \sim \sum_{k=0}^{\infty} 2^{k(\alpha-n)} = \begin{cases} <\infty & \alpha < n \\ +\infty & \alpha \geq n \end{cases}$$

Exponent is related to dimension

- More generally, for a compactly-supported, finite, Borel measure μ on $\mathbb{R}^n$, $t>0$ set

t-energy $$I_t(\mu) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1}{|x-y|^t}\, d\mu(x)\, d\mu(y)$$

"In dimension n, should be finite for $t<n$"

Prop: 1) Assume $\exists C$ $\forall x \in \mathbb{R}^n$ $\forall r>0$

$$\mu(B(x,r)) \leq C r^t$$

Then $I_{t-\varepsilon}(\mu) < \infty$ $\forall\, 0<\varepsilon<t$

2) Assume that $I_t(\mu) < \infty$. Then $\exists A \subseteq \mathbb{R}^n$, $\mu(A) \geq \frac{9}{10} \cdot \mu(\mathbb{R}^n)$ s.t.

$\nu := \mu|_A$ $\quad$ (i.e. $\nu(B) = \mu(A \cap B)$)

(*) $\quad \nu(B(x,r)) \leq C r^t \quad \forall x, r$

for some $C > 0$.

<u>Exercise:</u> We have to throw mass.

<u>Proof:</u>

(Trick: $f \geq 0$ $\quad \int_X f \, d\mu = \int_0^\infty \mu(\{f > t\}) \, dt$

Fubini, ...)

1) Set $s = t - \varepsilon$. Fix $x \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^s} = \int_0^\infty \mu\left(\left\{y ; \frac{1}{|x-y|^s} \geq u\right\}\right) du$$

$$= \int_0^\infty \mu\Big(B(x, \underbrace{u^{-1/s}}_{r})\Big) \, du$$

$r = u^{-1/s}$

$u = \frac{1}{r^s}$

$$= \int_0^\infty \frac{s}{r^{s+1}} \mu(B(x,r)) \, dr$$

$$\leq \mu(\mathbb{R}^n) \int_1^\infty \frac{s}{r^{s+1}} \, dr + s \int_0^1 C r^t \ldots$$

$$= \mu(\mathbb{R}^n) \cdot \int_1 \frac{\ldots}{r^{s+1}} \, dr + s \int_0 \frac{\ldots}{r^{s+1}} \, dr$$

$$= \mu(\mathbb{R}^n) + Cs \underbrace{\int_0^1 \frac{dr}{r^{1-\varepsilon}}}_{\frac{1}{\varepsilon}} = \mu(\mathbb{R}^n) + \frac{Cs}{\varepsilon}$$

$$I_s(\mu) \le \int_{\mathbb{R}^n} \left(\mu(\mathbb{R}^n) + \frac{Cs}{\varepsilon} \right) d\mu(x) = \mu(\mathbb{R}^n) \left(\mu(\mathbb{R}^n) + \frac{Cs}{\varepsilon} \right) < \infty$$

2) Know: $I_t(\mu) < \infty$. Define

$$f(x) = \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^t}$$

- f is μ-integrable.

$\Rightarrow$ f is finite μ. a.e.

$$\mu(\mathbb{R}^n) = \mu\Big(\underbrace{\bigcup_m \{ f < m \}}_{\text{here } f \text{ is finite, so full } \mu\text{-measure}} \Big)$$

$$\mu(\mathbb{R}^n) = \lim_{M\to\infty} \mu(\{f<M\})$$

Hence $\exists M>0$ s.t.

$$\mu(\{f<M\}) \geq \frac{9}{10}\cdot \mu(\mathbb{R}^n).$$

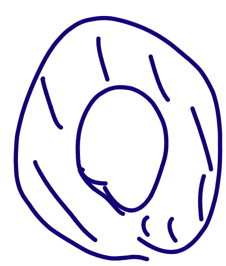
Set $A := \{f<M\}$

$$\nu := \mu|_A$$

$\forall x \in A$ $\quad f(x) = \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^t} \leq M.$

Let's see that ν has bounded ~~volume~~ measure growth.

For any $x\in\mathbb{R}^n$, $r>0$.



<u>Case 1:</u> $x\in A$.

$$\nu(B(x,r)) \leq \mu(B(x,r)) \leq \int_{B(x,r)} \frac{r^t}{|x-y|^t}\, d\mu(y)$$

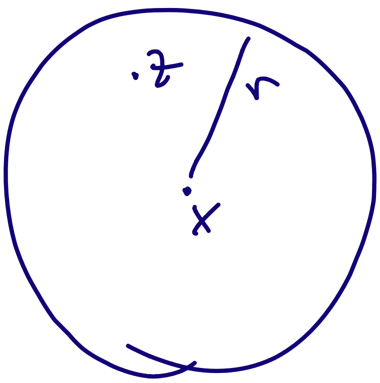
$B(x,r)$

$$\le r^t \cdot f(x) \le M r^t.$$

<u>Case 2:</u> $x \notin A$, $\nu(B(x,r)) > 0$.

Hence $\exists z \in A \cap B(x,r)$, and

$$\nu(B(x,r)) \le \nu(B(\underset{A}{z}, 2r)) \le M(2r)^t$$
$$= M \cdot 2^t \cdot r^t$$



Hence, $C = M \cdot 2^t$. ∎

"dimension of a measure"?

$$\mu(B(x,r)) \sim r^t$$

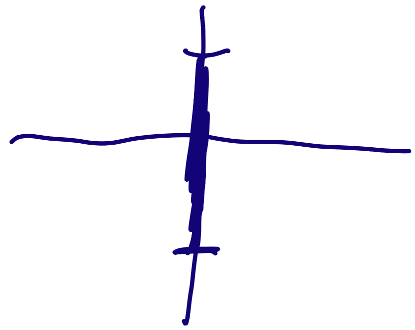
$$\mu(B(x,r)) \le C r^t$$

We saw, this is equivalent*

$$I_t(\mu) = \iint \frac{1}{|x-y|^t} \, d\mu(x) \, d\mu(y)$$

Examples: 0) Lebesgue measure on $\mathbb{R}^n$

1)



$\{0\}\times[-1,1]$
Length measure

Hausdorff measure in $\mathbb{R}^n$

- For a Borel set $A \subseteq \mathbb{R}^n$ and $s \geq 0$,

$$\mathcal{H}_s(A) = \lim_{\delta \to 0^+} \mathcal{H}_{s,\delta}(A)$$

$$\mathcal{H}_{s,\delta}(A) = \inf \left\{ \sum_k \underbrace{c_s r_k^s}_{\text{volume of a Euclidean ball of radius } r_k \text{ in dim } s} ; \; \begin{array}{l} A \subseteq \bigcup B(x_k, r_k), \; x_k \in \mathbb{R}^n \\ \forall k \;\; r_k \leq \delta \end{array} \right\}$$

not a measure

$$c_s = \frac{\pi^{s/2}}{\Gamma(1+s/2)}$$

Exercise: If is a Borel measure

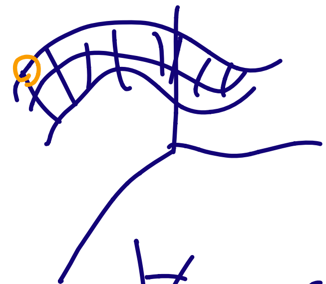
(it follows from Carathéodory's thm to be mentioned)

$\underline{s=0}$: $\mathcal{H}_0(A) = \#(A)$

· For integer $s = k$, $0 \le k \le n$,

$A \subseteq \mathbb{R}^n$ k-dimensional submanifold

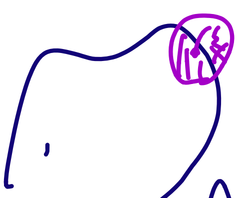
$\mathcal{H}_k(A) =$ k-dim. volume of A



· It is defined and makes sense $\forall\, 0 \le s \le n$.

<u>Remarks</u> The case $s \in \mathbb{Z}$ is nicer.

<u>Lebesgue's density thm:</u> For integer k, $A \subseteq \mathbb{R}^n$, $0 < \mathcal{H}_k(A) < \infty$, then for $\mathcal{H}_k$-almost-any $x \in A$,



$$\frac{\mathcal{H}_k(A \cap B(x,\varepsilon))}{C_k\, \varepsilon^k} \xrightarrow{\varepsilon \to 0^+} 1$$

C_k ← volume of k-dim. Euclidean ball.

· Almost never true for $s \notin \mathbb{Z}$.

Claim: If $\mathcal{H}_\alpha(A) < \infty$ then

$$\mathcal{H}_\beta(A) = 0 \quad \forall \beta > \alpha.$$

Proof:

$$\mathcal{H}_{\beta,\delta}(A) \le \frac{C_\alpha}{C_\beta}\,\delta^{\beta-\alpha}\,\mathcal{H}_{\alpha,\delta}(A)$$

$r_K < \delta$ $\quad C_\beta r_K^\beta \quad \le \frac{C_\alpha}{C_\beta}\,\delta^{\beta-\alpha}\,C_\alpha r_K^\alpha$ $\quad \delta \to 0^+$

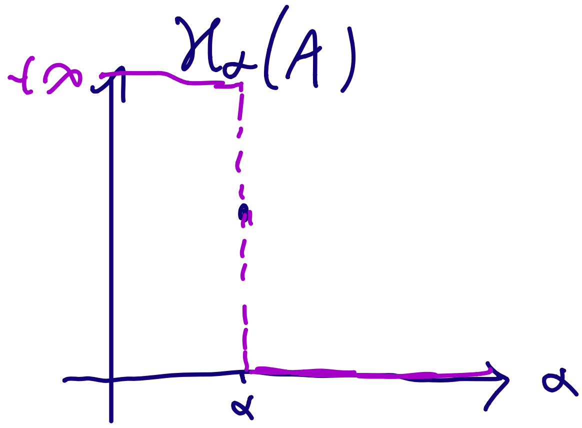
$$0 \qquad\qquad 0 \cdot \mathcal{H}_\alpha(A)$$

Claim: $s \mapsto \mathcal{H}_s(A)$ is non-increasing in s

Proof: $\delta < c(\alpha,\beta)$

$$\mathcal{H}_{\beta,\delta}(A) \le \frac{C_\alpha}{C_\beta}\,\delta^{\beta-\alpha}\,\mathcal{H}_{\alpha,\delta}(A)$$

Therefore,



Definition (Hausdorff dimension)

For a Borel set $A \subseteq \mathbb{R}^n$, define

$$\dim_H(A) = \sup\{\beta \geq 0 \,;\, \mathcal{H}_\beta(A) > 0\}$$

If $\#(A) = \infty$,

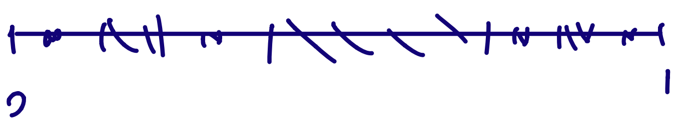
$$\dim_H(A) = \sup\{\beta \geq 0 \,;\, \mathcal{H}_\beta(A) = +\infty\}$$
$$= \inf\{\beta \geq 0 \,;\, \mathcal{H}_\beta(A) = 0\}.$$

Examples:

0) k-dim submanifold in $\mathbb{R}^n$ has dimension k.

1) The Cantor set

$$C = \left\{ \sum_{k=1}^{\infty} \frac{\alpha_k}{3^k} \,;\, \alpha_k \in \{0,2\} \right\}$$



2^n intervals each of length 3^{-n}

Claim: $\dim_H(C) = \frac{\log 2}{\log 3} \in (0,1)$

Proof: Upper bound: cover by union of balls

$$C \subseteq \bigcup_{k=1}^{2^n} I_k \quad , \quad I_k = \text{ball of radius } \frac{1}{2}\cdot 3^{-n}$$

$$\alpha = \frac{\log 2}{\log 3}$$

$$\mathcal{H}_{\alpha, 3^{-n}}(C) \le c_\alpha \cdot 2^n \cdot \left(\frac{3^{-n}}{2}\right)^\alpha$$

$$= c_\alpha \cdot$$

Then $\mathcal{H}_\alpha(C) \le c_\alpha < \infty$

Lower bound?

Frostman lemma: $\forall A$

$$\dim_H(A) = \sup\{ s \,;\, \exists\, 0 \not\equiv \mu \in \mathcal{M}(A),\ I_s(\mu) < \infty \}$$

all compactly-supp., finite, Borel meas. on A.

Corollary: In order to show that $\dim_H(A) \geq s$ enough to find a measure μ on A with

$$\forall x, r \quad \mu(B(x,r)) \leq C r^s$$

enough r which is 2^{-k} or geometric progression.

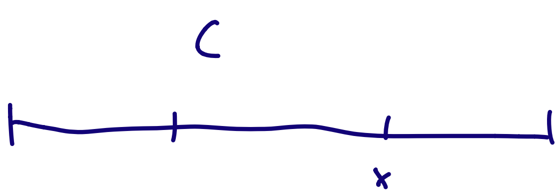
Example: For $C \subseteq \mathbb{R}$ Cantor,

$$\mu(A) := \mathbb{P}(X \in A)$$

$$X = \sum_{k=1}^{\infty} \frac{X_k}{3^k}$$

$X_1, X_2, \ldots$ are i.i.d random variables

$$\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 2) = \frac{1}{2}$$



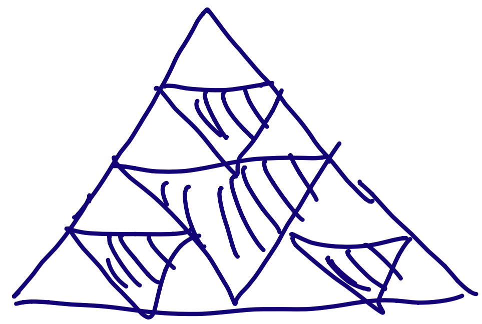
$$\mu(B(x, 3^{-n})) \leq C \cdot 2^{-n}$$

Frostman measure $\quad \alpha = \frac{\log 2}{\log 3}$

$$\mu(B(x, 3^{-n})) \le C \left(3^{-n}\right)^{\alpha}.$$

Exercises

- $\mu = c \cdot \mathcal{H}_\alpha$
- Sierpinski triangle



$$\dim = \frac{\log 3}{\log 2}$$

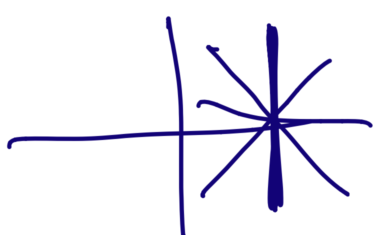
3^n triangles of radius 2^{-n}

$$3^n \cdot \left(2^{-n}\right)^{\alpha} = 1$$

- digital sundial (Falconer 1990s)

Thm (Besicovitch, Kakeya set)

There exists a compact $B \subseteq \mathbb{R}^2$ of zero Lebesgue measure that contains a translate of any unit interval (in any direction).



- Will prove: $\dim_H(B) = 2$.

A famous open problem: In $\mathbb{R}^3$?

Is it true $\dim_H = 3$?

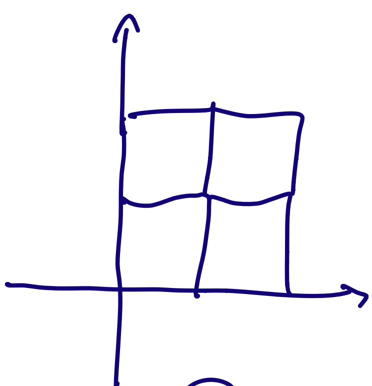
Proof of Frostman's lemma

(+ reminder of measure theory).

- How do we construct Borel measures on $\mathbb{R}^n$?

First, define them on dyadic cubes

$$Q = 2^{-l}\left([0,1)^n + \underset{\mathbb{Z}^n}{\overset{\in}{x}}\right), \quad l \in \mathbb{Z}$$



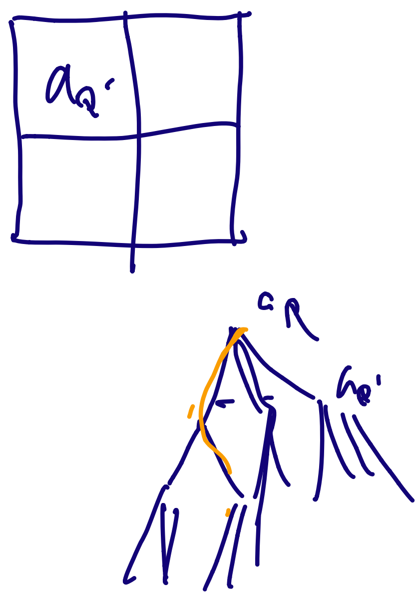
$$\delta_Q = 2^{-l} = \text{side length of } Q$$

Q = disjoint union of 2^n children.

- One way to construct a Borel measure is "top to bottom"

$$\mu\left(Q = \square\right) = a_Q \geq 0$$

$$(*) \quad a_Q = \underbrace{\sum_{Q' \text{ child of } Q}}_{2^n \text{ summands}} a_{Q'}$$



Consistency condition.

technical term: Finitely-additive measure on algebra of sets.

Thm (Caratheodory)

There exists a unique Borel measure on $\mathbb{R}^n$ satisfying $\mu(Q) = a_Q \geq 0$ if (*) holds, provided that Caratheodory's condition is satisfied:

(**) When $Q_1 \supsetneq Q_2 \supsetneq Q_3 \supseteq \cdots$ dyadic cubes $\boxed{\text{with } \bigcap_k Q_k = \emptyset}$,

$$a_{Q_k} \xrightarrow{k \to \infty} 0$$

OOPS! Wrong formulation! To be fixed next week

Example:

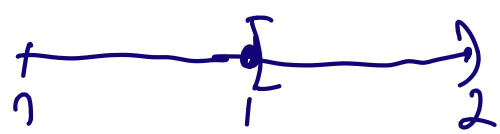
$$Q_k = \left[1 - \frac{1}{2^k}, 1\right)$$

$$\bigcap_{k=1}^{\infty} Q_k = \emptyset$$

Non-example:

$$a_Q = \begin{cases} 1 & \text{if } 1 \in \bar{Q} \setminus Q \\ 0 & \text{o/w} \end{cases}$$

violates



· It could have been proved in undergraduate measure theory class: · define exterior measure

: measurable sets

Frostman's lemma: $\forall A \subseteq \mathbb{R}^n$ compact

$$\dim_H(A) = \sup\{ s \,;\, \exists\, 0 \neq \mu \in M(A) \,;\, I_s(\mu) < \infty \}.$$

Proof: The easy direction.

Suppose that μ, a finite Borel measure on A, exists and non-zero, with $I_s(\mu) < \infty$.

By prop., can throw away a bit of the mass, and assume that

$$\forall x, r \quad \mu(B(x,r)) \leq C r^s$$

· Given any cover $A \subseteq \bigcup_{k=1}^{\infty} B(x_k, r_k)$

$$0 < c \le \mu(A) \le \sum_{k=1}^{\infty} \mu(B(x_k, r_k))$$

$$\underset{\substack{\| \\ \mu(A)}}{c} \le \sum_{k=1}^{\infty} C\, r_k^{s}$$

I.e., if $A \subseteq \bigcup B(x_k, r_k)$

then $\sum r_k^s \ge \frac{c}{C} = c' > 0.$

Hence $\mathcal{H}_{s,\delta}(A) \ge \tilde{c} > 0 \quad \forall \delta$

and $\mathcal{H}_s(A) \ge \check{c} > 0.$

<u>Proof of the hard direction</u> $\boxed{s > 0}$

Given $A \subseteq [0,1)^n = Q_0$

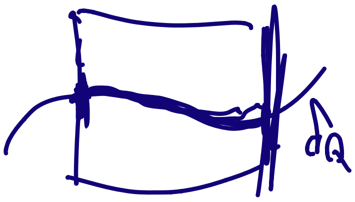
Assume $\mathcal{H}_s(A) > 0.$

Need to construct a measure μ on A

with $$\mu(B(x,r)) \le C r^s \quad \forall x, r$$

Step 1: For a dyadic cube $Q \subseteq Q_0$

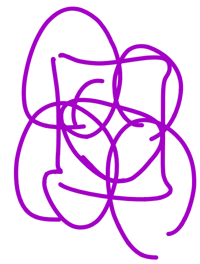
define
$$\boxed{\mu^+(Q) = \mathcal{H}_{s,\delta_Q}(A \cap Q)}$$



Q

Then Thm: $\mathcal{H}_{s,\delta_Q}(A)$ $\parallel$

0) $\mu^+(Q_0) > 0$



Provisional proof:

$$0 < \mathcal{H}_s(A) = \lim_{\delta \to 0} \mathcal{H}_{s,\delta}(A)$$

Hence for some $\delta_0 > 0$

$$\mathcal{H}_{s,\delta_0}(A) > 0$$

$A \subseteq \bigcup B(x_k, r_k)$, $\forall k\ r_k \le \delta_0$ — √

$$\Downarrow$$

$$\sum_{k=1}^{\infty} C_s r_k^s \ge \mathcal{H}_{s,\delta_0}(A) > 0$$

o/w, $\exists k$ $r_k \ge \delta_0$, and

$$\sum C_s r_k^s \ge C_s \delta_0^s$$

1) $\mu^+(Q) \le \sum_{Q' \text{ children}} \mu^+(Q')$ $\quad \delta_{Q'} = \frac{1}{2}\delta_Q$

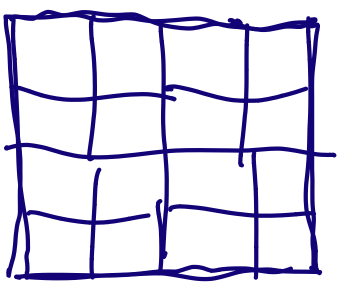
2) $\mu^+(Q) \le C \delta_Q^s$.

$Q \subseteq C$ balls of radius $\le \delta_Q$.

Step 2: Go on tree of dyadic cubes top to bottom, recursively reduce $\mu^+(Q)$ to create a_Q with

- $a_Q = \sum_{\substack{Q' \text{ children} \\ \text{of } Q}} a_{Q'}$

- $\mu^+(Q_0) = a_{Q_0} > 0$.



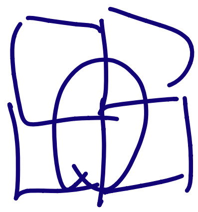
- $a_Q \le \mu^+(Q) \le C \delta_Q^s$

Step 3: By Caratheodory, there exists a Borel measure μ on $\mathbb{R}^n$ with $\mu(Q) = a_Q \le \mu^+(Q)$

(since $a_Q \le C \ell_Q^s \xrightarrow{\ell_Q \to 0} 0$)

1) $\forall Q$, $\mu(Q) \le C \ell_Q^s$

$\forall x \in \mathbb{R}^n, r > 0$



$$\mu(B(x,r)) \le \tilde{C} r^s$$

2) $\forall$ dyadic cube disjoint from A,

$$\mu(Q) = a_Q \le \mu^+(Q) = \mathcal{H}_{s,\ell_Q}(\overset{\emptyset}{\overset{\|}{A \cap Q}})$$

$$= 0$$

$$\mu\Big(\underbrace{\bigcup \text{dyadic cubes disjoint from } A}_{\mathbb{R}^n \setminus A}\Big) = 0$$

as A is compact.

Hence is supported on A. $\blacksquare$