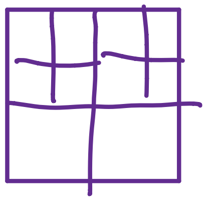
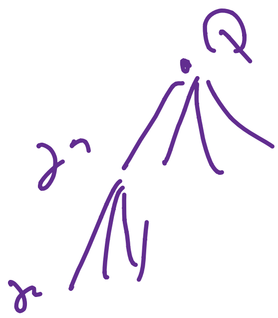
$Q =$ 

A correct formulation of the Caratheodory's condition.

$$Q = 2^{\ell}\left([0,1)^n + x\right) \qquad \text{dyadic sets}$$

$$\ell \in \mathbb{Z}, \quad x \in \mathbb{Z}^n$$

$Q =$ a disjoint union of 2^n children



In order to construct a Borel measure on $\mathbb{R}^n$, we provided $a_Q \geq 0 \ \forall Q$

$$(*) \qquad a_Q = \sum_{\substack{Q' \\ \text{a child} \\ \text{of } Q}} a_{Q'}$$

<u>Thm</u> (Caratheodory)

$\exists \mu$ a Borel measure μ on $\mathbb{R}^n$ with

$\mu(Q) = a_Q \ \forall Q$, provided that (*) holds and:

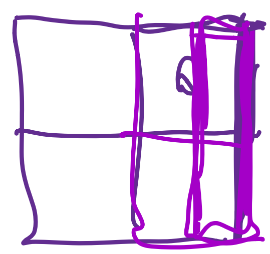
(**) When $B_1 \supset B_2 \supset \dots B_n \supset \dots$ are <u>basic sets</u> with $\bigcap_m B_m = \emptyset$, necessarily $a_{B_m} \xrightarrow{m\to\infty} 0$.

A basic set is a finite union of dyadic cubes

$$B = \dot{\bigcup_{i=1}^{m}} Q_i \quad \text{disjoint union}$$

$$a_B = \sum_{i=1}^{m} a_{Q_i}$$

<u>Example:</u> $n=2$



$$a_Q = \text{Length}\left(\left(\bar{Q}\setminus Q\right) \cap \left(\{1\}\times[0,1]\right)\right)$$

$$a_Q \le \delta_Q = \text{sidelength}$$

$$B_m = \left[1-\frac{1}{2^m}, 1\right) \times [0,1]$$

· We used Caratheodory for Frostman.

In place of Caratheodory, we use:

<u>Proposition</u>: Given $a_Q \geq 0$ satisfying $(*)$, then exists a Borel measure μ on $\mathbb{R}^n$, such that for any basic set $B \subseteq \mathbb{R}^n$

$$\mu(\operatorname{int} B) \leq a_B \leq \mu(\overline{B}).$$

$\uparrow$ interior of B

$$\left[\text{e.g. } \mu((0,1)^n) \leq a_{[0,1)^n} \leq \mu([0,1]^n)\right]$$

Complete the proof of Frostman's lemma.

Given $A \subseteq [0,1)^n = Q_0$ a compact set with

$$\mathcal{H}_s(A) > 0$$

we constructed $\forall$ dyadic cube

$$a_Q \leq \mu^+(Q) = \mathcal{H}_{s,\delta_Q}(A \cap Q)$$

satisfying (*), not all zero, s.t.

$\exists C \ \forall Q \qquad \boxed{a_Q \le C \delta_Q^s.}$

Use Proposition, and conclude that $\exists \mu$ Borel measure μ with

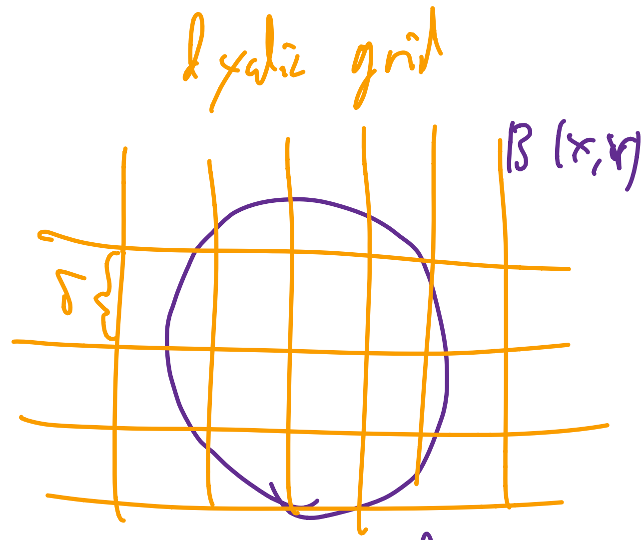
$$\mu(\operatorname{int} B) \le a_B \le \mu(\bar{B})$$

· Verify that μ has bounded measure growth.

$\forall x, r$

$B(x,r)$ is covered by C_n dyadic cubes of side length δ.

$$C_n \le (10\sqrt{n})^n$$



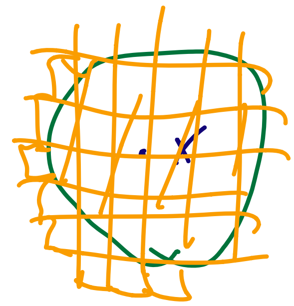
$$\delta = 2^{-l}$$

$$\frac{r}{2} < \delta \le r$$

(covered by the interior of this basic set$_B$).

$$\mu(B(x,r)) \le a_B \le C_n \le C_n C \delta^s \le \tilde{C} r^s.$$

· Why $\mu(\mathbb{R}^n \setminus A) = 0$?



$x \notin A$

$\Downarrow$

$x \in \underbrace{\operatorname{int}\left(\begin{array}{c}\text{basic set}\\ \text{avoiding } A\end{array}\right)}$

So $\mathbb{R}^n \setminus A$ is covered by

$$\mu\left(\operatorname{int}\left(\begin{array}{c}\text{basic set}\\ \text{avoiding } A\end{array}\right)\right) = 0$$

$$\mu(\mathbb{R}^n \setminus A) \le \sum_{k=1}^{\infty} \mu\left(\begin{array}{c}\text{basic set}\\ \text{avoiding } A\end{array}\right) = 0.$$

<u>Proof of Proposition</u>

What is a σ-additive, Borel measure?

<u>Riesz's answer</u>: A positive, bdd, linear functional on continuous functions.

Let $K \subseteq \mathbb{R}^n$ be compact, $\mu \in \mathcal{M}(K)$

finite Borel measures.

$$L_\mu(f) := \int f \, d\mu \qquad f \in C(K)$$

continuous functions

<u>bounded</u>: $\forall f \ |L_\mu(f)| \le C \cdot \|f\|_\infty$

$$C = \mu(K)$$

<u>positive</u>: $f \geq 0 \implies L_\mu(f) \geq 0.$

<u>Riesz's representation theorem</u>: $K \subseteq \mathbb{R}^n$ compact

$L : C(K) \to \mathbb{R}$ bdd, positive, linear functional. Then $\exists! \ \mu \in \mathcal{M}(K)$ s.t.

$$L = L_\mu$$

$$\forall f \quad L(f) = \int_K f \, d\mu.$$

· Topology on $\mathcal{M}(K)$ is usually weak*-topology, i.e.

$\mu_m \longrightarrow \mu$ if $\forall \varphi \in C_c(\mathbb{R}^n)$

$$\int \varphi \, d\mu_m \xrightarrow{m \to \infty} \int \varphi \, d\mu$$

Example: $\delta_{\frac{1}{m}} \xrightarrow{m\to\infty} \delta_0$

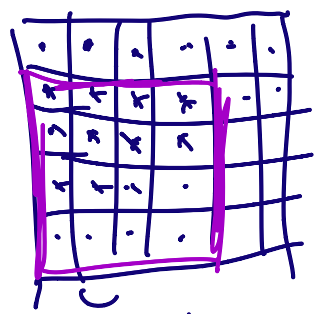
- The topology is metrizable, and we have compactness.

Thm (follows from Banach-Alaoglu)

Any sequence of Borel measures $(\mu_m)_{m\geq 1}$ on compact $K \subseteq \mathbb{R}^n$ with $\sup_m \mu_m(K) < \infty$ has a convergent subsequence.

Proof of Proposition: (In $Q_0 = [0,1)^n$).

$$\mu_k := \sum_{\substack{Q;\\ \delta_Q = 2^{-k}}} a_Q \, \delta_{x_Q}$$



$\delta = 2^{-k}$

where x_Q is the center of the dyadic cube Q.

- $\mu_k(Q) = a_Q \quad \forall Q$ with $\delta_Q \geq 2^{-k}$.

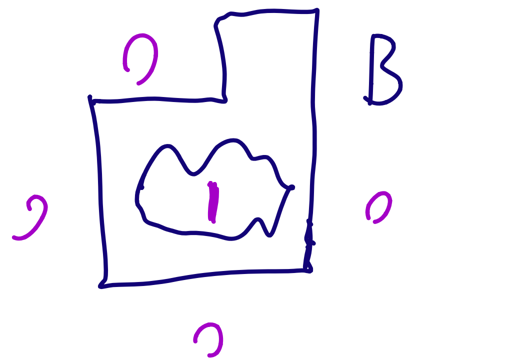
Pass to a convergent subsequence

$$\mu_{m_k} \longrightarrow \mu \in m(K)$$

<u>Need:</u> $\mu(\operatorname{int} B) \le a_B \quad \forall$ basic B.

Pick open U with $\overline{U} \subseteq \operatorname{int}(B)$

Then exists a cont. $f : \mathbb{R}^n \to [0,1]$ with



$$f = \begin{cases} 1 & \text{on } \overline{U} \\ 0 & \text{outside } B \end{cases}$$

$$\mu(U) \le \int f \, d\mu = \lim_{k \to \infty} \int f \, d\mu_{m_k}$$

$$\le \overline{\lim_{k\to\infty}} \; \underbrace{\mu_{m_k}(B)}_{\substack{\| \\ a_B}} = a_B$$

for a sufficiently large k.

$$\mu(\operatorname{int} B) = \lim \mu(U_m) \le a_B$$

$B = \bigcup_{m=1}^{\infty} U_m$ open sets with $\overline{U_m} \subset \operatorname{int}(B)$

Why $a_B \le \mu(\bar{B})$?

Take a closed F with $\mathrm{int}(F) \supseteq \bar{B}$

Take $f = \begin{cases} 1 & B \\ 0 & \text{outside } F \end{cases}$

and argue as above.

Fourier transform in $\mathbb{R}^n$

- For $f \in L^1(\mathbb{R}^n)$ its Fourier Transform is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} f(x)\,dx \quad (\xi \in \mathbb{R}^n)$$

$$= \langle f, e_\xi \rangle$$

where $e_\xi(x) = e^{i\langle \xi, x\rangle}$

- For a finite (complex-valued) Borel measure μ on $\mathbb{R}^n$

$\Big(\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$ where all are positive measures $\Big)$

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle}\, d\mu(x)$$

Why? E.g., the Fourier transform diagonalizes translation-invariant operators.

I.e., T commutes with translations

$$f_a(x) = f(x+a)$$

$$T(f_a) = (Tf)_a$$

Thus

$$\widehat{Tf}(\xi) = a(\xi)\,\hat{f}(\xi)$$

$\uparrow$ multiplier

E.g.

$$\widehat{\Delta f}(\xi) = -|\xi|^2 \hat{f}(\xi)$$

Claim: For any finite, complex-valued measure μ on $\mathbb{R}^n$ (i.e. $d\mu = f\,dx$, $f \in L^1$), $\hat{\mu} : \mathbb{R}^n \to \mathbb{C}$ is continuous and bounded with $\|\hat{f}\|_\infty \leq \|f\|_1$

$$\|\hat{\mu}\|_\infty \leq \|\mu\| = \sup_{\|\varphi\|_\infty \leq 1} \int \varphi \, d\mu$$

Proof: $$|\hat{f}(\xi)| = \left| \int_{\mathbb{R}^n} \underbrace{e^{-i\langle \xi, x\rangle}}_{\substack{\leq 1 \\ \text{in absolute value}}} f(x)\,dx \right|$$

$$\leq \int |f| = \|f\|_1$$

Why is $\hat{f}$ continuous? By dominated convergence,

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} f(x)\,dx$$

$$\xrightarrow{\xi \to \xi_0} \int_{\mathbb{R}^n} e^{-i\langle \xi_0, x\rangle} f(x)\,dx = \hat{f}(\xi_0)$$

Majorant: $|f|$ is an integrable majorant which bounds all involved function

$\square$

Examples $\qquad \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} f(x)\, dx$

0) $\widehat{\delta_0}(\xi) \equiv 1$

$\widehat{\delta_{x_0}}(\xi) = e^{-i\langle \xi, x_0\rangle}$

By the way, $\quad f_a(x) = f(x+a)$

$$\widehat{f_a}(\xi) = \int e^{-i\langle x, \xi\rangle} f(\underbrace{x+a}_{x})\, dx$$

$$= e^{i\langle a, \xi\rangle} \hat{f}(\xi)$$

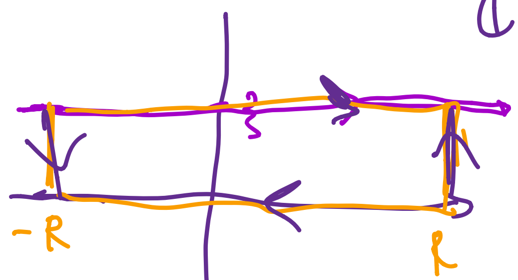
1) $\quad f(x) = (2\pi)^{-n/2} e^{-|x|^2/2} \quad$ in $\mathbb{R}^n$

$$\hat{f}(\xi) = e^{-|\xi|^2/2}$$

Why? In 1D,

$$\int_{-\infty}^{\infty} e^{-i\xi\cdot x} e^{-x^2/2}\, dx = \int_{-\infty}^{\infty} e^{-\frac{(x+i\xi)^2}{2} - \frac{\xi^2}{2}}\, dx$$

$$= e^{-\frac{\xi^2}{2}} \int_{-\infty}^{\infty} e^{-(x+i\xi)^2/2}\, dx$$



$$= e^{-\frac{\xi^2}{2}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx$$

$e^{-z^2/2}$

$$= e^{-\xi^2/2} \sqrt{2\pi}$$

In n-dimensional space

$$\widehat{\overbrace{f_1(x_1)\, f_2(x_2) \cdots f_n(x_n)}}(\xi)$$

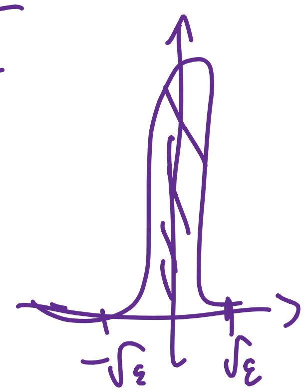
$$= \hat{f}_1(\xi_1)\, \hat{f}_2(\xi_2) \cdots \hat{f}_n(\xi_n)$$

2) scaling

$$\gamma_\varepsilon(x) = (2\pi\varepsilon)^{-n/2} e^{-\frac{|x|^2}{2\varepsilon}}$$

probability density,

Gaussian of variance ε



$$\hat{\gamma}_\varepsilon(\xi) = e^{-\frac{\varepsilon|\xi|^2}{2}}$$

In general

$$\widehat{f(\lambda x)}(\xi) = \frac{1}{\lambda^n} \hat{f}\left(\frac{\xi}{\lambda}\right)$$

Even more generally, $T : \mathbb{R}^n \to \mathbb{R}^n$ linear,

$$\widehat{f \circ T}(\xi) = \frac{1}{|\det T|} \hat{f} \circ T^{-*}$$

$$(T^*)_{ij} = \overline{T_{ji}}$$ $*$ – transpose and complex conjugate

$$T^{-*} = (T^{-1})^* = (T^*)^{-1}$$

$$\widehat{f \circ T}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(T(x))\, dx$$

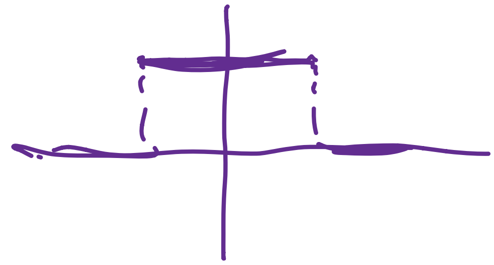
$y = T(x)$ $\qquad dy = |\det T|\, dx$

$$= \int e^{-\langle \xi, T^{-1}(y)\rangle} f(y) \frac{1}{|\det T|}\, dy$$

$$= \int_{\mathbb{R}^n} e^{-\langle T^{-*}\xi, y\rangle} f(y) \frac{dy}{|\det T|}$$

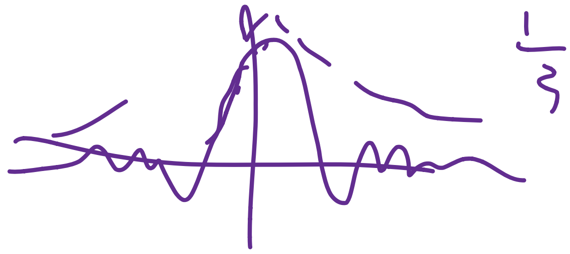
$$= \hat{f}(T^{-*}\xi) \cdot \frac{1}{|\det T|}.$$

3) $$f(x) = \mathbb{1}_{[-1,1]}$$



$$\hat{f}(\xi) = 2\frac{\sin\xi}{\xi} = 2\,\mathrm{sinc}(\xi)$$

$$\mathrm{sinc}(x) = \frac{\sin x}{x}$$



$$f(t) = e^{-|t|} \qquad \hat{f}(\xi) = \frac{2}{1+\xi^2}$$

Cauchy distribution

$$f(t) = \frac{1}{\cosh\left(\sqrt{\frac{\pi}{2}}\, t\right)}$$

$$\hat{f} = \sqrt{2\pi} \cdot f$$

Another probability distribution which is a fixed point (up to $\sqrt{2\pi}$) of FT.

- In all of these examples

$$\lim_{\xi \to \infty} \hat{f}(\xi) = 0$$

- The decay at ∞ can be just $\frac{1}{|\xi|^{\varepsilon}}$ for any $\varepsilon > 0$.

  I.e. $f \in L^1$, but might $\hat{f} \notin L^p$, $p > 0$

  $\int |\hat{f}|^{\varepsilon} = +\infty$.

- But, $\widehat{\mathbb{1}_{[-1,1]}}(\xi)$ goes to zero at ∞.

$$\widehat{\mathbb{1}_{[a,b]}}(\xi) \xrightarrow{\xi \to \infty} 0$$

- Hence any step function in $\mathbb{R}^n$

$$f = \sum_{k=1}^{n} \underset{\substack{\uparrow \\ \mathbb{C}}}{a_k}\, \chi_{\underbrace{\prod_{\ell=1}^{n} [a_\ell^{(k)}, b_\ell^{(k)}]}_{\text{box in } \mathbb{R}^n}}$$

Hence, $\lim_{\xi \to \infty} \hat{f}(\xi) = 0$.

<u>Corollary</u> (Riemann-Lebesgue lemma)

For any $f \in L^1(\mathbb{R}^n)$,

$$\lim_{\xi \to \infty} \hat{f}(\xi) = 0.$$

<u>Proof:</u> Fix $\varepsilon > 0$. There exists a step function g with

$$\|f - g\|_1 < \frac{\varepsilon}{2}.$$

Then $\exists R > 0$ s.t. $\forall |\xi| > R$

$$|\hat{g}(\xi)| < \frac{\varepsilon}{2}$$

(because g is a step function)

Therefore,

$\forall |\xi| > R,$

$$|\hat{f}(\xi)| \le |\hat{g}(\xi)| + |\widehat{f-g}(\xi)|$$
$$\le \frac{\varepsilon}{2} + \underbrace{\|f-g\|_1}_{\le \frac{\varepsilon}{2}} < \varepsilon$$

$\blacksquare$

Fourier inversion formula

" $\hat{\hat{f}}(x) = (2\pi)^n f(-x)$ "

(quotation marks: in the sense of formal distribution)

Hormander's notation: $\check{f}(x) = f(-x)$

" $\hat{\hat{f}} = (2\pi)^n \check{f}$ "

There are two spaces in which it's nicer to invert FT

1) $\widehat{L^2(\mathbb{R}^n)} = L^2(\mathbb{R}^n)$ FT is an isometry

2) $\mathcal{S}$ = Schwartz functions

$$\widehat{\mathcal{S}} = \mathcal{S}$$

"a small space"

Notation: • $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$
multi-index

• $\partial^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f$

• $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$

• $X^\alpha = \prod_{k=1}^{n} X_k^{\alpha_k}$

a monomial

Definition: A C^∞-smooth $f: \mathbb{R}^n \to \mathbb{C}$ is a Schwartz function (i.e., $f \in \mathcal{S}$) if $\forall \alpha, \beta \in \mathbb{Z}^n_{\geq 0}$

$$X^\alpha \partial^\beta f$$

is a bounded function in $\mathbb{R}^n$.

- I.e., any partial derivative $\partial^\beta f$ is bounded, and it decays at ∞ faster than $\frac{1}{P}$ for any polynomial P.

Examples:

- $e^{-x^2/2} \in \mathcal{S}$
- $\frac{1}{\cosh} \in \mathcal{S}$
- Any C^∞-smooth compactly-supported function

$$C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}$$

- $$\mathcal{S} \subseteq L^1(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n) \quad \forall 1 \le p \le \infty$$

- $\mathcal{S}$ is dense in $L^p(\mathbb{R}^n)$ $\forall\ 1 \le p < \infty$.

Topology in $\mathcal{S}$: Many semi norms, for $k \geq 0$

define a semi-norm $|\cdot|_k$ in $\mathcal{S}$

$$|f|_k = \sup \left\{ |x^\alpha \partial^\beta f(x)| \; ; \; \begin{array}{c} x \in \mathbb{R}^n \\ \alpha, \beta \in \mathbb{Z}^n_{\geq 0} \\ |\alpha| + |\beta| \leq k \end{array} \right\}$$

- $\mathcal{S}$ is a Frechét space

  A sequence $(f_m)_{m \geq 1}$ in $\mathcal{S}$ converges to $f \in \mathcal{S}$ if

$$\forall k \qquad |f_m - f|_k \xrightarrow{m \to \infty} 0$$

  Banach space – check just one norm.
  Frechét –"– – many norms.

- Topology is metritable

$$d(f,g) = \sum_{k=1}^{\infty} \frac{1}{k^2} \min \{ |f-g|_k, 1 \}$$

- $\mathcal{S}$ is a complete metric space.

Lemma: For $f \in \mathcal{S}$ also $\hat{f} \in \mathcal{S}$.

Moreover, for any $\alpha \in \mathbb{Z}^n_{\geq 0}$,

i) $\widehat{\partial^\alpha f}(\xi) = i^{|\alpha|} \xi^\alpha \hat{f}(\xi)$

ii) $\widehat{x^\alpha f}(\xi) = i^{|\alpha|} (\partial^\alpha \hat{f})(\xi)$

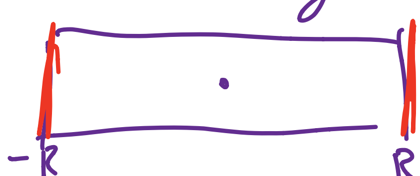
Proof: Begin with (i).
$\partial^\alpha f \in L^1$, in fact it's again a Schwartz function.

$$\widehat{\partial^\alpha f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} \partial^\alpha f(x)\, dx$$

(integrate by parts:

$$\int_{[-R,R]^n} \frac{\partial f}{\partial x_1} g = -\int_{[-R,R]^n} f \frac{\partial g}{\partial x_1} + \text{boundary term involving } fg \Bigg)$$

Integrating by parts,



$\leq R \cdot \sup$
$\leq R \cdot \frac{1}{R^{\cdot}}$

$$\widehat{\partial^\alpha f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle}\, \partial^\alpha f(x)\, dx$$ $\downarrow$ 0 $R\to\infty$

$$= \int_{\mathbb{R}^n} \Big(\underbrace{-\partial_x^\alpha\, e^{-i\langle \xi, x\rangle}}_{= i^{|\alpha|}\xi^\alpha e^{-i\langle \xi, x\rangle}}\Big) f(x)\, dx$$

$$= i^{|\alpha|}\xi^\alpha \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} f(x)\, dx$$

$$= i^{|\alpha|}\xi^{|\alpha|}\, \hat f(\xi).$$

ii) Differentiate under the integral sign

$$\hat f(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} f(x)\, dx$$

$$\partial^\alpha \hat f(\xi) = \int_{\mathbb{R}^n} \left(\partial_\xi^\alpha\, e^{-i\langle \xi, x\rangle}\right) f(x)\, dx$$

why?

$$= \int_{\mathbb{R}^n} \left((-i)^{|\alpha|} x^\alpha e^{-i\langle \xi, x\rangle}\right) f(x)\, dx$$

Hence

$$i^{|\alpha|}\partial^\alpha \hat{f}(\xi) = \int e^{-i\langle \xi, x\rangle} x^\alpha f(x)\,dx$$
$$= \widehat{x^\alpha f}(\xi).$$

$$\boxed{\text{If } f \in \mathcal{S} \text{ then } \partial^\alpha f,\ x^\alpha f \in \mathcal{S}}$$

Why $\hat{f} \in \mathcal{S}$? From (ii), $\hat{f}$ is smooth.

Once (i) and (ii) are proven,

$$\xi^\alpha \underbrace{\partial^\beta \hat{f}(\xi)}_{} = c \cdot \xi^\alpha \widehat{x^\beta f}$$

$$\underbrace{c' \widehat{\underbrace{\partial^\alpha(x^\beta f)}_{\in \mathcal{S} \subseteq L^1}}}_{\in L^\infty}$$

To summarize $\forall \alpha, \beta$

$$\xi^\alpha \partial^\beta \hat{f}(\xi)$$

is bounded. I.e., $\hat{f} \in \mathcal{S}$. $\blacksquare$

<u>Thm</u> (Fourier inversion in $\mathcal{S}$)

For any $f \in \mathcal{S}$, denoting $\check{f}(x) = f(-x)$, we have

$$\hat{\hat{f}} = (2\pi)^n \check{f}.$$

I.e., $\forall x \in \mathbb{R}^n$

$$f(-x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} \hat{f}(\xi)\, d\xi.$$

<u>Proof:</u> Fix $\varepsilon > 0$, and $x \in \mathbb{R}^n$.

$$\int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} \hat{f}(\xi)\, e^{-\varepsilon \frac{|\xi|^2}{2}} d\xi$$

$$\hat{f}(\xi) = \int f(y)\, e^{-i\langle \xi, y\rangle} dy$$

Fubini

$$\overset{?}{=} \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} f(y)\, e^{-\frac{\varepsilon|\xi|^2}{2}}\, dy\, d\xi$$

$$= \int_{\mathbb{R}^n} \Bigg(\underbrace{\int_{\mathbb{R}^n} e^{i\langle \xi, x-y\rangle} \underbrace{e^{-\frac{\varepsilon|\xi|^2}{2}}}_{\text{regularizing factor}} d\xi}_{\text{FT of Gaussian}} \Bigg) f(y)\, dy$$

$$= (2\pi)^n \int_{\mathbb{R}^n} \underbrace{\frac{(2\pi)^{-n/2}}{\varepsilon^{n/2}} \cdot e^{-\frac{|x-y|^2}{2\varepsilon}}}_{\text{probability density of a Gaussian r.v. of small variance } \varepsilon} f(y)\, dy$$

$$= (2\pi)^n \cdot \mathbb{E} f(x + \sqrt{\varepsilon}\, Z)$$

when Z is a standard Gaussian in $\mathbb{R}^n$

Next time: $\xrightarrow{\varepsilon \to n} f(x)$