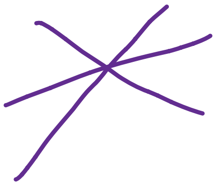
Construction of a Kakeya set

25/5/20

2D , $K \subseteq \mathbb{R}^2$, $\mathcal{H}^2(K) = 0$



$$K = \{ (x , f(t) + xt) ; x, t \in [0,1] \}$$

where

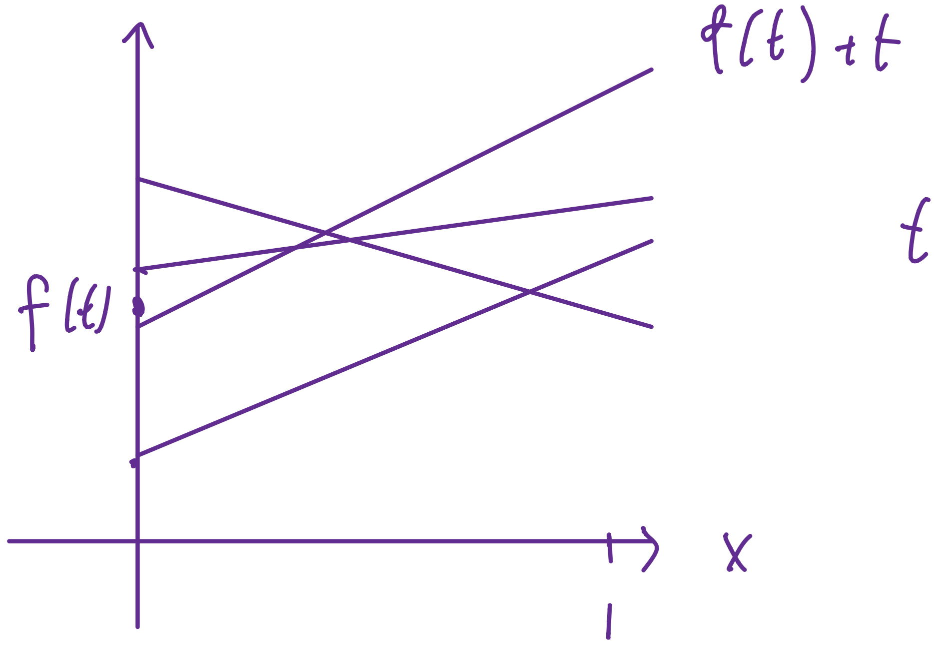
$f : [0,1] \to [0,1]$ is some function.

- Note that K contains a segment of length ≥ 1 in any slope $\in [0,1]$

- For any $t \in [0,1]$,

$$\text{graph}(x \mapsto \underbrace{f(t)}_{\text{const}} + \underbrace{xt}_{\text{a line segment of slope } t}) \subset K$$

$$K = \{ (x , f(t) + xt) ; x, t \in [0,1] \}$$



- 4 rotated copies K $(45^\circ, 90^\circ, 135^\circ, 0^\circ)$ contain a unit segment in each direction

- How would choose f so that $\mathcal{H}^2(K) = 0$?

[recall: f(t) is the point of intersection with the y axis of the line segment of slope t]

$$K = \{ (a , f(t) + at) ; a, t \in [0,1] \}$$

$$\mathcal{H}^2(K) \underset{\substack{\uparrow \\ \text{Fubini}}}{=} \int_0^1 \mathcal{H}^1\left(\{ f(t) + t ; t \in [0,1] \}\right) d$$

$$f_a(t) = f(t) + at$$

(a linear function plus f)

$$\mathcal{H}^2(K) = \int_0^1 \mathcal{H}^1\left(f_a([0,1]) \right) da$$

Q: Does then a measurable function $f : [0,1] \to [-3, 3]$ such that

- $\mathcal{H}^1(f([0,1])) = 0$?
  - constant
  - piecewise constant
- $\forall a \in [0,1]$,

  $f_a(t) = f(t) + at$ satisfies

$$\mathcal{H}^1\left(f_a([0,1])\right) = 0.$$

Lemma: YES!

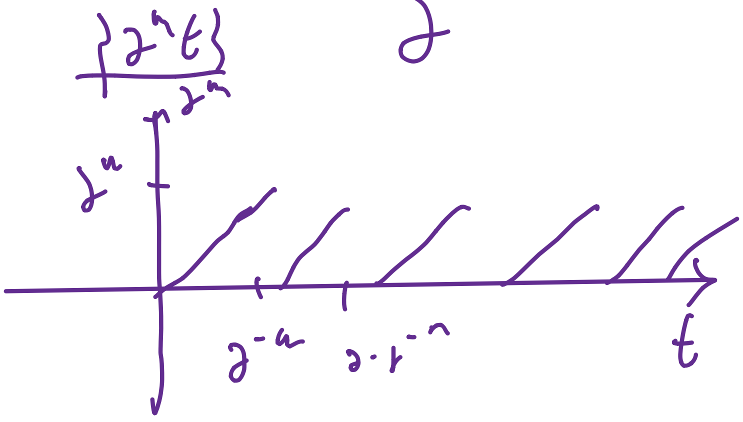
Let $(a_m)_{m\geq 1}$ dense in $[-1,2]$,

$a_0 = 0$, $|a_m - a_{m+1}| \leq \varepsilon_m \searrow 0$.

Our function is

$$f(t) = \sum_{m=1}^{\infty} (a_{m-1} - a_m) \frac{\{2^m t\}}{2^m}$$

$$\{t\} = t - \lfloor t \rfloor$$



Then f is well-defined, and $\forall a \in [0,1]$, $\operatorname{Im}(f_a)$ has zero 1D-measure.
and $\|f\|_\infty \leq 3$

Proof: For $k \geq 1$ s.t.

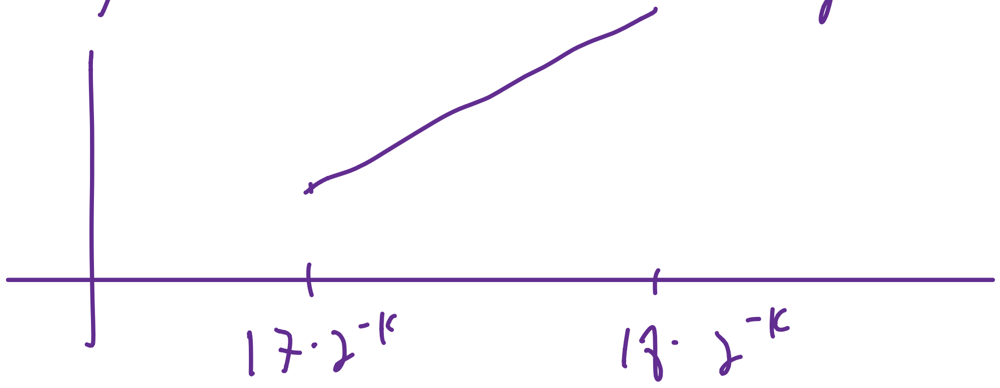
$$f_K(t) = \sum_{m=1}^{K} (a_{m-1} - a_m) \frac{\{2^m t\}}{2^m} \qquad \text{"}\frac{\sin(2^m t)}{2^m}\text{"}$$

$$|r_K(t)| = \left| \sum_{m=K+1}^{\infty} (a_{m-1} - a_m) \frac{\{2^m t\}}{2^m} \right|$$

$$\le \sum_{m=K+1}^{\infty} |\overset{\le \varepsilon_{m-1}}{a_{m-1} - a_m}| \frac{1}{2^m} \le \varepsilon_K \cdot \frac{1}{2^K}$$

Hence $f_K \longrightarrow f$ uniformly.

- f_K is piece linear: it is linear on dyadic interval of length 2^{-K}

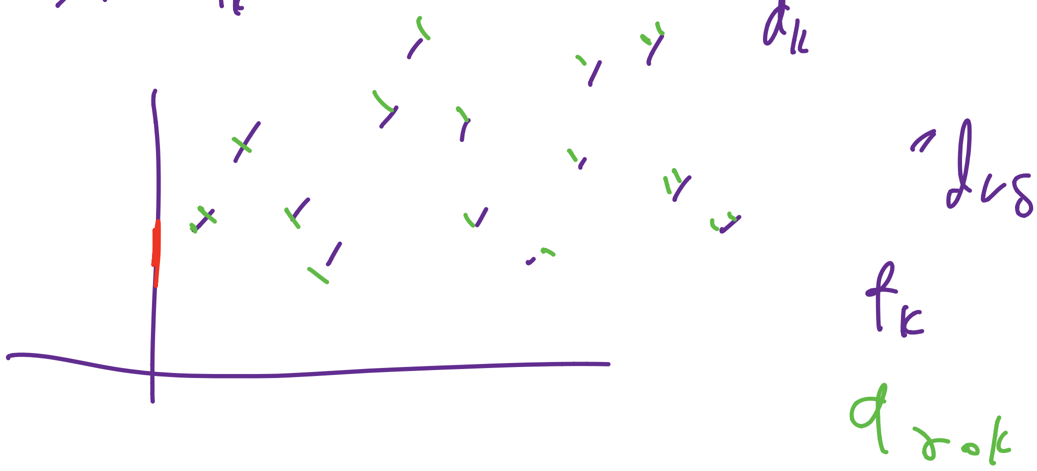


$\forall t$ s.t. $t \cdot 2^K \notin \mathbb{Z}$,

$$f_K'(t) = \sum_{m=1}^{K} (a_{m-1} - a_m) = -a_K$$

as $\frac{d}{dt}\{2^m t\} = 1$ whenever $2^m t \notin \mathbb{Z}$.

· $f = \lim f_k$

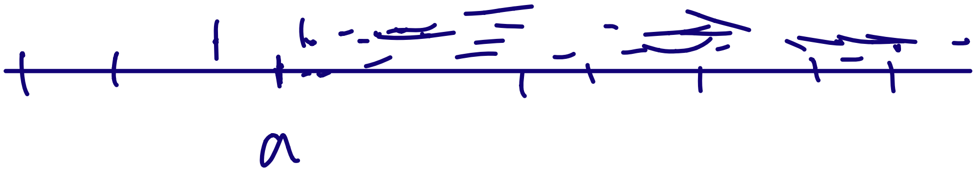


Fix $a \in [0,1]$ and try to understand $f_a([0,1])$.

· Pick k such that

$$|a_k - a| < \varepsilon_k$$

(pick a crossing from left to right of a)



Set $f_{k,a}(t) = f_k(t) + at$

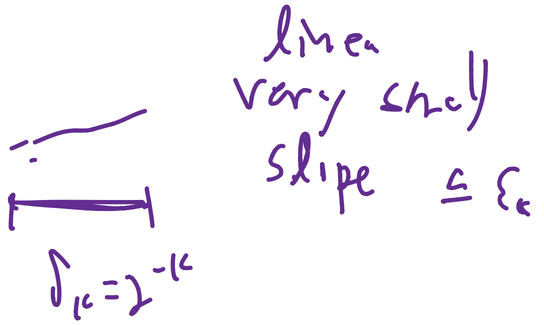
$f_a(t) = f(t) + at$

· In any dyadic interval of length 2^{-k},

$$f_{k,a}'(t) = f_k'(t) + a = -d_k + a$$

$$|f_{k,a}'(t)| \le \varepsilon_k$$

- Hence the image $f_{k,a}([0,1])$ is contained in the union of 2^k intervals of length most $\varepsilon_k \cdot 2^{-k}$.



- $f_a - f_{k,a} = r_k$, $\|r_k\|_\infty \le \varepsilon_k \cdot 2^{-k}$

- Therefore, the image of $f_a([0,1])$ is contained in the union of 2^k intervals of length at most

$$\underbrace{\varepsilon_k 2^{-k}}_{\text{before adding } r_k} + \underbrace{2\varepsilon_k 2^{-k}}_{\text{bound for contribution of } r_k}.$$

Hence

$$\mathcal{H}^1\left(f_a([0,1])\right) \leq \underset{\substack{\uparrow \\ \#\text{ intervals}}}{2^k} \cdot \left(3\varepsilon_k 2^{-k}\right)$$

$$\leq 3\varepsilon_k$$

true for any k with $|a - a_k| < \varepsilon_k$.

There are ∞ such k, so $\forall a \in [0,1]$

$$\mathcal{H}^1\left(f_a([0,1])\right) \leq 3\varepsilon_k \longrightarrow 0.$$

$\blacksquare$

Tempered Distributions

S^* = tempered distributions

= <u>continuous</u> linear functionals on S

= "generalized function"

· Any locally integrable / a Borel measure

$\exists N, C \quad |\mu|(B(0,R)) \leq CR^N$

is tempered.

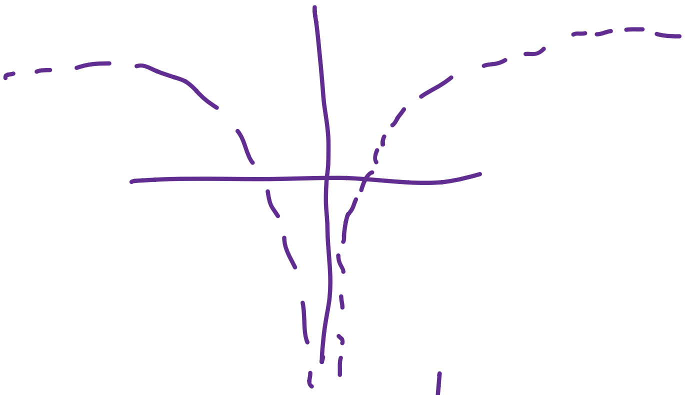
("anything that grows at most polynomially")

· Can always differentiate!

$F \in \mathcal{S}^* \implies \partial^\alpha F \in \mathcal{S}^*$

for any $\alpha \in \mathbb{Z}^n_{\geq 0}$.

Example: 1) $f(x) = \log|x|$ in $\mathbb{R}$



$|f(x)| \leq \frac{1}{|x|^\varepsilon}$ near 0

$\implies f \in L^1_{loc}(\mathbb{R}^n)$

$$\int_{[-R,R]} |f| \le C(R+1)\log(R+1) \le \tilde{C} R^2$$

$$\log|x| \in \mathcal{S}^*.$$

- What is f' ?

Classically, for $x \neq 0$

$$(\log|x|)' = \frac{1}{x} \rightsquigarrow \text{not locally integrable!}$$

Claim: In the sense of tempered distribution

$$\frac{d}{dx} \underset{\substack{\uparrow \\ \mathcal{S}^*}}{\log|x|} = pv\left(\frac{1}{x}\right) \quad \text{principal value}$$

where for $\varphi \in \mathcal{S}$

$$\boxed{L_{pv(1/x)}(\varphi) = \lim_{\varepsilon \to 0^+} \overset{\substack{\text{NOT ABS} \\ \text{CONVERGENT} \\ \downarrow}}{\int_{|x| \ge \varepsilon}} \varphi(x) \cdot \frac{1}{x}\, dx.}$$

NOT: $\lim \int_{x\in(-\varepsilon,2\varepsilon)}$, this is some thing.

Emphasize: $pv\left(\frac{1}{x}\right) \in \mathcal{S}^*$ is a 100% kosher tempered distribution.

reminder: $F \in \mathcal{S}^*$, $\varphi \in \mathcal{S}$

$$F'(\varphi) := -F(\varphi').$$

Proof: Why in sense of tempered, $\frac{d}{dx}\log|x| = pv\left(\frac{1}{x}\right)$?

By def., $\forall \varphi \in \mathcal{S}$, such $F = \overset{\mathcal{S}^*}{\overset{\in}{\log|x|}}$

$$F'(\varphi) = -F(\varphi') = -\int_{-\infty}^{\infty} \underbrace{\log|x|\, \varphi'(x)}_{\substack{\uparrow \\ L^1(\mathbb{R})}} dx$$

$$\overset{\text{dominated converge}}{=} -\lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \log|x| \, \varphi'(x) \, dx$$

$$= -\lim_{\varepsilon \to 0} \int_{-\infty}^{-\varepsilon} \underset{v}{\log(-x)} \, \underset{u'}{\varphi'(x)} \, dx + \int_{\varepsilon}^{\infty} \log(x) \, \varphi'(x) \, dx$$

$$= -\lim_{\varepsilon \to 0} \; -\int_{-\infty}^{-\varepsilon} \frac{1}{x} \varphi(x) \, dx + \log(-x) \, \varphi(x) \Big|_{-\infty}^{-\varepsilon}$$

$$- \int_{\varepsilon}^{\infty} \frac{1}{x} \varphi(x) \, dx + \log x \, \varphi(x) \Big|_{\varepsilon}^{\infty}$$

$$-2\varepsilon \, \varphi'(0) + o(\varepsilon)$$

$$= -\lim_{\varepsilon \to 0} \left[\log \varepsilon \cdot \left(\varphi(-\varepsilon) - \varphi(\varepsilon) \right) - \int_{|x| \geq \varepsilon} \frac{1}{x} \varphi(x) \, dx \right]$$

$$= \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{1}{x} \varphi(x) \, dx$$

$$\boxed{\begin{aligned} \varphi(\varepsilon) &= \varphi(0) \\ &+ \varepsilon \varphi'(0) + o(\varepsilon) \end{aligned}}$$

$$= L_{pr(\frac{1}{x})}(\varphi). \qquad \triangleright$$

Remark: $\lim \int_{\substack{x \geq \varepsilon \\ \text{or } x \leq -2\varepsilon}}$ we get $pr(\frac{1}{x})$

plus some δ at 0.

2) $F = \sum_{x \in \mathbb{Z}^n} \delta_x \in \mathcal{S}^*$

Main theory

· A C^∞-function f on $\mathbb{R}^n$ is of moderate growth if $\forall \alpha\ \exists C_\alpha, N_\alpha$

$$|\partial^\alpha f(x)| \leq C_\alpha (1 + |x|^{N_\alpha})$$

(Typically (say in 1D), $N_\alpha = N - |\alpha|$)

- $$\underset{\substack{\uparrow \\ \mathcal{S}^{*}}}{F} * \underset{\substack{\uparrow \\ \mathcal{S}}}{\psi}\ (\varphi) := F(\check{\psi} * \varphi)$$

Prop 1: $F \in \mathcal{S}^{*}$, $\psi \in \mathcal{S}$, then $F * \psi$ is C^∞ of moderate function

Proof: Last week we proved that if is continuous,

$$(F * \psi)(x) = F(\tau_x \check{\psi})$$

when $\tau_x \varphi(y) = \varphi(y - x)$.

We showed that

$$x \longmapsto F(\tau_x \psi) \quad \text{is cont.}$$

Why is $\frac{\partial}{\partial x_1} F(\tau_x \psi)$ exist?

$e_1 = (1, 0, \dots 0)$

$$\frac{1}{\varepsilon}\left(F(\tau_{x+\varepsilon e_1}\psi) - F(\tau_x \psi)\right) \qquad \forall \psi \in \mathcal{S}$$

$$= F\left(\frac{\tau_{x+\varepsilon e_1}\psi - \tau_x \psi}{\varepsilon}\right)$$

$$\Big\downarrow \ \varepsilon \to 0$$

$$-\partial^1 \tau_x \psi \qquad \text{in } \mathcal{S}$$

Hence when $\varepsilon \to 0$

(x) $$\frac{1}{\varepsilon}\left[F(\tau_{x+\varepsilon e_1}\psi) - F(\tau_x \psi)\right] \longrightarrow -F(\partial^1 \tau_x \psi)$$

equivalently, $\forall x \in \mathbb{R}^n$

$$\frac{\partial}{\partial x_1} F(\tau_x \psi) = -F\left(\frac{\partial}{\partial x_1} \tau_x \psi\right)$$

$$\Downarrow$$

$$\partial_x^\alpha F(\tau_x \psi) = (-1)^{|\alpha|} F(\partial^\alpha \tau_x \psi)$$

Hence $x \mapsto F(\tau_x \psi)$ is C^∞.

· Why is it of moderate growth?

Since $F \in S'^{*}$, then exists N, C s.t.

$$|F(\varphi)| \le C\,|\varphi|_N \quad \forall \varphi \in S'$$

· Hence, $\forall \alpha, x$

$$|\partial^\alpha F(\tau_x \psi)| = |F(\partial^\alpha \tau_x \psi)|$$

$$\le C\,|\partial^\alpha \tau_x \psi|_N$$

$$= C \cdot \sup_{\substack{y \in \mathbb{R}^n \\ |\gamma|+|\beta| \le N}} |y^\gamma \partial^{\beta+\alpha} \underbrace{(\tau_x \psi)(y)}_{\psi(y-x)}|$$

$$= C \cdot \sup_{\substack{y \in \mathbb{R}^n \\ |\gamma|+|\beta| \le N}} |\underbrace{(x+y)^\gamma}_{\substack{\wedge \\ C_N(|x|^n + |y|^n)}} \partial^{\alpha+\beta} \psi(y)|$$

$$\leq \tilde{C}\left(1+|x|^N\right) \cdot \sup_{\substack{y \in \mathbb{R}^n \\ |\gamma|+|\beta| \leq N}} \Big| y^\gamma \partial^{\alpha+\beta} \underset{\overset{\uparrow}{\mathcal{S}}}{\psi(y)} \Big|$$

$$\leq \tilde{C}_\alpha \left(1+|x|^N\right).$$

$\square$

· **Topology in $\mathcal{S}^*$**: We say that

$\mathcal{S}^* \ni F_m \xrightarrow{m\to\infty} \underset{\overset{\uparrow}{\mathcal{S}^*}}{F}$ converges weakly if

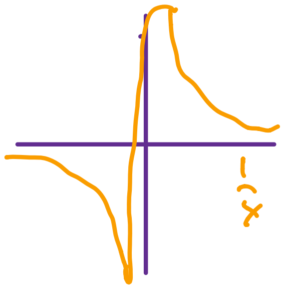
$\forall \varphi \in \mathcal{S}$ (fixed)

$$F_m(\varphi) \xrightarrow{m\to\infty} F(\varphi)$$

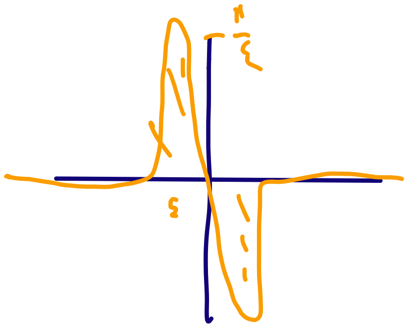
Prop 2: The space of C^∞ functions of moderate growth is dense in $\mathcal{S}^*$ in the weak topology.

· Useful way to think about $\mathcal{S}^*$:

$\mathrm{pv}\left(\frac{1}{x}\right)$



δ_0'



__Proof:__ Pick $\eta \in C_c^\infty(\mathbb{R}^n)$, $\eta \geq 0$, $\int_{\mathbb{R}^n} \eta = 1$, s.t. η even $\check{\eta} = \eta$

$$\eta_\varepsilon(x) = \varepsilon^{-n} \eta(x/\varepsilon) \quad \text{mollifier.}$$

$\cap \mathcal{S}'$

Then, $\forall \varphi \in \mathcal{S}$

$$\varphi * \eta_\varepsilon \xrightarrow{\varepsilon \to 0} \varphi \quad \text{in } \mathcal{S}$$

Therefore, by definition, $\forall F \in \mathcal{S}^*$

$$\underset{\substack{\uparrow \\ C^\infty \text{ moderate growth}}}{F *} \eta_\varepsilon \xrightarrow{\varepsilon \to 0^+} F$$

Why? Check that $\forall \varphi \in \mathcal{S}$,

However

$$\begin{tikzcd}
F*\eta_\varepsilon(\varphi) \arrow[r, "\varepsilon\to 0"] \arrow[d, equal] & F(\varphi) \arrow[dd, equal] \\
F(\check{\eta}_\varepsilon * \varphi) & \\
F(\check{\eta}_\varepsilon * \varphi) & F(\varphi)
\end{tikzcd}$$

Since $\eta_\varepsilon * \varphi \longrightarrow \varphi$ in $\mathcal{S}$,

by continuity of F

$$F(\eta_\varepsilon * \varphi) \xrightarrow{\varepsilon\to 0} F(\varphi),$$

as desired. ∎

$$(F*\psi)(\varphi) = F(\check{\psi} * \varphi)$$

Fourier transform of tempered distribution

Given any continuous operator

$$T : \mathcal{S} \to \mathcal{S}$$

we may define

$$(T^* F)(\varphi) = F(T\varphi)$$

Lemma: ("continuity of FT in $\mathcal{S}$")

$\forall \varphi \in \mathcal{S}$ (in $\mathbb{R}^n$)

$\forall N$

$$|\underset{\overset{\uparrow}{\mathcal{S}}}{\hat{\varphi}}|_N \le C_{N,n} |\varphi|_{N+n+1}$$

Proof: $\forall \varphi \in \mathcal{S}$

$$\|\varphi\|_1 \le C_n |\varphi|_{n+1} \quad ✓$$

$$\left(\text{If } |\varphi|_{n+1} \le M \text{ then } |\varphi(x)| \le \frac{CM}{1+|x|^{n+1}} \right)$$

(which is integrable)

Now, $\forall N$ $\forall \alpha, \beta$ $|\alpha| + |\beta| \leq N$

$$\|x^{\alpha} \partial^{\beta} \hat{\varphi}\|_{\infty} = \|\widehat{\partial^{\alpha} (\xi^{\beta} \varphi)}\|_{\infty}$$

$$\leq \|\partial^{\alpha}(\xi^{\beta}\varphi)\|_{1} \leq C_{n} |\partial^{\alpha}(\xi^{\beta}\varphi)|_{n+1}$$

$$\leq C_{n,\alpha,\beta} |\varphi|_{n+|\alpha|+|\beta|+1}$$

$$\leq C_{N,n} |\varphi|_{n+N+1} .$$ ∎

Recall: We identify a function f and the functional

$$L_{f}(\varphi) = \int f \varphi$$

$$\int \hat{f} \varphi = \int f \hat{\varphi}$$

Def: For $F \in \mathcal{S}'^{*}$, its Fourier transform $\hat{F} \in \mathcal{S}'^{*}$ defined via

$$\hat{F}(\varphi) := F(\hat{\varphi}) \quad \forall \varphi \in \mathcal{S}.$$

This coincides with the usual F.T. in $L^1(\mathbb{R}^n)$.

A well-defined distribution, tempered.

$$\underset{\mathcal{S}}{\underset{\uparrow}{\varphi_k}} \longrightarrow \underset{\mathcal{S}}{\underset{\uparrow}{\varphi}} \Rightarrow \hat{\varphi}_k \longrightarrow \hat{\varphi} \Rightarrow$$

$$\hat{F}(\varphi_k) \longrightarrow \hat{F}(\varphi).$$

- Hence, $\forall$ L^p-function, smooth function of moderate growth – the Fourier transform is well-defined, in the sense of distributions.

Examples: 1) $\hat{\delta}_0(\xi) = 1$.

Why? $\hat{\delta}_0(\varphi) = \delta_0(\hat{\varphi}) = \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi$

$= L_1(\varphi).$

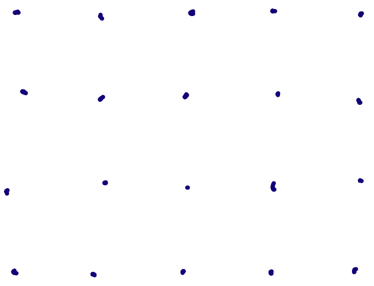
2) If $F \equiv 1 \in \mathcal{S}'$ then $\hat{F} = (2\pi)^n \delta_0$

$$\hat{F}(\varphi) = F(\hat{\varphi}) = \int_{\mathbb{R}^n} \hat{\varphi} = \hat{\hat{\varphi}}(0) =$$

$$= (2\pi)^n \check{\varphi}(0) = (2\pi)^n L_{\delta_0}(\varphi).$$

In this case, $\hat{\hat{F}} = (2\pi)^n F$.

3) $F = \sum_{m \in \mathbb{Z}^n} \delta_m$



$$\hat{F} = (2\pi)^{n/2} F$$

Poisson summation formula! $\forall \varphi \in \mathcal{S}$

$$\underbrace{\sum_{m \in \mathbb{Z}^n} \hat{\varphi}(m)}_{\hat{F}(\varphi)} = (2\pi)^{n/2} \underbrace{\sum_{m \in \mathbb{Z}^n} \varphi(m)}_{F(\varphi)}$$

Proposition (Fourier inversion formula)

1) $\forall\ F \in \mathcal{S}'^{*}$

$$\hat{\hat{F}} = (2\pi)^n\ \check{F}$$

when $\check{F}(\varphi) = F(\check{\varphi})$

2) $\forall\ \alpha$

$$\widehat{\partial^\alpha F} = i^{|\alpha|} \xi^\alpha\ \hat{F}(\xi)$$

$$\widehat{X^\alpha F} = i^{|\alpha|}\ \partial_\xi^\alpha\ \hat{F}$$

Proof: 1) For any $\varphi \in \mathcal{S}'$

important result, Fourier inversion.

$$\hat{\hat{F}}(\varphi) = \hat{F}(\hat{\varphi}) = F(\hat{\hat{\varphi}}) \swarrow = F((2\pi)^n\ \check{\varphi}) = (2\pi)^n\ \check{F}(\varphi).$$

2) $$\widehat{\partial^\alpha F}(\varphi) = \partial^\alpha F(\hat{\varphi})$$

$$= (-1)^{|\alpha|} F(\partial^\alpha \hat{\varphi})$$

$$= (-1)^{|\alpha|} F\left((-i)^{|\alpha|} \widehat{x^\alpha \varphi}\right)$$

$$= i^{|\alpha|} \hat{F}(x^\alpha \varphi)$$

$$= i^{|\alpha|} \underbrace{(x^\alpha \hat{F})}(\varphi)$$

the product of a moderate growth function by $\hat{F}$.

<u>Examples:</u>

1) In 1D

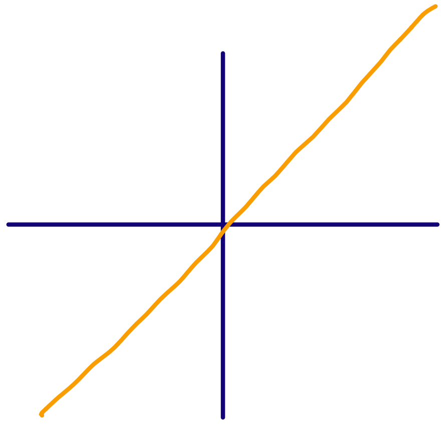
$$\widehat{\delta_0'}(\xi) = i\xi \hat{\delta}_0 = i\xi$$

$$\Vert$$

$$\widehat{\frac{\partial}{\partial \xi}\delta_0}$$

$$\widehat{\delta_0'} =$$

function



$\boxed{\text{in 1D}}$

2) $$\underset{\substack{\uparrow\\ L^1}}{\widehat{e^{-|x|}}}(\xi) = \underset{\substack{\uparrow\\ L^1}}{\frac{2}{1+\xi^2}}$$

↘ Cauchy distribution

$$\widehat{\frac{1}{1+x^2}}(\xi) = \pi \cdot e^{-|\xi|}$$

$\in L^1$

Hence, in the sense of tempered distributions

$$\widehat{\frac{x}{1+x^2}}(\xi) = i \frac{\partial}{\partial \xi}\left(\pi e^{-|\xi|}\right)$$

↑ classically, not diff. at 0.

Claim: In the sense of tempered distributions,

$$\frac{\partial}{\partial \xi} e^{-|\xi|} = -\operatorname{sgn}(\xi)\, e^{-|\xi|} \in L^1$$

In general, for any continuous, piecewise

C^1 - function, its derivative in the sense of distribution coincides with the classical derivative (no δ measure at the non-diff. points).

Proof: $\forall \varphi \in \mathcal{S}$

$$\frac{\partial}{\partial \xi} e^{-|\xi|}(\varphi) = -\int_{\mathbb{R}} e^{-|\xi|} \varphi'$$

$$= -\int_{-\infty}^{0} e^{+\xi} \varphi'(\xi)\, d\xi - \int e^{-\xi} \varphi'(\xi)\, d\xi$$

$$= +\int_{-\infty}^{0} e^{\xi} \varphi(\xi)\, d\xi - \int e^{-\xi} \varphi(\xi)\, d\xi$$

$+$ boundary terms at ∞ (vanish)

$+ \; e^0 \varphi(0) - e^0 \varphi(0) \; (=0)$

$$= -\int_{-\infty}^{\infty} \operatorname{sgn}(\xi)\, e^{-|\xi|} \varphi(\xi)\, d\xi$$

Hence

$$\widehat{\frac{x}{1+x^2}}(\xi) = -i\pi \, \mathrm{sgn}(\xi)\, e^{-|\xi|}.$$

3) Repeat with ε

$$f(t) = e^{-\varepsilon|t|}$$

$$\hat{f}(\xi) = \frac{1}{\varepsilon}\,\frac{2}{1+(\xi/\varepsilon)^2} = \frac{2\varepsilon}{\xi^2+\varepsilon^2}$$

Hence, by Fourier inversion:

$$\widehat{\frac{1}{x^2+\varepsilon^2}}(y) = \frac{\pi}{\varepsilon}\cdot e^{-\varepsilon|y|}$$

Multiply by x,

$$\widehat{\frac{x}{x^2+\varepsilon^2}}(y) = i\,\frac{\pi}{\varepsilon}\cdot\left(-\varepsilon\, \mathrm{sgn}(y)\right)\cdot e^{-\varepsilon|y|}$$

$$= -i\pi\cdot \mathrm{sgn}(y)\, e^{-\varepsilon|y|}$$

Claim 1: In the sense of distribution,

$$\underset{\underset{\mathcal{S}'}{\uparrow}}{pv\left(\frac{1}{x}\right)} = \lim_{\varepsilon\to 0} \underset{\underset{\mathcal{S}'}{\uparrow}}{\frac{x}{x^2+\varepsilon^2}} \quad \text{weakly}$$

Claim 2: The Fourier transform in $\mathcal{S}'$ is continuous in the weak topology.

Proof of 2: $F_k \longrightarrow F$ in $\mathcal{S}'$

i.e., $\forall \varphi \in \mathcal{S}$

$$F_k(\varphi) \longrightarrow F(\varphi)$$

Hence $\forall \varphi \in \mathcal{S}$, also $\hat{\varphi} \in \mathcal{S}$, hence

$$\hat{F}_k(\varphi) = F_k(\hat{\varphi}) \longrightarrow F(\hat{\varphi}) = \hat{F}(\varphi)$$

Then $\hat{F}_k \longrightarrow \hat{F}$ in $\mathcal{S}'$.

. Similarly, derivative is cont. in $\mathcal{S}'$.

Proof of 1: We need to prove

$$\begin{array}{ccc} \mathrm{pr}\left(\frac{1}{x}\right) & = & \lim\limits_{\varepsilon\to 0} \frac{x}{x^2+\varepsilon^2} \\ \| & & \| \quad \text{classically} \\ \frac{\partial}{\partial x}\log|x| & & \frac{\partial}{\partial x}\log\sqrt{x^2+\varepsilon^2} \in C^\infty \end{array}$$

Enough to prove

$$\log|x| = \lim_{\varepsilon\to 0} \log\sqrt{x^2+\varepsilon^2}$$

This converges in $L^1([-1,1])$ and uniformly in $\mathbb{R}\setminus[-1,1]$, hence as tempered distribution.

Corollary:

$$\widehat{\mathrm{pr}\left(\frac{1}{x}\right)} = \lim_{\varepsilon\to 0} \widehat{\frac{x}{x^2+\varepsilon^2}}$$

$$= \lim_{\varepsilon \to 0} -i\pi \quad \mathrm{sgn}(y)\, e^{-\varepsilon |y|}$$

$$= -i\pi \quad \mathrm{sgn}(y)$$

$$\widehat{pv\left(\frac{1}{x}\right)}(y) = C \qquad \mathrm{sgn}(y)$$