1/6/2020

Last week:

$$F = (pv)\frac{1}{x} = pv\left(\frac{1}{x}\right) \in \mathcal{S}^*$$

defined as: $\forall \varphi \in \mathcal{S}$,

$$F(\varphi) = \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx$$

- How do generalize this example of $pv\left(\frac{1}{x}\right)$?
- Suppose $E \subseteq \mathcal{S}$ is a subspace (either dense, or its closure is of finite codim). Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is measurable, and

$\forall \varphi \in E$

$$L_f(\varphi) = \int_{\mathbb{R}^n} f\varphi$$ is absolutely convergent

and furthermore, $\exists N, C$ s.t.

$$|L_f(\varphi)| \le C\, |\varphi|_N \qquad \forall \varphi \in E$$

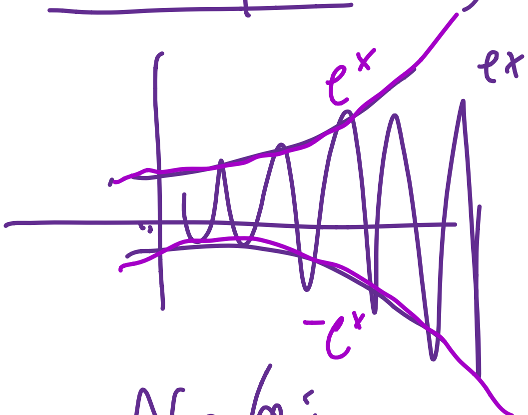
↖ semi-norm

- By Hahn-Banach $\exists$ an extension of $L_f$ for $L \in \mathcal{S}^*$ with

$$|L(\varphi)| \le C\, |\varphi|_N \qquad \forall \varphi \in \mathcal{S}$$

and $L = L_f$ on $E$.

Examples: 1) $f(x) = e^x \cos(e^x)$ on $\mathbb{R}$




exponentially

$$E = C_c^\infty(\mathbb{R})$$

Note: $f(x) = g'(x) \quad g(x) = \sin(e^x)$

Hence $\forall \varphi \in E$,

$$\left|\int_{\mathbb{R}} f\varphi\right| = \left|\int_{\mathbb{R}} g'\varphi\right| = +\left|\int_{-\infty}^{\infty} \underbrace{\sin(e^x)}_{\text{bdd}}\ \underbrace{\varphi'(x)}_{\text{decays fast, a Schwartz function}}\,dx\right|$$

$$\leq 100 \cdot |\varphi|_N \qquad N=3$$

$$|\varphi'(x)| \leq 100 \frac{|\varphi|_3}{1+|x|^2}$$

- $E \subset \mathcal{S}$ is a dense subspace

(dense? $\quad \eta(0)=1, \quad \eta \in C_c^\infty(\mathbb{R}^n)$

$f \in \mathcal{S} \quad \eta(\varepsilon x) f(x) \in C_c^\infty(\mathbb{R}^n)$

$$\varepsilon \eta'(\varepsilon x) f(x) + \eta(\varepsilon x) f'(x)$$

$\downarrow$

$0$ )

Hence $\exists!$ $L \in \mathcal{S}^*$ s.t. $\forall \varphi \in E$,

$$L(\varphi) = \int_{-\infty}^{\infty} e^x \cos(e^x)\, \varphi(x)\, dx$$

- Recall: If convolve any $f \in \mathcal{S}^*$ with a tiny bump function, we get a

smooth function of $\underline{\text{moderate growth}}$
(poly. growth with all derivatives).

Hence if $\eta \in C_c^\infty(\mathbb{R}^n)$, then
$\eta * (e^x \cos(e^x))$ grows at most polynomially.

2) $E = \{\varphi \in \mathcal{S} ;\ \varphi(0) = 0\}$.

$f(x) = \frac{1}{x}$.

$\varphi_0(x) = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}}$

Any $\varphi \in \mathcal{S}$ is $\varphi = \underset{\in E}{\varphi_1} + \underset{\in \mathbb{R}}{c}\, \varphi_0$

$\forall \varphi \in E$,

$$L_f(\varphi) = \int_{-\infty}^{\infty} \underbrace{\frac{1}{x}\varphi(x)}_{\text{bdd}}\, dx$$ is absolutely convergent

Here there are many extension to $L \in \mathcal{S}^*$. E.g.

$\mathrm{pv}\left(\frac{1}{x}\right)$ or $\mathrm{pv}\left(\frac{1}{x}\right) + 17 \cdot \delta_0$

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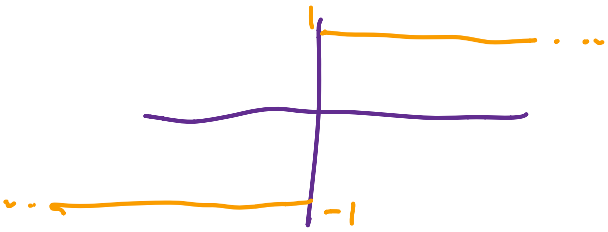
- We proved that
$$\mathrm{pv}\left(\frac{1}{x}\right) = \lim_{\varepsilon \to 0^+} \frac{x}{x^2+\varepsilon^2} \quad \text{weakly in } \mathcal{S}^*$$
$$\frac{x}{x^2+\varepsilon^2} \in \mathcal{S}^*$$
$$\mathrm{pv}\left(\frac{1}{x}\right) = \frac{d}{dx} \log|x| \qquad \frac{x}{x^2+\varepsilon^2} = \frac{d}{dx} \log\sqrt{x^2+\varepsilon^2}$$

- A computation showed that
$$\widehat{\frac{x}{x^2+\varepsilon^2}}(y) = c \,\mathrm{sgn}(y)\, e^{-\varepsilon|y|}$$

- Fourier transform is weakly continuous in $\mathcal{S}^*$, hence
$$\boxed{\widehat{\mathrm{pv}\left(\frac{1}{x}\right)}(y) = c \;\mathrm{sgn}(y)}$$

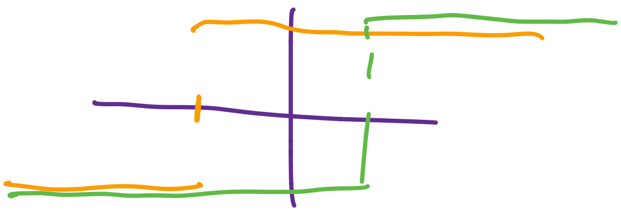
· Fourier inversion in $\mathcal{S}^*$:

$$\widehat{\mathrm{sgn}(y)} = \frac{1}{c}\,\widehat{\widehat{\mathrm{pv}\left(\frac{1}{x}\right)}} = \frac{2\pi}{c}\,\mathrm{pv}\left(\frac{1}{x}\right)^{\vee}$$




$$= -\frac{2\pi}{c}\,\mathrm{pv}\left(\frac{1}{x}\right)$$

## Sanity check (and determine $c$)



2. $2 \cdot \mathbb{1}_{[-1,1]}(y) = \mathrm{sgn}(y+1) - \mathrm{sgn}(y-1)$

These are elements of $\mathcal{S}^*$. Hence

$$\widehat{2\,\mathbb{1}_{[-1,1]}} = \widehat{\mathrm{sgn}(y+1)} - \widehat{\mathrm{sgn}(y-1)}$$

$$\parallel$$

$$4\,\frac{\sin(x)}{x} = -e^{+ix}\cdot\frac{2\pi}{c}\,\mathrm{pv}\left(\frac{1}{x}\right) + e^{-ix}\,\frac{2\pi}{c}\,\mathrm{pv}\left(\frac{1}{x}\right)$$

$$4\,\frac{\sin x}{x} = \frac{4\pi}{c}\left(\frac{-e^{-ix}+e^{ix}}{2}\right)\operatorname{pv}\left(\frac{1}{x}\right)$$

$$4\,\frac{\sin x}{x} = -i\,\frac{4\pi}{c}\,\sin x\cdot \operatorname{pv}\left(\frac{1}{x}\right)$$

<u>Claim:</u> $\underbrace{\sin x}_{\text{smooth moderately growing}}\cdot\ \operatorname{pv}\left(\frac{1}{x}\right) = \frac{\sin x}{x}$

$\operatorname{pv}\left(\frac{1}{x}\right) \in S'$

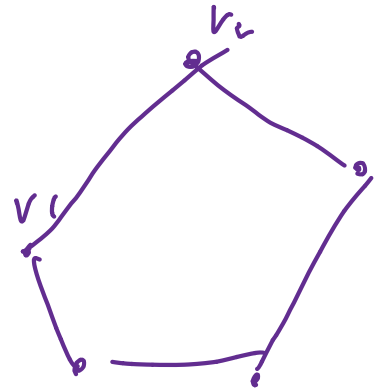
$$\left(\forall \varphi \in S,\ \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\sin x\cdot\frac{1}{x}\cdot\varphi(x)\,dx = \int_{\mathbb{R}}\frac{\sin x}{x}\,\varphi(x)\right)$$

$$4 = -i\,\frac{4\pi}{c} \Rightarrow \boxed{c = -i\pi.}$$

- A glimpse into Brion's formula, which is a higher-dim. generalization of the sanity check.

- $K \subseteq \mathbb{R}^n$ is a convex polytope

  $v_1, \dots, v_N \in \mathbb{R}^n$ vertices

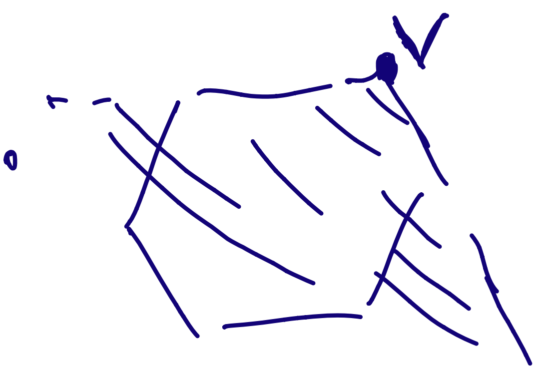



## An elementary version of Brion:

$$\widehat{\mathbb{1}_K}(\xi) = \sum_{v \text{ vertex}} q_v(\xi)\, e^{-i\langle \xi, v\rangle}$$

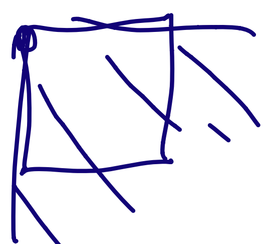
when $q_v(\xi)$ is a function, homogeneous of order $-n$, on which we know many things.

- $\mathbb{1}_K \in L^1 \Rightarrow \widehat{\mathbb{1}_K}$ is cont, bdd function (real analytic)

- 


$$K_v = v + \bigcup_{t>0} t(K-v)$$

a cone

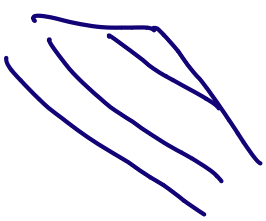


Brion: $$\underbrace{\widehat{\mathbb{1}_K}}_{C^\infty} = \sum_{\substack{v \\ \text{vertex}}} \underbrace{\widehat{\mathbb{1}_{K_v}}}_{\mathcal{S}'} + E$$

where $E$ is some tempered distribution supported on a finite union of hyperplanes.

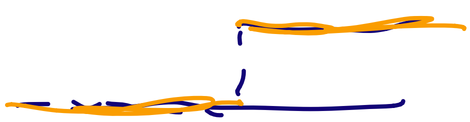
- $\widehat{\mathbb{1}_{K_v}}$ can be written explicitly in 2D, or in n-D if $K$ is a simplex.

$$K_v = T(\mathbb{R}^n_+) \qquad T \text{ affine}$$



$$\widehat{\mathbb{1}_{K_v}} = \widehat{\mathbb{1}_{\mathbb{R}^n_+}} \circ T \cdot |\det T|$$

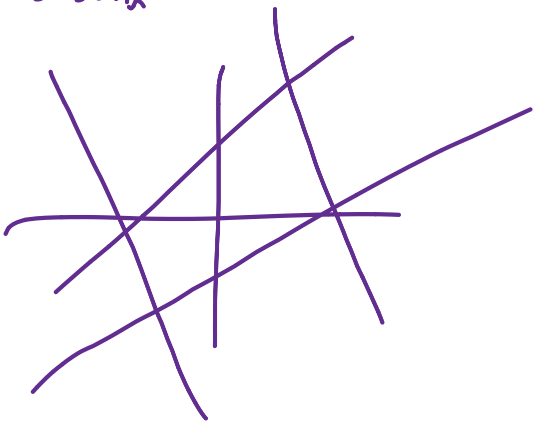
$$\widehat{\mathbb{1}_{\mathbb{R}_+}}(\xi) = c \cdot \mathrm{pv}\left(\frac{1}{\xi}\right) + E$$

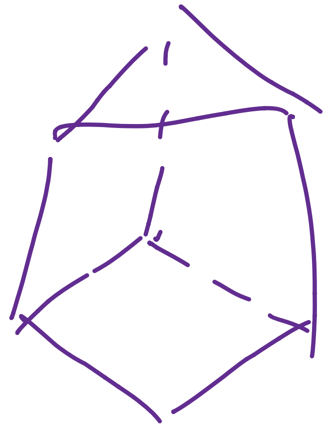


$(E =$ some $\delta$ at zero$)$

- Barvinok's book "A course of Convexity"

- hyperplane in $\mathbb{R}^n$: An affine subspace of codimension





**Lemma:** $\forall \xi \in \mathbb{R}^n$, $\forall \lambda \in \mathbb{R}^n$, $\langle \lambda, \xi \rangle \neq 0$,

$$\int_K e^{-i\langle \xi, x\rangle}\, dx = i \sum_{F \text{ facets}} \frac{\langle \lambda, N\rangle}{\langle \lambda, \xi\rangle} \int_F e^{-i\langle \xi, x\rangle}$$

**Proof:** Apply the divergence thm to

$$V(x) = \frac{\lambda}{\langle \lambda, \xi\rangle} e^{-i\langle \xi, x\rangle}$$

$$\mathrm{div}(V) = \sum_{j=1}^{n} \frac{\lambda_j(-i\xi_j)}{\langle \lambda, \xi\rangle} e^{-i\langle \xi, x\rangle}$$

$$= -i\, e^{-i\langle \xi, x\rangle}$$

Hence,

$$\hat{\mathbb{1}}_K(\xi) = i \cdot \int_K \operatorname{div}(V) = i \int_{\partial V} V \cdot N$$

outer normal

$$= i \sum_{F \text{ facet}} \frac{\langle \lambda, N \rangle}{\langle \lambda, \xi \rangle} \int_F e^{-i\langle \xi, x \rangle}$$

∎

Proof of the elementary version of Brion:

By induction:

$$\int_K e^{-i\langle x, \xi \rangle} = \sum_{F\ (n-1)\text{-dim faces}} \frac{\langle \lambda, N \rangle}{\langle \lambda, \xi \rangle} \int_F e^{-i\langle x, \xi \rangle}$$

$$= \ldots \sum_{(n-2)\text{-dim faces}} \frac{\langle \lambda, N \rangle \langle \lambda_F, N_F \rangle}{\langle \lambda, \xi \rangle \langle \lambda_F, \xi \rangle} \cdot \int_{F_2} e^{-i\langle x, \xi \rangle}$$

$$= \cdots = \sum_{v \text{ vertices}} \underbrace{q_v(\xi)}_{\substack{\text{product of } n \\ \text{terms, each of} \\ \text{the form } \frac{\langle \lambda, N\rangle}{\langle \lambda, \xi\rangle} \\ \text{for various } \lambda, N}} \cdot e^{-i\langle \xi, v\rangle}$$

It is a homogeneous function in $\xi$.

(The $\lambda$ are locally const. in $\xi$).

$\hat{\mathbb{1}}_K(y)$ is defined and holomorphic $y \in \mathbb{C}^n$.

$$\hat{\mathbb{1}}_K(iy) = \int_K e^{\langle x, y\rangle}\, dx$$ ← Laplace transform

$$\lim_{M\to\infty} \int_K e^{\langle x-v, My\rangle} \Big/ \int_{K_v} e^{\langle x-v, My\rangle} = 1$$


for $y \in$ normal cone to $K$ at $x$

## Homogeneous Tempered Distributions

**Def:** 0) A function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is $\lambda$-homogeneous if

$$\forall a > 0 \qquad f(ax) = a^\lambda f(x)$$

Denote $\quad f_a(x) = f(ax)$, scaling of $f$.

1) For tempered distribution $F \in \mathcal{S}^*$, $a > 0$ set

$$\forall \varphi \in \mathcal{S} \quad F_a(\varphi) = F\left(a^{-n} \varphi_{1/a}\right)$$

Why? because

$$\int f(ax)\,\varphi(x)\,dx \underset{y = xa}{=} \int f(y)\,\varphi(y/a)\,a^{-n}\,dy$$

$$= a^{-n} \int f\, \varphi_{1/a}$$

2) $F \in \mathcal{S}^*$ is $\lambda$-homogeneous if

$\forall a > 0 \quad F_a = a^{\lambda} F.$

Examples: 0) $\frac{1}{x}$ is a $(-1)$-homo. function

1) $F = pv\left(\frac{1}{x}\right)$ is - - - - - - dist.

because odd

$$F_a(\varphi) = \frac{1}{a} F(\varphi_{1/a})$$

$$= \frac{1}{a} \lim_{\varepsilon \to 0^+} \int_{|x| > \varepsilon} \varphi(x/a) \frac{dx}{x}$$

$$= \frac{1}{a} \lim_{\varepsilon \to 0^+} \int_{|x| > \varepsilon} \varphi(x) \frac{dx}{x} = \frac{1}{a} F(\varphi)$$

2) $F = \delta_0$ is $(-1)$-homogeneous dist.

$$F_a(\varphi) = \frac{1}{a} F(\varphi_{1/a}) = \frac{1}{a} \varphi\left(\frac{0}{a}\right) = \frac{1}{a} F(\varphi)$$

- $\delta_0'$ is $(-2)$-homogeneous.

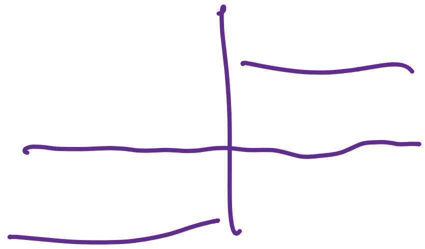
<u>Lemma:</u> If $F \in \mathcal{S}'^*$ is $\lambda$-homogeneous then $\hat{F} \in \mathcal{S}'^*$ is $(-n-\lambda)$-homogeneous.

<u>Proof:</u>
$$\hat{F}_a(\varphi) = \frac{1}{a^n} \hat{F}(\varphi_{1/a})$$
$$= \frac{1}{a^n} F(\widehat{\varphi_{1/a}}) = \frac{1}{a^n} F(a^n (\hat{\varphi})_a)$$
$$= F((\hat{\varphi})_a) = a^{-n} F_{1/a}(\hat{\varphi})$$
$$= a^{-n}\left(\frac{1}{a}\right)^{\lambda} F(\hat{\varphi}) = a^{-n-\lambda} \hat{F}(\varphi)$$
$$(\hat{F})_a = a^{-n-\lambda} \hat{F}. \qquad \blacksquare$$

- Sum of the two homogeneities [illegible] is $-n$.

<u>Examples:</u> 1) $\widehat{\mathrm{pv}\left(\frac{1}{x}\right)}$ is 0-homogeneous

$$= c\cdot \mathrm{sgn}(\xi)$$



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2) $\delta_0$ is $(-1)$-homo.

$\hat{\delta}_0$ is 0-homo

$$\hat{\delta}_0 \equiv 1$$

<u>Proposition:</u> Let $0 < \lambda < n$. Then

$$\widehat{\frac{1}{|x|^{\lambda}}} = C_{n,\lambda}\,\frac{1}{|x|^{n-\lambda}}$$

<u>Remarks:</u>

1) If $0 < \lambda < n$, $\frac{1}{|x|^{\lambda}} \in L^1_{loc}(\mathbb{R}^n)$

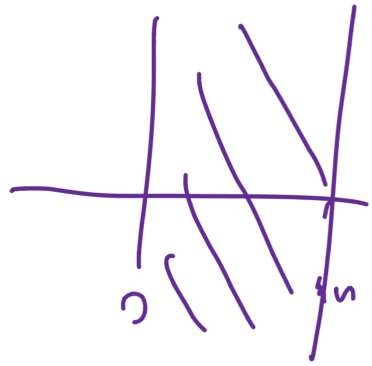
$\frac{1}{|x|^{n-\lambda}} \in L^1_{loc}(\mathbb{R}^n)$

Hence, for $C_c^\infty(\mathbb{R}^n)$, $F = \frac{1}{|x|^{\lambda}}$

$$F(\varphi) = \int_{\mathbb{R}^n} \frac{1}{|x|^\lambda}\,\varphi(x)\,dx \quad \text{is absolutely convergent}$$

2) If $\lambda \in \mathbb{C}$, then our proof works for

$$0 < \mathrm{Re}(\lambda) < n$$



Unless $\lambda \notin \mathbb{Z}$ (negative integers below $-n+1$)

then is a tempered distribution corresponding to $\frac{1}{|x|^\lambda}$ by analytic continuation.

<u>Proof:</u> We will compute directly, using the trick

$$\frac{1}{A^\lambda} = \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-tA}\, t^\lambda \,\frac{dt}{t}$$

Hence

$$\frac{1}{|x|^\lambda} = \overline{C}_\lambda \left(\frac{1}{|x|^2/2}\right)^{\lambda/2}$$

$$\frac{1}{|x|^\lambda} = C_\lambda \int_0^\infty e^{-t|x|^2/2}\, t^{\lambda/2}\, \frac{dt}{t}$$

By definition, if $F = \frac{1}{|x|^\lambda} \in \mathcal{S}^*$

$$\hat{F}(\varphi) = F(\hat{\varphi}) \overset{0<\lambda<n}{=} \int_{\mathbb{R}^n} \frac{1}{|x|^\lambda}\, \underset{\substack{\uparrow \\ \mathcal{S}}}{\hat{\varphi}}(x)\, dx$$

$$= C_\lambda \int_{\mathbb{R}^n} \left( \int_0^\infty e^{-\frac{t|x|^2}{2}}\, t^{\frac{\lambda}{2}} \frac{dt}{t} \right) \hat{\varphi}(x)\, dx$$

$$\underset{\substack{\text{(by} \\ \text{absolute} \\ \text{convergence)}}}{\overset{\text{Fubini}}{=}} C_\lambda \int_0^\infty \left( \int_{\mathbb{R}^n} e^{-\frac{t|x|^2}{2}}\, \hat{\varphi}(x)\, dx \right) t^{\frac{\lambda}{2}} \frac{dt}{t}$$

$$= C_\lambda \int_0^\infty \left( \int_{\mathbb{R}^n} \widehat{e^{-\frac{t|x|^2}{2}}}\, \varphi(x)\, dx \right) t^{\frac{\lambda}{2}} \frac{dt}{t}$$

$$= \tilde{C}_\lambda \int_0^\infty \int_{\mathbb{R}^n} t^{-n/2} e^{-\frac{|x|^2}{2t}}\, \varphi(x)\, dx\, t^{\frac{\lambda}{2}} \frac{dt}{t}$$

$$\underset{\substack{\text{(absolutely}\\ \text{convergent)}}}{\overset{\text{Fubini}}{=}} \tilde{c}_\lambda \int_{\mathbb{R}^n} \left( \int_0^\infty t^{\frac{\lambda - n}{2}} e^{-\frac{|x|^2}{2t}} \frac{dt}{t} \right) \varphi(x)\,dx$$

$$\overset{s = \frac{1}{t}}{=} \tilde{\tilde{c}}_\lambda \int_{\mathbb{R}^n} \left( \int_0^\infty s^{\frac{n-\lambda}{2}} e^{-\frac{s|x|^2}{2}} \frac{ds}{s} \right) \varphi(x)\,dx$$

$$= \hat{c}_\lambda \int_{\mathbb{R}^n} \frac{1}{|x|^{n-\lambda}} \varphi(x)\,dx$$

By that formula:

$$\frac{1}{|x|^\lambda} = c_\lambda \int_0^\infty e^{-t|x|^2/2}\, t^{\lambda/2} \frac{dt}{t}. \quad \blacksquare$$

$$\cdot\ \widehat{\frac{1}{|x|^\lambda}} = c_{n,\lambda} \frac{1}{|x|^{n-\lambda}} \qquad 0 < \lambda < n$$

Recall the $t$-dim. energy of a compactly-supported Borel measure $\mu$,

$$I_t(\mu) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1}{|x-y|^t}\, d\mu(x)\, d\mu(y)$$

**Thm** ("t-energy through Fourier transform")

$$I_t(\mu) = c_{n,t} \int_{\mathbb{R}^n} \frac{1}{|y|^{n-t}} |\hat{\mu}(y)|^2 \, dy$$

- Need to somewhat extend the scope of some parts of our tool box.

**Thm:** (Plancherel) $\forall f \in L^2(\mathbb{R}^n)$,

$$\|\hat{f}\|_2 = (2\pi)^{n/2} \|f\|_2$$

in particular $\hat{f} \in L^2$.

**Proof:** Previously for $f \in L^1 \cap L^2$ (or $\mathcal{S}$?)

How to prove it in general? For $f \in L^2$ pick

$$\mathcal{S} \ni f_m \longrightarrow f \quad \text{in } L^2 \text{ sense.}$$

$$\|\hat{f}_m\|_2 = (2\pi)^{n/2} \|f_m\|_2 \longrightarrow (2\pi)^{n/2} \|f\|_2$$

• $\hat{f}_m$ is a Cauchy sequence in $L^2$,

$$\|\hat{f}_m - \hat{f}_k\|_2 = c_n \| f_m - f_k \|_2 \xrightarrow{m,k\to\infty} 0$$

Hence $\exists\, g \in L^2(\mathbb{R}^n)$ with $\|g\|_2 = (2\pi)^{n/2}\|f\|_2$

$$(1) \quad \hat{f}_m \longrightarrow g \quad \text{in } L^2(\mathbb{R}^n)$$

but

$$f_m \longrightarrow f \quad \text{weakly in } \mathcal{S}^*$$

hence

$$(2) \quad \hat{f}_m \longrightarrow \hat{f} \quad - \text{"} -$$

Both (1) & (2) are weak convergences in $\mathcal{S}^*$, hence $g = \hat{f} \in L^2$

$$\|\hat{f}\|_2 = \|g\|_2 = (2\pi)^{n/2} \|f\|_2 .$$

$\blacksquare$

<u>Corollary:</u> $\forall f, g \in L^2(\mathbb{R}^n)$

$$\langle f, g \rangle = (2\pi)^{-n} \langle \hat{f}, \hat{g} \rangle$$

$$\begin{array}{cccccc} \uparrow & \uparrow & \qquad\qquad & \uparrow & \uparrow \\ L^2 & L^2 & & L^2 & L^2 \end{array}$$

- There are also less symmetric versions:

$F \in \mathcal{S}^*, \quad \varphi \in \mathcal{S}$

$$(*) \quad F(\check{\varphi}) = (2\pi)^{-n} F(\hat{\hat{\varphi}}) = (2\pi)^{-n} \hat{F}(\hat{\varphi})$$

- Recall if $F \in L^1 \cap \mathcal{S}^*$ then

$$L_F(\varphi) = \int_{\mathbb{R}^n} F \varphi \qquad \boxed{\text{without bar}}$$

<u>Convenient notation:</u> $F \in \mathcal{S}^*, \ \varphi \in \mathcal{S}$

define

$$\langle F, \varphi \rangle := F(\bar{\varphi})$$

$$\left( \langle \varphi, F \rangle = \bar{F}(\varphi) \right)$$

Now, replace $(*)$ by: For any

$F \in \mathcal{S}^{*}$, $\varphi \in \mathcal{S}$,

$$\langle \underset{\mathcal{S}^{*}}{\underset{\uparrow}{F}}, \underset{\mathcal{S}}{\underset{\uparrow}{\varphi}} \rangle = (2\pi)^{-n} \langle \underset{\mathcal{S}^{*}}{\underset{\uparrow}{\hat{F}}}, \underset{\mathcal{S}}{\underset{\uparrow}{\hat{\varphi}}} \rangle$$

$\psi = \check{\varphi}$

<u>Proof:</u>

$$\langle F, \psi \rangle = F(\check{\bar{\varphi}}) = (2\pi)^{-n} F(\hat{\hat{\bar{\varphi}}})$$

$$= (2\pi)^{-n} \hat{F}(\hat{\bar{\varphi}}) = (2\pi)^{-n} \hat{F}(\overset{\cancel{\wedge}}{\check{\varphi}})$$

$$= (2\pi)^{-n} \langle \hat{F}, \hat{\psi} \rangle .$$

<u>Convolution:</u> $F \in \mathcal{S}^{*}$, $\varphi \in \mathcal{S}$ then

$$\widehat{F * \varphi} = \hat{\varphi} \cdot \hat{F}$$

<u>Proof:</u>

$$\widehat{F * \psi}(\varphi) = F * \psi(\hat{\varphi}) = F(\check{\psi} * \hat{\varphi})$$

$$= (2\pi)^{-n} F\left(\underbrace{\hat{\hat{\psi}} * \hat{\varphi}}_{\text{Schwartz}}\right) = F\left(\hat{\hat{\psi}} \cdot \varphi\right)$$

$$= (2\pi)^{-n} \hat{F}\left(\hat{\psi} \cdot \varphi\right) = \left(\hat{F} \cdot \hat{\psi}\right)(\varphi). \quad \square$$

<u>One more fact:</u> $\varphi \in \mathcal{S}$, $\mu$ is a compactly-supported Borel measure (or a compactly-supported distribution) then

$$\varphi * \mu \in \mathcal{S}$$

<u>Proof:</u> $$(\mu * \varphi)(x) = \int_{\mathbb{R}^n} \tau_x \check{\varphi} \, d\mu$$

as we saw. Since $\mu$ is compactly-supported

$$x \longmapsto \int_{\mathbb{R}^n} \tau_x \check{\varphi} \, d\mu(x)$$

decays fast enough at $\infty$, and same for the derivatives.

$$\partial^\alpha (\mu * \varphi)(x) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \tau_x (\partial^\alpha \check{\varphi})\, d\mu(x)$$

which again decays fast. ∎

<u>Thm</u> ("$t$-energy through Fourier transform")

$$I_t(\mu) = c_{n,t} \int_{\mathbb{R}^n} \frac{1}{|y|^{n-t}} |\hat{\mu}(y)|^2\, dy$$

<u>Proof:</u> For any $K \in \mathcal{S}'$, even function $(\check{K} = K)$, real-valued, we have

$$I_t(\mu) = \iint \frac{1}{|x-y|^t}\, d\mu(x)\, d\mu(y)$$

Now

$$\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} K(x-y)\, d\mu(y) \right) d\mu(x)$$

$$= \int_{\mathbb{R}^n} \underset{\substack{\uparrow \\ \mathcal{S}'}}{(K * \mu)}(x) \, \underset{\substack{\uparrow \\ \mathcal{S}'^*}}{d\mu(x)}$$

our notation

$$= \langle \mu, \underset{\substack{\uparrow \\ \mathcal{S}}}{K * \mu} \rangle$$

$$= (2\pi)^{-n} \langle \underset{\substack{\uparrow \\ \mathcal{S}'^*}}{\hat{\mu}}, \widehat{K * \mu} \rangle$$

actually a bdd function

$$\widehat{K * \mu} = \underbrace{\hat{K} \cdot \hat{\mu}}_{\text{Schwartz function}}$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\mu}(y) \, \overline{\hat{K}(y) \hat{\mu}(y)} \, dy$$

even real-valued

To summarize

$$(*) \quad \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y) \, d\mu(x) \, d\mu(y) = c \int_{\mathbb{R}^n} \hat{K}(y) \, |\hat{\mu}(y)|^2 \, dy$$

We need $K(x) = \dfrac{1}{|x|^t}$ so $\hat{K}(y) = c \cdot \dfrac{1}{|y|^{n-t}}$.

But $\frac{1}{|x|^t} \notin \mathcal{S}$. So, define

$$K_\varepsilon(x) = \left(\frac{1}{|x|^t} * \gamma_\varepsilon\right) e^{-\varepsilon|x|^2/2} \in \mathcal{S}$$

$$\gamma_\varepsilon(x) = \frac{1}{(2\pi\varepsilon^2)^{n/2}} e^{-\frac{|x|^2}{2\varepsilon^2}}$$

In $\mathcal{S}'$, $K_\varepsilon \to \frac{1}{|x|^t}$ weakly.

$$\widehat{K_\varepsilon} = c\left(\frac{1}{|x|^{n-t}} \cdot e^{-\frac{\varepsilon|x|^2}{2}}\right) * \gamma_\varepsilon \in \mathcal{S}$$

$\widehat{K_\varepsilon}$ again converges to $\frac{1}{|x|^{n-t}}$ as $\varepsilon \to 0$.

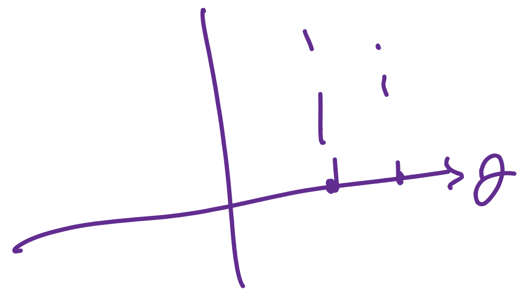
<u>Exercise:</u> Plug in $K_\varepsilon$ in $(*)$, let $\varepsilon \to 0$, and conclude the thm.

## Applications to projections of fractals

For $\theta \in S^{n-1}$ denote

$$P_\theta(x) = \langle x, \theta \rangle .$$

- If $K \subseteq \mathbb{R}^n$, $P_\theta(K)$ is the orthogonal projection of $K$ to the line $\mathbb{R}\theta$ or in direction $\theta$.




<u>Thm</u> (Marstrand '54, Kaufman '69)

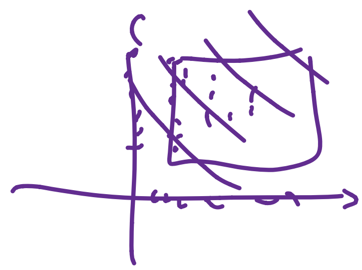
Suppose $K \subseteq \mathbb{R}^n$, $\dim_H(K) \geq 1$. Then for almost any $\theta \in S^{n-1}$,

$$\mathcal{H}^1\Big(\underset{\mathbb{R}}{\underset{\cap}{P_\theta(K)}}\Big) > 0 .$$

<u>Example:</u> $K \subseteq \mathbb{R}^2$, $K = \text{Cantor} \times \text{Cantor}$

$$P_\theta(K) = ?$$

$\theta = e_1$ or $e_2$



$$|P_\theta(K)| = 0$$

But in a typical direction, you a get a set of positive Lebesgue measure.

<u>Thm</u> (Falconer '99, Peres-Schlag '00)

If $K \subseteq \mathbb{R}^n$ has $\dim(K) > 2$, then for almost any $\theta \in S^{n-1}$,

$$P_\theta(K) \subseteq \mathbb{R}$$

contains an interval of positive length.

<u>Example:</u> Why is 2 sharp?

Take a Kakeya set $K \subseteq B(0, 10) \subseteq \mathbb{R}^2$

$K_\delta = \delta$-nbhd of $K$

$$\mathcal{H}^2(K_\delta) \longrightarrow 0 \quad \text{as } \delta \to 0$$

We take $C = B(0,\frac{1}{2}) \setminus \left( \bigcup_{m=1}^{\infty} K_{\delta_m} + q_m \right)$

$\mathbb{Q}^2 = \{ q_m \}_{m=1,2,\dots}$ for some $\delta_m \searrow 0$

so that
$$\sum_{m=1}^{\infty} \mathcal{H}^2 (K_{\delta_m}) \le \frac{1}{10}.$$

Hence $\mathcal{H}^2(C) > 0$.


Hence $P_\theta(C)$ always have an empty interior.