15/6/20

<u>Last time:</u> Littlewood-Paley decomposition

$$\underset{\mathcal{S}^{\prime *}}{\overset{\mathcal{S}'}{L^2(\mathbb{R}^n)}} \ni f = \sum_{k=0}^{\infty} \underset{\underset{C^\infty \text{ moderate growth}}{\uparrow}}{P_k} f$$

<u>$k \geq 1$:</u> $\operatorname{supp}\left(\widehat{P_k f}\right) \subseteq \{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$

<u>$k=0$:</u> $\operatorname{supp}\left(\widehat{P_0 f}\right) \subseteq \{|\xi| \leq 2\}$

<u>Recall:</u> $\widehat{P_k f} = \psi_k \hat{f}$

Fourier multiplier: $\widehat{P_k f}(\xi) = \psi_k(\xi)\hat{f}(\xi)$ for some function ψ_k.

- Fourier multipliers always commute (just multiplication after FT)
- For example of a Fourier-multiplier

write $D^\alpha = (-i)^{|\alpha|} \partial^\alpha$

$$D_j = -i \frac{\partial}{\partial x_j}$$

This way, $\widehat{D_j f}(\xi) = \xi_j \hat{f}(\xi)$

Somewhat standard notation: $p \in \mathcal{S}'$

$$\widehat{p(D) f}(\xi) = p(\xi) \hat{f}(\xi)$$

Recall: $\widehat{P_K f} = \psi_K \hat{f}$

- $\psi_K \in \mathcal{S}$
- This means that

$$P_K f = \tilde{\psi}_K \underset{\wedge \mathcal{S}}{*} f$$

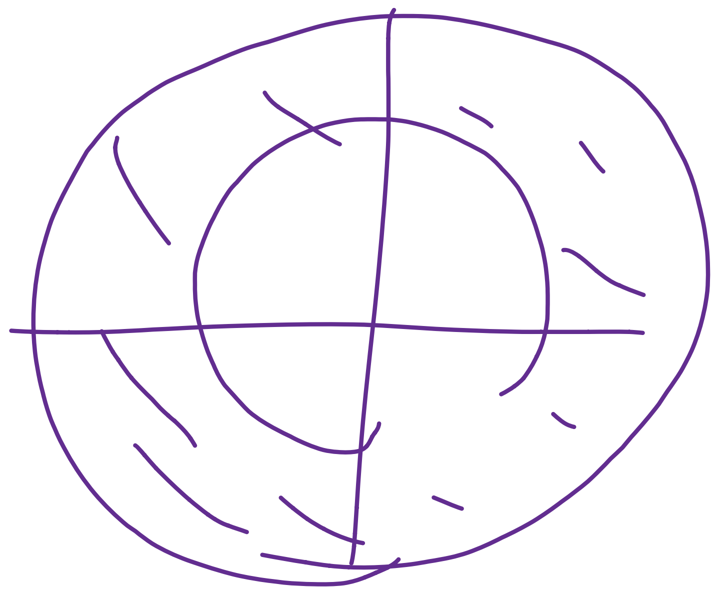
when $\widehat{\tilde{\psi}_K} = \psi_K$, the

inverse Fourier transform

$\widetilde{\psi}_k = (2\pi)^{-n} \widehat{\check{\psi}_k}$

ξ

$\widehat{P_k f}$



- The ψ_k were a partition of unity of $\mathbb{R}^n$,

$$\sum_{k=0}^{\infty} \psi_k(\xi) \equiv 1$$

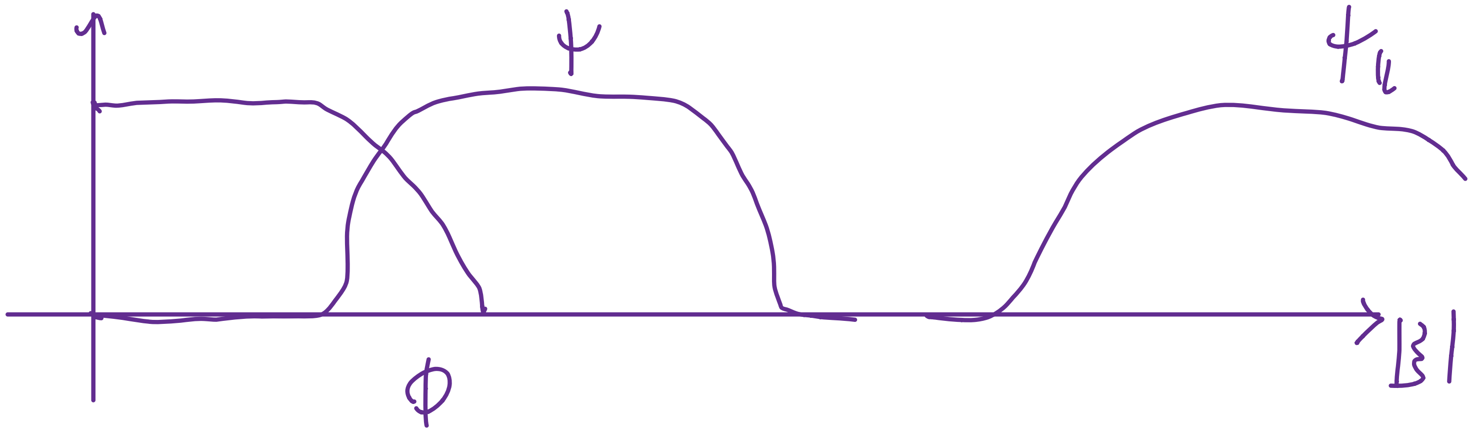
Our specific construction:

$$\begin{cases} \psi_k(\xi) = \psi(2^{-k}\xi) & k \geq 1 \\ \psi_0(\xi) = \phi(\xi) & k = 0 \end{cases}$$

where $C_c^\infty \ni \phi(\xi) = \begin{cases} 1 & |\xi| \leq 1 \\ 0 & |\xi| \geq \frac{3}{2} \end{cases}$

ϕ radially decreasing

and $\psi(\xi) = \phi(\xi) - \phi(2\xi) \geq 0$



"Localization in frequency space".

Recall: $\|f\|_{C^s} = \|f\|_\infty + \sup_{x,y\in\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^s}$

$|x-y|\leq 1$

not crucial, but yields equivalent norm

Thm: $\forall f \in C^s \quad 0<s<1,$

$$\|f\|_{C^s} \sim \sup_{k\geq 0} 2^{ks}\|P_k f\|_\infty$$

where $A\sim B$ means $cA \leq B \leq CA$, and here the implied constants depend on n and s.

· I.e., $\|f\|_{C^s} < \infty \iff \exists M > 0$

s.t. $\| P_k f \|_\infty \le M \cdot \frac{1}{(2^k)^s}$

$P_k f = \psi_k(D) f$

order of magnitude of frequencies in $P_k f$

<u>Remark:</u> C^s_{loc} = space of functions that are locally Hölder

$f \in C^s_{loc} \iff \forall \theta \in C_c^\infty(\mathbb{R}^n)$
$f\theta \in C^s$

We will see from proof that if

$$\sup_{k \ge 1} 2^{ks} \| P_k f \|_\infty < \infty \Rightarrow f \in C^s_{loc}$$

($k=0$ is only for "global" behaviour).

Lemma: (Bernstein): If $\mathrm{Supp}(\hat{f}) \subseteq B(0,R)$

then $\|\nabla f\|_\infty \lesssim R \cdot \|f\|_\infty$

where $A \lesssim B$ means $A \leq CB$,

here the implied constant depends on n.

Proof of thm: Assume first $f \in C^s$.

$$P_k f = \tilde{\Psi}_k * f \in C^\infty.$$

Need to bound L^∞-norms of $P_k f$.

$k=0$: $\|P_0 f\|_\infty = \sup_x \left| \int_y \phi(x-y) f(y) dy \right|$

$$\leq \sup_x \int_{\mathbb{R}^n} |\phi(x-y)| dy \cdot \|f\|_\infty$$

$$= \|\phi\|_1 \cdot \|f\|_\infty \leq C \|f\|_{C^s}$$

$k \geq 1$: $\Psi_k(0) = 0$, hence $\int_{\mathbb{R}^n} \tilde{\Psi}_k = \Psi_k(0) = 0$

Hence; for any $x \in \mathbb{R}^n$,

$$|P_k f(x)| = \left| \int_{\mathbb{R}^n} \tilde{\psi}_k(y) f(x-y)\, dy \right|$$

$$= \left| \int_{\mathbb{R}^n} \tilde{\psi}_k(y) \underbrace{[f(x-y) - f(x)]}_{|y|^s} dy \right|$$

$$\lesssim \int_{\mathbb{R}^n} |\tilde{\psi}_k(y)|\, |y|^s\, dy \cdot \|f\|_{C^s}$$

Recall $\psi_k(\xi) = \psi(2^{-k}\xi)$

hence $\tilde{\psi}_k(\xi) = 2^{nk}\tilde{\psi}(2^k \xi)$

$$= \|f\|_{C^s}\, 2^{nk} \int_{\mathbb{R}^n} |\tilde{\psi}(2^k y)|\, |y|^s\, dy$$

$$= \|f\|_{C^s}\, 2^{-ks} \int_{\mathbb{R}^n} |\tilde{\psi}(2^k y)| \cdot |2^k y|^s (2^{kn} dy)$$

$z = 2^k y$

$$= 2^{-ks} \|f\|_{C^s} \int_{\mathbb{R}^n} |\tilde{\psi}(y)| \cdot |y|^s\, dy$$

$$= C_s \, 2^{-2ks} \|f\|_{C^s}.$$

For the other side: Assume that

$$M := \sup_k 2^{ks} \|P_k f\|_\infty < \infty$$

Then

$$f = \sum_{k=0}^\infty P_k f$$

actually convergent in L^∞: $\Longrightarrow$ f is continuous

$$\|P_k f\|_\infty \le 2^{-ks} \cdot M$$

So,

$$\|f\|_\infty \le \sum_{k=0}^\infty 2^{-ks} M = C_s \cdot M.$$

· Fix $x, y \in \mathbb{R}^n$, $x \neq y$, $|x-y| < 1$.

Need to prove

$$|f(x) - f(y)| \le C_s \, M \, |x-y|^s.$$

Fix $N \ge 1$ s.t.

$$2^{-N} \le |x-y| \le 2^{-N+1}.$$

Now,

$$|f(x)-f(y)| \le \sum_{k=0}^{\infty} |P_k f(x) - P_k f(y)|$$

$$\le \sum_{k=0}^{N} |P_k f(x) - P_k f(y)| + \sum_{k=N+1}^{\infty} 2 \cdot M\, 2^{-ks}$$

$$\sum_{k=N+1}^{\infty} 2 \cdot M\, 2^{-ks} = C_s \cdot M \cdot 2^{-Ns} \le \tilde{C}_s M\, |x-y|^s$$

For $k \le N$, use Bernstein:

$$\|\nabla P_k f\|_\infty \lesssim 2^k \; \|P_k f\|_\infty$$

Hence, by mean value theorem, $\forall x, y$

$$|P_k f(x) - P_k f(y)| \le |x-y| \cdot \|\nabla P_k f\|_\infty$$

$$\lesssim |x-y| \cdot 2^k \|P_k f\|_\infty$$

. To summarize, $|x-y| \sim 2^{-N}$

$$|f(x)-f(y)| \le C_s M |\cancel{x}-y|^s + \sum_{k=0}^{N} |P_k f(x) - P_k f(y)|$$

$$\lesssim M|x-y|^s + \sum_{k=0}^{N} |x-y| 2^k \|P_k f\|_\infty$$

$$\lesssim M|x-y|^s + |x-y| \sum_{k=0}^{N} 2^k \cdot (M 2^{-ks})$$

$$\lesssim M|x-y|^s + |x-y| \cdot M \cdot \underbrace{2^{N(1-s)}}_{\left(\frac{1}{|x-y|}\right)^{1-s}}$$

$$\lesssim M|x-y|^s.$$

$\blacksquare$

$$\sum_{k=0}^{N} 2^{k(1-s)} \approx 2^{N(1-s)}$$

Remarks about L^p theory (no details)

- Bernstein lemma: $\mathrm{Supp}(\hat{f}) \subseteq \{\frac{R}{2} \le |\xi| \le 2R\}$

$$\Downarrow$$

$$\|\nabla f\|_p \sim R \|f\|_p$$

- The function

$$Sf = \sqrt{\sum_{k=0}^{\infty} |P_k f|^2}$$

is called the square function of P-W.

$$\|f\|_p \sim \|Sf\|_p \qquad 1<p<\infty$$

For $p=2$ - trivial equality (Plancherel)

$$\begin{array}{ccc} |f| & & Sf \\ \| & & \\ \left|\sum P_k f\right| & \sim & \sqrt{\sum |P_k f|^2} \end{array}$$

$$\left(\left|\sum \overset{\text{random}}{\pm} a_i\right| \sim \sqrt{\sum a_i^2} \text{ with prob} \geq \frac{9}{10} \right)$$

different frequencies are roughly un-correlated.

$$\int \left|\sum P_k f\right|^2 = \int \sum |P_k f|^2$$

i.e., cross terms do not contribute, little correlation

Remarks:

1) $\|f\|_{C^s} \sim \sup_k 2^{ks} \|P_k f\|_\infty$

$$\|f\|_{H^s} \sim \sqrt{\sum 2^{2ks} \|P_k f\|_2^2}$$

2) What about C^s_{loc}?

If $\sup_{k\geq 1} 2^{ks} \|P_k f\|_\infty < \infty$

then $\sum_{k=1}^{\infty} P_k f$ is C^s

and $f = P_0 f + \underbrace{\sum_{k=1}^{\infty} P_k f}_{C^s}$

and $P_0 f$ is a C^∞-function of moderate growth

$$P_0 f = \tilde{\phi} * f$$

$\mathcal{S}$

Hence $f \in C^s_{loc}$ (C^∞-function plus C^s)

· Recall:

· $\widehat{\beta(D) f}(\xi) = \beta(\xi)\hat{f}(\xi)$

· $\beta_k(\xi) = \beta(2^{-k}\xi)$

" β lives on $|\xi| \sim 1$, then β_k lives near $|\xi| \sim 2^k$ ".

<u>Basic lemma</u>: $\forall \beta \in \mathcal{S}$, $\beta(0) = 0$, $\forall f \in C^s$ $\forall k \geq 1$

$$\|\beta_k(D) f\|_\infty \lesssim 2^{-ks} \|f\|_{C^s}$$

when the implied constant depends on n, s and $\underbrace{\beta}_{\text{the function}}$.

Proof: β_k functions as ψ_k from thm:

$$\beta_k(f) = \tilde{\beta}_k * f \quad , \quad \int \tilde{\beta}_k = \beta_k(0) = 0$$

hence for any $x \in \mathbb{R}^n$,

$$\left| \beta_k(0) f(x) \right| = \left| \int_{\mathbb{R}^n} \tilde{\beta}_k(y) f(x-y) \, dy \right|$$

$$= \left| \int_{\mathbb{R}^n} \tilde{\beta}_k(y) \left(f(x-y) - f(x) \right) dy \right|$$

$$\leq \int_{\mathbb{R}^n} \left| \tilde{\beta}_k(y) \right| \cdot |y|^s \, dy \cdot \|f\|_{C^s}$$

$$= \|f\|_{C^s} \cdot 2^{nk} \int_{\mathbb{R}^n} |\tilde{\beta}(2^k y)| \, |y|^s \, dy$$

$$= \|f\|_{C^s} \cdot 2^{-ks} \int_{\mathbb{R}^n} |\tilde{\beta}(y)| \cdot |y|^s \, dy$$

$$\lesssim 2^{-ks} \|f\|_{C^s}. \quad \blacksquare$$

. Basic lemma is just about scaling (or homogeneity) in frequency space.

Elliptic regularity of the Laplacian

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

$\forall \varphi \in \mathcal{S}$ $\quad \widehat{\Delta\varphi}(\xi) = \sum_{j=1}^{n} (-i\xi_j)(-i\xi_j)\hat{\varphi}(\xi)$

$$\widehat{\Delta\varphi}(\xi) = -|\xi|^2 \hat{\varphi}(\xi)$$

. The Laplacia is a Fourier multipliers.

<u>Thm</u>: Suppose $u \in \mathcal{S}'$ solves $\Delta u = r$.

Then

$$r \in C^s \implies u \in C^{2,s}_{loc}$$

infact, u equals a C^∞-function of moderate growth plus a $C^{2,s}$- function.

Proof: We need to show that
$\partial^\alpha u \in C^s_{loc}$ for $|\alpha| \leq 2$

Enough for $|\alpha| = 2$ ($\Rightarrow$ first derivation are C^1-smooth)

Need to prove that for $i,j = 1,..,n$

$$D_i D_j u \in C^s_{loc}$$

$$- \partial_{ij} u$$

By thm, enough to show that
$\forall k \geq 1$

$$(*) \quad \| P_k (D_i D_j u) \|_\infty \leq 2^{-ks} \cdot \|v\|_{C^s}$$

Recall: $D_i D_j u = \sum_{k=1}^\infty P_k (D_i D_j u) + P_0 (D_i D_j u)$

where $P_0 (D_i D_j u) = D_i D_j u * \tilde{\Phi}_0$
is C^∞-smooth of moderate growth.

How to prove $(*)$? $\quad \hat{V}(\xi) = -|\xi|^2 \hat{u}(\xi)$

$$\widehat{P_k(D_iD_j)u} = \psi_k(\xi)\,\xi_i\xi_j\,\hat{u}(\xi)$$

$$= \underbrace{-\psi_k(\xi)\frac{\xi_i\xi_j}{|\xi|^2}}_{\shortparallel}\,\hat{V}(\xi).$$

$$\beta_k(\xi) = \beta(2^{-k}\xi)$$

$$\psi_k(\xi) = \psi(2^{-k}\xi)$$

when $\quad \beta(\xi) = -\psi(\xi)\dfrac{\xi_i\xi_j}{|\xi|^2}$

where $\beta \in \mathcal{S}$ with $\beta(0) = 0$.

Hence, from basic lemma, $\forall k \geq 1$

$$\|P_k(D_iD_j)u\|_\infty = \|\beta_k(D)V\|_\infty$$

$$\lesssim 2^{-ks}\cdot\|V\|_{C^s}$$

with implied constant depending on β, hence only on n and s. ∎

<u>Recall:</u> $\|u\|_{C^{2,s}} = \max_{|\alpha|\le 2} \|\partial^\alpha f\|_{C^s}$.

(Next weeks: ψDOs, local issues of elliptic regularity.)

<u>Remark:</u> If $v \in C^{k,s}$ and $\Delta u = v$

then $u \in C^{k+2,s}_{loc}$

because $\Delta u = v$ $\quad |\alpha| = k$

$$\partial^\alpha \Delta v = \Delta(\partial^\alpha u) = \partial^\alpha v$$

$\Downarrow$ then $\quad \partial^\alpha v \in C^s$

$C^{2,s}$

· If you use Sobolev norms, elliptic regularity of Laplacian is easier

<u>Prop:</u> Let $u \in L^2$ satisfies $\Delta u = v$ (in $\mathcal{S}^*$) then $\forall s > 0$

$$v \in H^s \implies u \in H^{s+2}$$

<u>Proof:</u> Recall $\|f\|_{H^s}^2 = \int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi$

Since $\Delta u = v$, $-|\xi|^2 \hat{u}(\xi) = \hat{v}(\xi)$, and

$$\|u\|_{H^{s+2}}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^{s+2} |\hat{u}(\xi)|^2 d\xi$$

$$\lesssim \int \left(1+|\xi|^{2(s+2)}\right) |\hat{u}(\xi)|^2 d\xi$$

$$\lesssim \|u\|_2^2 + \int_{\mathbb{R}^n} |\xi|^{2(s+2)} |\hat{u}(\xi)|^2 d\xi$$

$$= \|u\|_2^2 + \int_{\mathbb{R}^n} |\xi|^{2s} \cdot \left||\xi|^2 \hat{u}(\xi)\right|^2 d\xi$$

$$= \|u\|^2 + \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{v}(\xi)|^2 d\xi$$

$$\lesssim \|u\|^2 + \|v\|_{\dot{H}^s}^2 < \infty. \quad \square$$

· $\Delta u = v$, u is 2 derivatives smoother than v in Sobolev sense and in Hölder sense.

<u>Thm</u> (Sobolev embedding, Morrey's lemma)

$\forall\, 0 < s < 1 \quad \forall f \in H^{n/2+s}$

$$\|f\|_{C^s} \lesssim \|f\|_{H^{\frac{n}{2}+s}}$$

or

$$H^{\frac{n}{2}+s} \subseteq C^s$$

<u>Proof:</u> Need $\forall\, k \geq 0$

$$\|\psi_k(D) f\|_\infty = \|P_k f\|_\infty \lesssim 2^{-ks} \|f\|_{H^{s+\frac{n}{2}}}$$

<u>For $k \geq 1$:</u>

$$\|\psi_k(D) f\|_\infty \leq \int_{\mathbb{R}^n} \left|\psi_k(\xi)\, \hat{f}(\xi)\right| d\xi$$

$$= \int_{2^{k-1} \le |\xi| \le 2^{k+1}} |\psi_k(\xi)| \, |\hat{f}(\xi)| \, d\xi$$

$$\le \int_{2^{k-1} \le |\xi| \le 2^{k+1}} |\hat{f}(\xi)|$$

$$\overset{C-S}{\le} \sqrt{\int_{2^{k-1} \le |\xi| \le 2^{k+1}} |\hat{f}(\xi)|^2 d\xi} \cdot \sqrt{2^{kn}}$$

$$\lesssim 2^{-ks} \sqrt{\int_{2^{k-1} \le |\xi| \le 2^{k+1}} |\xi|^{2\left(\frac{n}{2}+s\right)} |\hat{f}(\xi)|^2 d\xi}$$

$$\le 2^{-ks} \|f\|_{H^{s+n/2}}. \qquad \blacksquare$$

Remarks:

1) $0 < s < 1$, $k \ge 0$ $\quad H^{\frac{n}{2}+s+k} \subseteq C^{k,s}$

Proof: $\forall \, |\alpha| \le k$

$$\partial^\alpha f \in H^{\frac{n}{2}+s} \Rightarrow \partial^\alpha f \in C^s.$$

2) How can we recover the exponents in Sobolev inequality.

$$\|f\|_{\dot{C}^s} = \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^s} \lesssim \|f\|_{H^{s+\frac{n}{2}}}$$

$$\sigma_r f(x) = f(x/r)$$

$$\|\sigma_r f\|_{\dot{C}^s} = r^s \|f\|_{\dot{C}^s}$$

$$\|\sigma_r f\|_{H^{s+\frac{n}{2}}} = r^s \|f\|_{H^{s+\frac{n}{2}}}$$

" One order relates to the s "

3) $\Delta u \in C^s \Rightarrow u \in C^{2,s}_{loc}$

only for $0 < s < 1$.

An example that $\Delta u \in L^\infty$ but $\|\partial^\alpha f\|_\infty = +\infty$ for $|\alpha| = 2$

Exercise: $\Delta u \in C_c(\mathbb{R}^n)$ but $u \notin C^2$

An example

Example: $u(x,y) = (x^2-y^2)\log(x^2+y^2)$

$$\Delta(uv) = (\Delta u)\cdot v + (\Delta v)\cdot u + 2\langle \nabla u, \nabla v\rangle$$

Hence

$$\Delta u = 2\cdot(2x,-2y)\cdot\frac{(2x,2y)}{x^2+y^2}$$

$$= 8\,\frac{x^2-y^2}{x^2+y^2} \in (-8,8)$$

is in L^∞

but

$$u_{xx} = 2\log(x^2+y^2) + 2\cdot 2x\cdot\frac{2x}{x^2+y^2} + (x^2-y^2)\cdot\frac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2}$$

$$= 2\log(x^2+y^2) + \frac{-2(x^2-y^2)^2 + 8x^2(x^2+y^2)}{(x^2+y^2)^2}$$

and bounded. bdd

- Another example of a multiplier operator in addition to Laplacian

Hilbert transform

- $F = \mathrm{pv}\left(\frac{1}{x}\right) \in \mathcal{S}'$

$\forall \varphi \in \mathcal{S}$ $\quad F(\varphi) = \lim\limits_{\varepsilon \to 0^+} \int\limits_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx$

$$\hat{F}(\xi) = -i \cdot \pi \,\mathrm{sgn}(\xi)$$

<u>Def:</u> For $\varphi \in \mathcal{S}$, define the Hilbert transform

$$H\varphi = \frac{1}{\pi}\, \mathrm{pv}\left(\frac{1}{x}\right) * \varphi$$

$$\widehat{H\varphi}(\xi) = -i\, \mathrm{sgn}(\xi)\, \hat{\varphi}(\xi)$$

(convolution of $\mathcal{S}'$ with $\mathcal{S}$ goes to product).

<u>Claim:</u> $\|H\varphi\|_2 = \|\varphi\|_2 \quad \forall \varphi \in \mathcal{S}$

and $H^2 = -Id.$

<u>Proof:</u>

$$\|H\varphi\|_2^2 = (2\pi)^{-n/2} \int |i \,\mathrm{sgn}(\xi)\, \hat{\varphi}(\xi)|^2 d\xi$$

$$= (2\pi)^{-n/2} \int |\hat{\varphi}(\xi)|^2 d\xi = \|\varphi\|_2^2$$

<u>Corollary:</u> H is extended by continuity to an isometry $H: L^2 \to L^2$

Why $H^2 = -Id$

because $\forall \varphi \in \mathcal{S}$

$$\widehat{H^2\varphi} = -i\,\mathrm{sgn}(\xi) \cdot (-i\,\mathrm{sgn}(\xi))\, \hat{\varphi}$$

$$= -\hat{\varphi}.$$

<u>Claim:</u> $\forall f \in \mathcal{S}, \quad 0 < s < 1$

$$\|Hf\|_{C^s} \lesssim \|f\|_{C^s} + \|f\|_1$$

Hence $H : C^s \cap L^1 \longrightarrow C^s$ is well-defined and continuous.

<u>Proof:</u> By the Hölder norm thm, need to prove that $\forall k \geq 0$,

$$\| P_k H f \|_\infty \leq 2^{-sk} \left(\|\varphi\|_{C^s} + \|\varphi\|_1 \right).$$

$\forall k \geq 1$,

$$\| P_k H f \|_\infty = \| \underbrace{\psi_k(D) H}_{\substack{\| \\ \beta_k(D)}} f \|_\infty$$

$\beta_k(D) f$

where $\beta_k(\xi) = \psi_k(\xi) \cdot (-i \operatorname{sgn}(\xi))$ and the notation fits with the basic lemma:

$$\psi_k(\xi) = \psi(2^{-k} \xi)$$

So if we set

$\exists$ $\beta(\xi) = -i\,\mathrm{sgn}(\xi)\,\psi(\xi)$, $\beta(0)=0$

$\boxed{\beta \text{ vanishes in a nbhd of zero.}}$

0 homogeneous

$$\mathrm{sgn}(\xi) = \begin{cases} 1 & \xi > 0 \\ 0 & \xi = 0 \\ -1 & \xi < 0 \end{cases}$$

and indeed
$$\beta_k(\xi) = \beta(2^{-k}\xi) = \psi_k(\xi)\,(-i\,\mathrm{sgn}(\xi))$$

Thus

$$\| P_k H f \|_\infty = \| \beta_k(D) f \|_\infty \lesssim 2^{-ks} \|f\|_s$$

from basic lemma

For $k=0$:

$$\| P_0 H f \|_\infty \lesssim \int \left| \widehat{P_0 H f} \right|$$
$$= \int_{-\infty}^{\infty} \phi_0(\xi) \, \cancel{|\mathrm{sgn}(\xi)|} \cdot |\hat{f}(\xi)| \, d\xi$$

$$= \int_{-2}^{2} \underbrace{\phi_0(\xi)}_{\leq 1} \, |\hat{f}(\xi)| \, d\xi$$

$$\leq 4 \cdot \|\hat{f}\|_\infty \lesssim \|f\|_1 .$$

∎

<u>Corollary:</u> For any $f \in C^5 \cap L^1$, and any $x \in \mathbb{R}$,

$$(*) \qquad Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} \, dy$$

is well-defined, and $Hf : \mathbb{R} \to \mathbb{C}$ is in C^5.

<u>Proof:</u> Why is $(*)$ correct for $f \in \mathcal{S}$?

$$Hf = F * f \qquad \forall f \in \mathcal{S}.$$

Therefore, $F = \frac{1}{\pi} \cdot pv\left(\frac{1}{x}\right)$

$$(Hf)(x) = F\left(\tau_x \check{f}\right) \qquad \begin{array}{l} \tau_x f(y) \\ f(y-x) \end{array}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(x-y)}{y} \, dy.$$

Hence $(*)$ holds true for $f \in \mathcal{S}$.

For the general case: Hf is continuous in f in $C^s \cap L^1$-norm.

We need to show that also the RHS of $(*)$ is continuous in $C^s \cap L^1$-norm.

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|y| > \varepsilon} \frac{f(x-y)}{y} \, dy$$

Fix $x \in \mathbb{R}$, say $x = 0$. Need:

$$f \longmapsto \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{f(y)}{y}$$ is cont. $C^s \cap L^1$

$$\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{f(y)}{y} = \lim_{\varepsilon\to 0}\underbrace{\int_{1>|y|>\varepsilon}\frac{f(y)}{y}} + \underbrace{\int_{|y|>1}\frac{f(y)}{y}\,dy}_{\text{conv. in } L^1 \text{ because } \frac{1}{y} \text{ is odd}}$$

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<1}\frac{f(y)}{y}\,dy$$

$$=\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|y|<1}\frac{f(y)-f(0)}{y}\,dy$$

integrable

$$\leq \|f\|_{C^s}\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|y|\leq 1}\frac{|y|^s}{y}\,dy$$

$$\leq C_s\,\|f\|_{C^s}.$$

∎

Why is Hilbert transform useful?

Prop: Suppose f is a holomorphic function in $\{z \in \mathbb{C} ; \operatorname{Im}(z) \geq 0\}$, and

$$|f(z)| = O\left(\frac{1}{1+|z|^2}\right). \quad \text{as } z \to \infty.$$

Then on $\mathbb{R}$,

$$H(\operatorname{Re} f) = \operatorname{Im} f.$$

$$H(-\operatorname{Im} f) = \operatorname{Re} f.$$