Harmonic Analysis - Assignment 1

Matan Eilat

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1. Verify that the Sierpinski triangle has Hausdorff dimension $\frac{\log 3}{\log 2}$.

   Solution: Denote the Sierpinski triangle by T. Recall that T is obtained by repeated removal of triangular subsets from an equilateral triangle T_1. Denote the barycenter of T_1 by x_1.

   Suppose that $0 < r < R$ are such that $B(x_1, r) \subset T_1 \subset B(x_1, R)$, and set $\alpha = \frac{\log 3}{\log 2} = \log_2 3$.

Claim 1: $\mathcal{H}_\alpha(T) < \infty$

Proof: By the construction process of T described above, it is clear that for any $n \geq 0$

$$T \subset \bigcup_{k=1}^{3^n} B\left(x_k, \frac{R}{2^n}\right)$$

where $\{x_k\}_{k=1}^{3^n}$ are the barycenters of the triangles at step n of the construction process. Therefore

$$\mathcal{H}_{\alpha, \frac{R}{2^n}}(T) \leq C_\alpha \cdot 3^n \cdot \left(\frac{R}{2^n}\right)^\alpha = C_\alpha \cdot R^\alpha$$

Hence $\mathcal{H}_\alpha(T) \leq C_\alpha \cdot R^\alpha < \infty$. □

Claim 2: There exists a compactly-supported, finite, Borel measure μ on T such that

$$\mu\left(B\left(x, \frac{r}{2^n}\right)\right) \leq C \cdot (2^{-n})^\alpha \qquad \text{for all } x \in T, n \geq 1$$

Proof: For any Borel subset $A \subset T$ define

$$\mu(A) = \mathbb{P}(X \in A)$$

where X is a random variable obtained in the following way:

Notice that a point in T is completely determined by the sequence of choices made in the construction process, i.e. at each step one has to choose one out of three possible sub-triangles to decend to.

Let $(x_k)_{k\geq 1}$ be the i.i.d. random variables representing these choices, giving equal probability, which is 1/3, to each choice. Now we let the random variable X be the point obtained by the sequence of choices $(x_k)_{k\geq 1}$.

Using this, we see that for any $x \in T$ and $n \geq 1$

$$\mu\left(B\left(x, \frac{r}{2^n}\right)\right) \leq C \cdot 3^{-n} = C \cdot (2^{-n})^\alpha$$

By Claim (1) we see that

$$\dim_{\mathcal{H}}(T) = \sup\{\beta \geq 0 \,:\, \mathcal{H}_\beta(T) = +\infty\} \leq \alpha$$

By Claim (2), Frostman's Lemma and a proposition from class, we see that

$$\dim_{\mathcal{H}}(T) = \sup\{s\,;\, \exists 0 \neq \mu \in \mathcal{M}(T)\ ,\ I_s(\mu) < \infty\} \geq \alpha$$

Hence $\dim_{\mathcal{H}}(T) = \alpha$. $\square$

2. Let $X_1, X_2, ...$ be independent, identically distributed random variables with

$$\mathbb{P}(X_k = 0) = \mathbb{P}(X_k = 2) = \frac{1}{2}$$

Prove that for any Borel subset A of the Cantor set $C \subset [0,1]$

$$\mathbb{P}\bigg(\sum_{k=1}^{\infty} \frac{X_k}{3^k} \in A\bigg) = c \cdot \mathcal{H}_\alpha(A)$$

for $\alpha = \log 2/\log 3$ and some constant $c > 0$.

Solution: Since $\dim_{\mathcal{H}}(C) = \alpha$, there exists some contant $c > 0$ such that

$$\mathcal{H}_\alpha(C) = \frac{1}{c}$$

In order words,

$$\mathbb{P}\bigg(\sum_{k=1}^{\infty} \frac{X_k}{3^k} \in C\bigg) = 1 = c \cdot \mathcal{H}_\alpha(C)$$

For any $z \in C$, write the ternary expansion of z by

$$z = \sum_{k=1}^{\infty} \frac{z_k}{3^k} = 0.z_1 z_2...$$

where $z_k \in \{0, 2\}$. Let $\mathcal{P}$ be the set of cylinder sets, i.e.

$$[a_1 a_2 .... a_n] = \{z = \sum_{k=1}^{\infty} \frac{z_k}{3^k} \in C\ :\ z_i = a_i\ \forall 1 \leq i \leq n\}$$

The cylinders sets are:

- A basis of the topology of C.
- A π- system.

Claim 1: For any $A = [a_1 .... a_n] \in \mathcal{P}$ we have

$$\mathbb{P}\bigg(\sum_{k=1}^{\infty} \frac{X_k}{3^k} \in A\bigg) = c \cdot \mathcal{H}_\alpha(A)$$

Proof: It is clear from the recursive structure of C that

$$\mathbb{P}\bigg(\sum_{k=1}^{\infty} \frac{X_k}{3^k} \in A\bigg) = \frac{1}{2^n} = \frac{\mathcal{H}_\alpha(A)}{\mathcal{H}_\alpha(C)} = c \cdot \mathcal{H}_\alpha(A)$$

Now we use the following proposition from measure theory:

Proposition 1: Let (Ω, Σ) be a measurable space, and let $\mathcal{P}$ be a π-system which generates Σ. Suppose that μ_1 and μ_2 are two measures on Σ with the property that $\mu_1(\Omega) = \mu_2(\Omega) < \infty$ and

$$\mu_1(A) = \mu_2(A) \qquad \text{for all } A \in \mathcal{P}$$

Then $\mu_1 = \mu_2$.

Since the cylinders sets form a countable basis of the topology, they generate the Borel σ-algebra.

Using Proposition (1) combined with Claim (1), we see that the equality holds. $\square$

3. For any $0 < t < n$, find a compactly-supported, finite Borel measure μ on $\mathbb{R}^n$ with a finite t-energy, i.e.,

$$I_t(\mu) = \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^{-t} d\mu(x) d\mu(y) < \infty$$

yet for any $M > 1$ there exist $x \in \mathbb{R}^n$, $r > 0$ with

$$\mu(B(x,r)) > Mr^t$$

Solution: For $k \geq 1$, let $x_k = 2^{-k} e_1$, let $r_k = 2^{-(k+2)}$ and let

$$B_k = B(x_k, r_k) = B(2^{-k} e_1, 2^{-(k+2)})$$

Let $d\mu = f dx$ where dx is the Lebesgue measure, $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} c_k & x \in B_k \\ 0 & \text{Otherwise} \end{cases}$$

and c_k is given by

$$c_k = k \cdot r_k^{t-n} = k \cdot 2^{(n-t)(k+2)}$$

Notice that since

$$\frac{1}{2^k} - \frac{1}{2^{k+2}} - \left(\frac{1}{2^{k+1}} + \frac{1}{2^{k+3}} \right) = \frac{1}{2^{k+3}}$$

we have that any $B(x, 2^{-(k+3)}) \cap B_l = \emptyset$ for any $x \in B_k$ and $l \neq k$.

Write ω_n for the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Claim 1: Let $k \geq 1$. Then:

(a) $\mu(B_k) = k\omega_n r_k^t$

(b) For any $x \in B_k$ and $0 < r < 2^{-(k+3)}$ we have that $\mu(B(x,r)) \leq c_k \omega_n r^n$

Proof: We have that

$$\mu(B_k) = \int\limits_{B_k} f dx = c_k \cdot \mathrm{Leb}(B_k) = c_k \omega_n r_k^n = k\omega_n r_k^t$$

and

$$\mu(B(x,r)) = \int\limits_{B(x,r)} f dx \leq c_k \cdot \mathrm{Leb}(B(x,r)) = c_k \omega_n r^n$$

Claim 2: For any $m \geq 1$ we have $\sum_{k=1}^{\infty} k^m 2^{-tk} < \infty$.

Proof: Let $z = 2^t > 1$. Then
$$\sum_{k=1}^{\infty} k^m 2^{-tk} = \sum_{k=1}^{\infty} \frac{k^m}{z^k}$$
Letting $a_k = \frac{k^m}{z^k}$ we see that
$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^m}{z^{k+1}} \cdot \frac{z^k}{k^m} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^m \cdot \frac{1}{z} = \frac{1}{z} < 1$$
and by the ratio test the series converges. $\square$

Corollary 1: $\mu(\mathbb{R}^n) < \infty$

Proof: Using the claim we see that
$$\mu(\mathbb{R}^n) = \sum_{k=1}^{\infty} \mu(B_k) = 2^{-2t}\omega_n \cdot \sum_{k=1}^{\infty} k 2^{-tk} < \infty$$

Claim 3: Let
$$f(x) = \int\limits_{\mathbb{R}^n} |x-y|^{-t} d\mu(y)$$
Then for any $x \in B_k$ we have
$$f(x) \leq C_1 k + C_2$$
Proof: We split the integral into two:
$$f(x) = \int\limits_{\mathbb{R}^n} |x-y|^{-t} d\mu(y) = \underbrace{\int\limits_{B(x,2^{-(k+3)})} |x-y|^{-t} d\mu(y)}_{I} + \underbrace{\int\limits_{\mathbb{R}^n \setminus B(x,2^{-(k+3)})} |x-y|^{-t} d\mu(y)}_{II}$$
We bound the second part as follows:
$$II \leq 2^{-t(k+3)} \cdot \mu(\mathbb{R}^n) < \mu(\mathbb{R}^n) =: C_2$$
Moreover,
$$I = \int\limits_{\mathbb{R}^n} |x-y|^{-t} \cdot \mathbb{1}_{B(x,2^{-(k+3)})}(y) d\mu(y) = \int\limits_0^{\infty} \mu(\{y \,:\, |x-y|^{-t} \cdot \mathbb{1}_{B(x,2^{-(k+3)})}(y) > u\}) du$$
Noticing that
$$\{y \,:\, |x-y|^{-t} \cdot \mathbb{1}_{B(x,2^{-(k+3)})}(y) \geq u\} = B(x, u^{-\frac{1}{t}}) \cap B(x, 2^{-(k+3)}) = \begin{cases} B(x, u^{-\frac{1}{t}}) & u \geq 2^{t(k+3)} \\ B(x, 2^{-(k+3)}) & u < 2^{t(k+3)} \end{cases}$$
we have that
$$I = \underbrace{\int\limits_0^{2^{t(k+3)}} \mu(B(x, 2^{-(k+3)})) du}_{A} + \underbrace{\int\limits_{2^{t(k+3)}}^{\infty} \mu(B(x, u^{-\frac{1}{t}})) du}_{B}$$

Clearly

$$\mu(B(x, 2^{-(k+3)})) \leq \mu(B(x_k, 2^{-(k+3)})) \leq \mu(B_k) = k\omega_n r_k^t = k\omega_n 2^{-t(k+2)}$$

and so we obtain the following bound for A

$$A \leq 2^{t(k+3)} k\omega_n 2^{-t(k+2)} = k\omega_n 2^t$$

For B we have that

$$B = \left[r = u^{-\frac{1}{t}} ; \ \frac{du}{dr} = -\frac{t}{r^{t+1}} \right] = \int\limits_0^{2^{-(k+3)}} \frac{t}{r^{t+1}} \mu(B(x,r)) dr \leq$$

$$\leq c_k \omega_n \cdot t \int\limits_0^{2^{-(k+3)}} r^{n-t-1} dr = \frac{\omega_n t}{n-t} \cdot c_k \cdot \left(r^{n-t} \Big|_0^{2^{-(k+3)}} \right) = \frac{\omega_n t}{n-t} \cdot c_k \cdot 2^{-(n-t)(k+3)} = \frac{\omega_n t}{(n-t)2^{n-t}} \cdot k$$

In total we obtain

$$I = A + B \leq k \cdot \omega_n \left(2^t + \frac{t}{(n-t)2^{n-t}} \right) =: C_1 k$$

and the claim is proved. □

Corollary 2: We have that $I_t(\mu) < \infty$.

Proof:

$$I_t(\mu) = \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} |x-y|^{-t} d\mu(y) d\mu(x) = \int\limits_{\mathbb{R}^n} f(x) d\mu(x) = \sum_{k=1}^{\infty} \int\limits_{B_k} f(x) d\mu(x) \leq$$

$$\leq \sum_{k=1}^{\infty} (C_1 k + C_2) \cdot \mu(B_k) = \sum_{k=1}^{\infty} (C_1 k + C_2) \cdot \mu(B_k) = \sum_{k=1}^{\infty} (C_1 k + C_2) \cdot k\omega_m 2^{-t(k+2)} =$$

$$= 2^{-2t} \omega_n \left[C_1 \sum_{k=1}^{\infty} k^2 2^{-tk} + C_2 \sum_{k=1}^{\infty} k 2^{-tk} \right] < \infty$$

All in all, μ is a compactly-supported, finite Borel measure on $\mathbb{R}^n$ with finite t-energy.
However, for any $M > 1$ we may choose k such that $k > \frac{M}{\omega_n}$, and we obtain that

$$\mu(B_k) = \mu(B(x_k, r_k)) = k\omega_n r_k^t > M r_k^t$$

and we are done. □

4. Write $\mathcal{S}$ for the space of Schwartz functions in $\mathbb{R}^n$. Let $T : \mathcal{S} \to \mathcal{S}$ be a continuous, translation invariant linear operator. Prove that $\mathcal{F}^{-1} \circ T \circ \mathcal{F}$ is a multiplication operator, where $\mathcal{F}$ is the Fourier transform.

Solution: For a function $f \in \mathcal{S}$, write $\overline{f} \in \mathcal{S}$ for the function given by $\overline{f}(x) = f(-x)$.

Claim 1: For any $\Phi \in \mathcal{S}$ we have that

$$\Phi * T\varphi = (\overline{T^*(\overline{\Phi})}) * \varphi$$

where T^* is the adjoint operator of T.

Proof:

$$(\Phi * T\varphi)(x) = \int_{\mathbb{R}^n} \Phi(y) \cdot (T\varphi)(x-y)dy = \int_{\mathbb{R}^n} \Phi(y) \cdot (T\varphi)_x(-y)dy = \int_{\mathbb{R}^n} \overline{\Phi}(y) \cdot (T\varphi_x)(y)dy = \langle \overline{\Phi}, T\varphi_x \rangle =$$

$$= \langle T^*(\overline{\Phi}), \varphi_x \rangle = \int_{\mathbb{R}^n} (T^*(\overline{\Phi}))(y) \cdot \varphi_x(y)dy = \int_{\mathbb{R}^n} (\overline{T^*(\overline{\Phi})})(y) \cdot \varphi_x(-y)dy =$$

$$= \int_{\mathbb{R}^n} (\overline{T^*(\overline{\Phi})})(y) \cdot \varphi(x-y)dy = ((\overline{T^*(\overline{\Phi})}) * \varphi)(x)$$

Let $\Phi \in \mathcal{S}$ such that $\mathcal{F}^{-1}(\Phi) > 0$. Using the claim we have that

$$\mathcal{F}^{-1}(\Phi) \cdot \mathcal{F}^{-1}(T\varphi) = \mathcal{F}^{-1}(\Phi * T\varphi) = \mathcal{F}^{-1}(\overline{T^*(\overline{\Phi})} * \varphi) = \mathcal{F}^{-1}(\overline{T^*(\overline{\Phi})}) \cdot \mathcal{F}^{-1}(\varphi)$$

Thus

$$\mathcal{F}^{-1}(T\varphi) = \frac{\mathcal{F}^{-1}(\overline{T^*(\overline{\Phi})})}{\mathcal{F}^{-1}(\Phi)} \cdot \mathcal{F}^{-1}(\varphi)$$

Defining $a \in \mathcal{S}$ to be $a = \mathcal{F}^{-1}(\overline{T^*(\overline{\Phi})})/\mathcal{F}^{-1}(\Phi)$ we have that

$$(\mathcal{F}^{-1} \circ T \circ \mathcal{F}^{-1})(\varphi) = \mathcal{F}^{-1}(T(\mathcal{F}^{-1}(\varphi))) = a \cdot \mathcal{F}^{-1}(\mathcal{F}(\varphi)) = a \cdot \varphi$$

and we are done. □

5. Prove that for any $\varepsilon > 0$, the function $f(\xi) = (1 + |\xi|^2)^{-\varepsilon}$ on $\mathbb{R}$ is the Fourier transform of an L^1- function g.

Solution: Notice that

$$\mathcal{F}(e^{-\frac{x^2}{4t}}) = \sqrt{4\pi t}e^{-\xi^2 t}$$

where $\mathcal{F}$ denotes the Fourier transform. Define

$$g(x) = \int_0^\infty e^{-\frac{x^2}{4t}} \varphi(t)dt$$

where $\varphi(t) = \frac{e^{-t}t^\alpha}{\sqrt{4\pi}\Gamma(\varepsilon)}$ and $\alpha = -\frac{3}{2} + \varepsilon$. Notice that

$$\varphi(t) \cdot \sqrt{4\pi t} = \frac{e^{-t}t^{\varepsilon-1}}{\Gamma(\varepsilon)}$$

Then by Fubini we have that

$$\hat{g}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} g(x)dx = \int_{\mathbb{R}} e^{-i\xi x}\Big(\int_0^\infty e^{-\frac{x^2}{4t}}\varphi(t)dt\Big)dx = \int_0^\infty \varphi(t)\Big(\int_{\mathbb{R}} e^{-i\xi x}e^{-\frac{x^2}{4t}}dx\Big)dt =$$

$$= \int_0^\infty \varphi(t)\mathcal{F}(e^{-\frac{x^2}{4t}})dt = \int_0^\infty \varphi(t)\sqrt{4\pi t}e^{-\xi^2 t}dt = \int_0^\infty \frac{e^{-t}t^{\varepsilon-1}}{\Gamma(\varepsilon)}e^{-\xi^2 t}dt = \frac{1}{\Gamma(\varepsilon)}\int_0^\infty t^{\varepsilon-1}e^{-t(1+\xi^2)}dt =$$

$$= \Big[s = (1+\xi^2)t\Big] = \frac{(1+\xi^2)^{-\varepsilon}}{\Gamma(\varepsilon)}\int_0^\infty s^{\varepsilon-1}e^{-s}ds = (1+\xi^2)^{-\varepsilon}$$

6. Prove that if $f \in L^1$ is continuous, bounded and with a non-negative Fourier transform, then $\hat{f} \in L^1$.

Solution: Let $\theta(x) = e^{-|x|^2}$ and for $R > 0$ define θ_R to be

$$\theta_R(x) := \theta(x/R) = e^{-\frac{|x|^2}{R^2}}$$

Claim 1: $\|\theta_R \cdot \hat{f}\|_1 \leq (2\pi)^n \|f\|_\infty$.

Proof: The Fourier transform of θ_R is given by

$$\hat{\theta}_R(x) = R^n \pi^{\frac{n}{2}} e^{-\frac{|\xi|^2 R^2}{4}}$$

Since

$$\int_{\mathbb{R}^n} e^{-\frac{|x|^2 R^2}{4}} dx = \frac{2^n}{R^n} \int_{\mathbb{R}^n} e^{-\frac{|x|^2 R^2}{4}} \cdot \frac{R^n}{2^n} dx = \left[y = \frac{R}{2} x \right] = \frac{2^n}{R^n} \int_{\mathbb{R}^n} e^{-|y|^2} dy = \frac{2^n}{R^n} \pi^{\frac{n}{2}}$$

we obtain that

$$\|\hat{\theta}_R\|_1 = R^n \pi^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2 R^2}{4}} d\xi = (2\pi)^n$$

By Fubini, we have that

$$\|\theta_R \cdot \hat{f}\|_1 = \left[\hat{f} \geq 0 \right] = \int_{\mathbb{R}^n} \theta_R(\xi) \cdot \hat{f}(\xi) d\xi = \int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{R^2}} \cdot \left(\int_{\mathbb{R}^n} e^{-i\langle \xi, x \rangle} f(x) dx \right) d\xi =$$

$$= \int_{\mathbb{R}^n} f(x) \left(\int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{R^2}} e^{-i\langle \xi, x \rangle} d\xi \right) dx = \int_{\mathbb{R}^n} f(x) \hat{\theta}_R(x) dx \leq \|f\|_\infty \|\hat{\theta}_R\|_1 = (2\pi)^n \|f\|_\infty$$

For any $x \in \mathbb{R}^n$ we have that $R \mapsto \theta_R(x)$ is non-decreasing, and that

$$\lim_{R \to \infty} \theta_R(x) = 1$$

Thus the monotone convergence theorem implies that

$$\|\hat{f}\|_1 = \lim_{R \to \infty} \|\theta_R \cdot \hat{f}\|_1 \leq (2\pi)^n \|f\|_\infty < \infty$$

which means that $\hat{f} \in L^1$. $\square$