9/12/20

Steiner point and the problem of chasing convex bodies

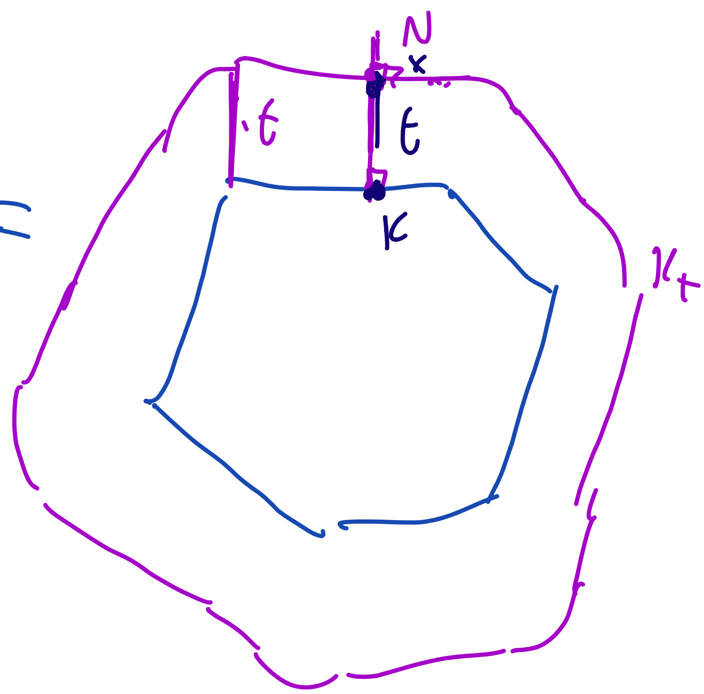
Steiner Point

Given $K, T \subseteq \mathbb{R}^n$,

$$K + T = \left\{ x + y \;;\; \begin{matrix} x \in K \\ y \in T \end{matrix} \right\}$$

$K + t B^n$ =

unit ball



Theorem (Steiner, 180 years ago)

$\forall$ convex $K \subseteq \mathbb{R}^n$, $t \geq 0$

$$\text{barycenter}\left(K + tB^n\right) \in K$$

Proof: Enough to show that $\forall t > 0$

$$\text{barycenter}\left(\partial\big(\underbrace{K + tB^n}_{K_t}\big)\right) \in K$$

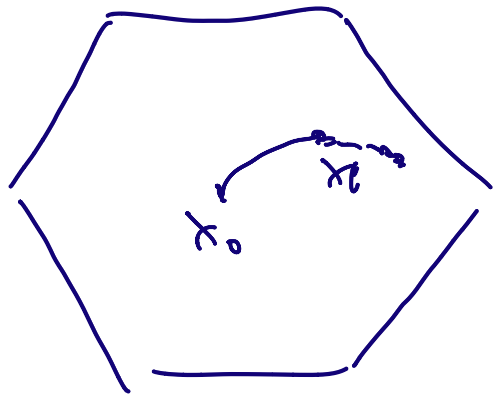
$$\frac{1}{\mathrm{vol}_{n-1}(\partial K_t)} \int_{\partial K_t} x \, dx = \int_{\partial K_t} \underbrace{\mathrm{Proj}_K x}_{\text{contribution lies in } K} + \underbrace{(x - \mathrm{Proj}_K x)}_{= tN} \frac{dx}{|\partial K_t|}$$

It remains to show that

$$\int_{\partial K_t} N = 0$$

(just divergence thm, no flux for constant vector field).

$\square$



$x_t = \text{barycenter}(K_t)$

$K \subseteq \mathbb{R}^n$

<u>Def:</u> The Steiner point $s(K)$ is

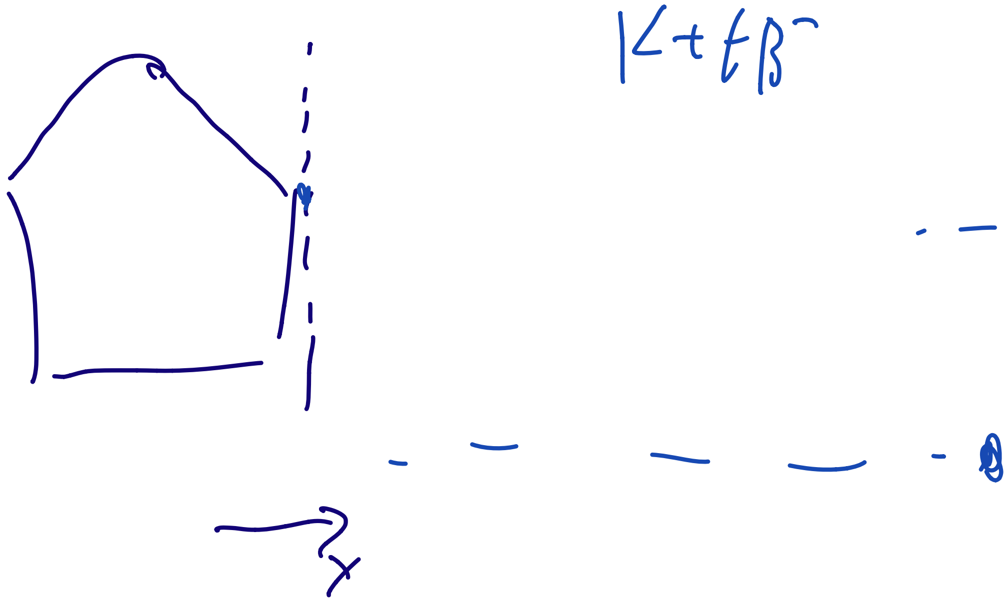
$$s(K) = \lim_{t \to \infty} \text{barycenter}(K + tB^n)$$

The limit exists, and proof implies

$$s(K) = \int_{S^{n-1}} \nabla h_K(x) \, d\sigma_{n-1}(x)$$

when $h_K(x) = \sup_{y \in K} \langle x, y \rangle$ is the supporting functional.

$\nabla h_K(x) =$ point when sup is attained. $\in K$



Two important properties:

1) $s(K+T) = s(K) + s(T)$

2) Lipschitz dependence on K w.r.t Hausdorff metric.

"Dual formula": $$s(K) = n \int_{S^{n-1}} h_K(x)\, x \, d\sigma_{n-1}(x)$$

$$= \int_{S^{n-1}} \nabla h_K(x) \, d\sigma_{n-1}(x)$$

$\exists \theta \in S^{n-1}$ (divergence thm).

$$|s(K)-s(T)| = \langle s(K)-s(T), \theta \rangle$$

$$= n \int_{S^{n-1}} \big(h_K(x) - h_T(x)\big) \langle x, \theta \rangle \, d\sigma_{n-1}(x)$$

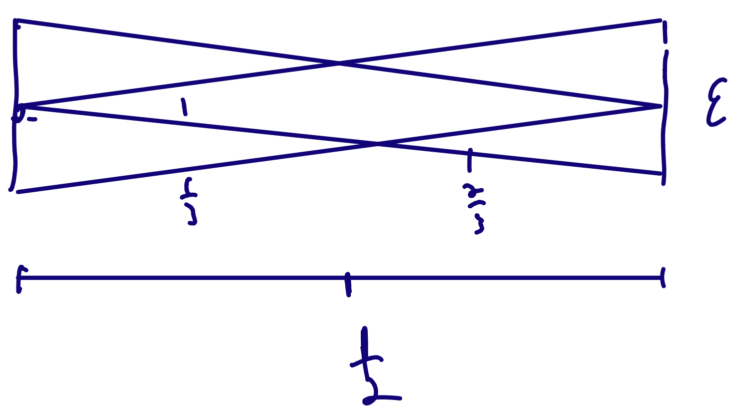
$$\leq n \cdot \underbrace{\sup_{S^{n-1}} |h_K - h_T|}_{\substack{\parallel \\ d_{\text{Hausdorff}}(K,T)}} \cdot \int_{S^{n-1}} |\langle x, \theta \rangle| \, d\sigma_{n-1}(x)$$

$$\int_{S^{n-1}} |\langle x, \theta \rangle| \, d\sigma_{n-1}(x) \simeq \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}}$$

Here it's L_n-Lipschitz, for

$$L_n \approx \sqrt{\frac{2}{\pi}} \sqrt{n}$$

<u>Comparison to barycenter</u> (not stable)



$\rightsquigarrow$ Recently, has led to progress in online optimization

<u>Linear programming</u> (1940s)

convex set $K \subseteq \mathbb{R}^n$ $\left\{ \begin{array}{l} \text{Given linear inequalities} \\ \langle x, u_i \rangle \leq b_i \quad (x \in \mathbb{R}^n) \\ \qquad\qquad\qquad\qquad i = 1, \dots, T \end{array} \right.$

<u>Goal:</u> Find $x \in K$.

<u>An online variant of linear programming</u>

(Friedman - Linial '92)

The inequalities come one by one with time.

For each $t = 1, \dots, T$ we are given $K_t \subseteq \mathbb{R}^n$

<u>Our goal:</u> Find $x_t \in K_t$ online,

so as minimize

$$\text{Cost} = \sum_{t=1}^{T} |X_{t+1} - X_t| + \text{diameter}(K_1)$$

<u>Important remark:</u> Can see the past but not the future. When we choose x_t, only know $K_1, \ldots, K_t \subseteq \mathbb{R}^n$.

<u>Easier variant:</u> Nested $K_{t+1} \subseteq K$

<u>Harder variant:</u> Any convex sets, say in $\mathbb{R}^n$.

<u>What is a good online algorithm?</u>

<u>Standard answer:</u> Cost not much larger than cost of best possible algorithm that <u>can see the future</u>.

$$\text{Cost}_T(\text{OPT}) = \inf_{\forall t\ x_t \in K_t} \sum_{t=1}^{T} |X_{t+1} - X_t| + \text{diam}(K_1)$$

An online algorithm:

$$X_t = f_t(K_1, \ldots, K_t)$$

(w/o the future)

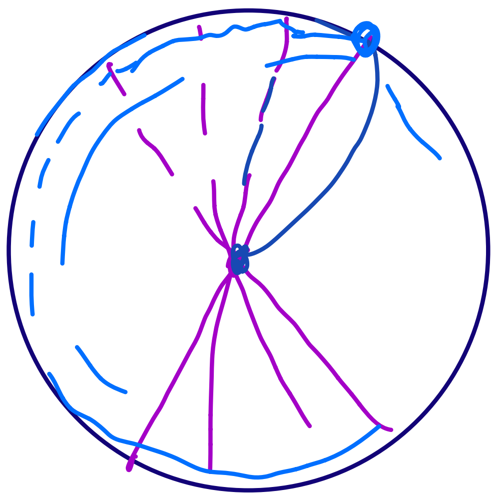
Thm (Friedman-Linial '93)

In 2D $\exists$ online algorithm with

$$\forall T \quad \underbrace{\frac{\mathrm{Cost}_T(ALG)}{\mathrm{Cost}_T(OPT)}}_{\text{competitive ratio}} \leq C$$

for $C > 0$ universal.

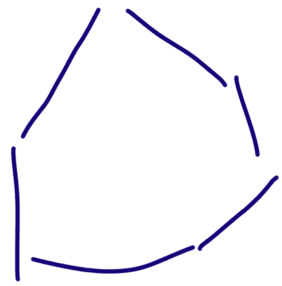
$$K_\theta = [-e^{i\theta}, e^{i\theta}]$$



Bubeck-Lee-Li-Sellke '19 $\forall n$ $\exists$ online algorithm, competitive ratio $\Theta(e^{cn})$

in $\mathbb{R}^n$.

- Same exponential in n even in nested case.
- Use of barycenter

K_t 

- <u>A new idea</u>: Use Steiner point in place of barycenter.

<u>Bubeck–Klartag–Lee–Li–Sellke '20</u>:
In the nested case, $\exists$ online algorithm with competitive ratio $n+1$ in $\mathbb{R}^n$.

<u>Algorithm</u>: $X_t = s(K_t)$

- Two teams independently used Steiner point, essentially reducing matters to the nested case (functional version)

and got $O(n)$ in the general case
$n+1$

- Sellke
- Argue, Gupta, Guruganesh, Tang

} best paper award in SODA '20

<u>Proof in nested case:</u> diameter$(K_1)=1$

$$\mathrm{Cost}_T(\mathrm{OPT}) \in [1, 2]$$

$$\mathrm{COST}_T(\mathrm{ALG}) =$$

$$= 1 + \sum_{t=1}^{T} \left| s(K_{t+1}) - s(K_t) \right|$$

$$\leq 1 + n \cdot \sum_{t=1}^{T} \left| \int_{S^{n-1}} x \left(h_{K_{t+1}} - h_{K_t} \right) dx \right|$$

$$\leq 1 + n \sum_{t=1}^{T} \int_{S^{n-1}} \left| h_{K_t} - h_{K_{t+1}} \right| d\sigma_{n-1}$$

$$= 1 + n \int_{S^{n-1}} \left(h_{K_1} - h_{K_T} \right) \le n+1 .$$

In the non-nested case

- Work function: For $x \in \mathbb{R}^n$

$W_T(x)$ = minimum cost of any legal path ending at x

$$W_T(x) = \inf_{x_i \in K_i} \sum_{t=1}^{T-1} |x_{t+1} - x_t| + |x - x_T|$$

$$W_{t+1}(x) = \inf_{y \in K_{t+1}} \left[W_t(y) + |x - y| \right]$$

- Convex function, 1-Lip, $W_{t+1} \ge W_t$

"nested"

Algorithm:

x_t = functional Steiner point of W_t

$$s(W_t) = n \int_{S^{n-1}} W_t^*(\theta)\, \theta \, d\sigma_{n-1}(\theta)$$

where $W_t^*(y) = \sup_{y \in \mathbb{R}^n} \left[\langle x, y \rangle - W_t(x) \right]$

<u>Claim 1:</u> $X_t \in K_t$

<u>Claim 2:</u>

$$\mathrm{Cost}_T(\mathrm{OPT}) = \inf_x W_T(x)$$

$$= - W_T^*(0)$$

$$\sum \left| s(W_{t+1}) - s(W_t) \right| \leq (n+1) \left[-W_T^*(0) \right]$$

as $W_t^* \geq W_{t+1}^*$.

$f_t : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ convex

<u>Service costs</u> $\sum f_t(x_t)$

movement cost : $\sum |x_{t+1} - x_t|$