Harmonic Analysis, homework assignment no. 1

Please submit your solution in pdf format by May 11 at 2PM at the link:
`https://www.dropbox.com/request/nlF3cz6aMrh5dHpfMGRs`

1. (a) Verify that the Sierpinski triangle (see wikipedia) has Haussdorf dimension $\log 3/\log 2$.

   (b) Watch a video on a construction of a Besicovitch set at
   `https://www.youtube.com/watch?v=j-dce6QmVAQ`

2. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with $\mathbb{P}(X_k = 0) = \mathbb{P}(X_k = 2) = 1/2$. Prove that for any Borel subset A of the Cantor set $C \subseteq [0,1]$,
$$\mathbb{P}\left(\sum_{k=1}^{\infty} \frac{X_k}{3^k} \in A\right) = c \cdot \mathcal{H}_\alpha(A)$$
for $\alpha = \log 2/\log 3$ and some constant $c > 0$.

3. For any $0 < t < n$, find a compactly-supported, finite Borel measure μ on $\mathbb{R}^n$ with a finite t-energy, i.e., $I_t(\mu) = \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^{-t} d\mu(x) d\mu(y) < \infty$, yet for any $M > 1$ there exist $x \in \mathbb{R}^n, r > 0$ with
$$\mu(B(x,r)) > Mr^t.$$

4. Write $\mathcal{S}$ for the space of Schwartz functions in $\mathbb{R}^n$. Let $T : \mathcal{S} \to \mathcal{S}$ be a continuous, translation-invariant linear operator (i.e., $T(f_a) = (Tf)_a$ when $f_a(x) = f(x+a)$). Prove that $\mathcal{F}^{-1} \circ T \circ \mathcal{F}$ is a multiplication operator, where $\mathcal{F}$ is the Fourier transform.

5. Prove that for any $\varepsilon > 0$, the function $f(\xi) = (1 + |\xi|^2)^{-\varepsilon}$ on $\mathbb{R}$ is the Fourier transform of an L^1-function g. Hint: Express g as a mixture of Gaussians, try
$$g(x) = \int_0^\infty e^{-x^2/t} \varphi(t) dt.$$
say with $\varphi(t) = e^{-t} t^\alpha$.

6. Prove that if $f \in L^1$ is continuous, bounded and with a non-negative Fourier transform, then $\hat{f} \in L^1$. Hint: Maybe look at $\int \hat{f}(\xi)\theta(\varepsilon\xi) d\xi$.
(By the way, the boundness and continuity of f are not necessary here).