Harmonic Analysis, homework assignment no. 2

Please submit your solution in pdf format by Monday, June 1st at 2PM at the link:
https://www.dropbox.com/request/faRqeNGFJKtwgyd4YYkY

You are asked to solve at least 6 questions.

1. Suppose that $f, \hat{f} \in L^1(\mathbb{R}^n)$. Prove that by modifying f on a set of measure zero we may obtain a continuous function that vanishes at infinity.

2. Write Z for the Cauchy random variable, with density $\rho(t) = 1/[\pi(1+t^2)]$ in $\mathbb{R}$. For a compactly-supported, continuous function $f : \mathbb{R} \to \mathbb{C}$ set
$$F(x,y) = \mathbb{E} f(x+yZ) \qquad\qquad (y > 0).$$
Prove that F is a harmonic function in the upper half plane, and that $F(\cdot, y)$ tends uniformly to f as $y \to 0^+$.

3. (a) Prove that the convolution of two Schwartz functions, is a Schwartz function.
   (b) Prove that the convolution of a Schwartz function with a compactly-supported Borel measure, is a Schwartz function.

4. Let $n \geq 5$ and let $f \in L^1(\mathbb{R})$ satisfy $\int_{-\infty}^{\infty} |\hat{f}(t)|^2 t^{n-1} dt < \infty$. Prove that f is a C^k-function for $k = \lfloor (n-3)/2 \rfloor$.

5. Let $n \geq 5$, let $f \in L^1 \cap L^2(\mathbb{R}^n)$, and recall the Radon transform
$$Rf(\theta, t) = \int_{P_{\theta,t}} f$$
for $P_{\theta,t} = \{x \in \mathbb{R}^n, ; \langle x, \theta \rangle = t\}$. Prove that for almost any $\theta \in S^{n-1}$, the function $Rf(\theta, t)$ is C^k-smooth in the t variable for $k = \lfloor (n-3)/2 \rfloor$.

6. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be a non-negative function of integral one. Set $\psi_k(x) = k^n \psi(kx)$. Prove that for any $\varphi \in \mathcal{S}$,
$$\psi_k * \varphi \longrightarrow \varphi$$
in the topology of the Schwartz space.

7. Prove that for $f \in C^k \cap \mathcal{S}^*$, the classical derivative $\partial^\alpha f$ for $|\alpha| \leq k$ coincides with the distributional derivative $\partial^\alpha f$ when acting on smooth, compactly-supported functions.

8. Prove that for $F \in \mathcal{S}^*$ and $\varphi, \psi \in \mathcal{S}$,
$$F(\varphi * \psi) = \int_{\mathbb{R}^n} F(\tau_y \psi) \varphi(y) dy.$$
where $(\tau_y \psi)(x) = \psi(x-y)$.