# Harmonic Analysis, homework assignment no. 3

Please submit your solution in pdf format by Monday, June 22 at 2PM at the link:
`https://www.dropbox.com/request/40YmvQmDqBf1mbxCpuCU`

You are asked to solve at least 4 questions.

1. A tempered distribution $F \in \mathcal{S}^*$ is supported in a closed set $F \subseteq \mathbb{R}^n$ if $F(\varphi) = 0$ for all $\varphi \in \mathcal{S}$ with $\overline{Supp(\varphi)} \subseteq \mathbb{R}^n \setminus F$.

   Prove that if $F \in \mathcal{S}^*$ is supported at $\{x_0\}$ for some $x_0 \in \mathbb{R}^n$, then $F$ takes the form $F = \sum_{|\alpha| \leq N} a_\alpha \partial^\alpha \delta_{x_0}$ for some $N \geq 0$ and coefficients $a_\alpha \in \mathbb{C}$.

2. A tempered distribution $F \in \mathcal{S}^*$ is $\mathbb{Z}^n$-periodic if $F(\varphi) = F(\tau_k \varphi)$ for all $k \in \mathbb{Z}^n, \varphi \in \mathcal{S}$ where $\tau_k \varphi(x) = \varphi(x - k)$.

   Prove that $F \in \mathcal{S}^*$ is $\mathbb{Z}^n$-periodic if and only if $\hat{F} = \sum_{k \in \mathbb{Z}^n} a_k \delta_{2\pi k}$ for numbers $a_k \in \mathbb{C}$ $(k \in \mathbb{Z}^n)$ called the Fourier coefficients.

3. Prove the Poisson summation formula: For any $f \in \mathcal{S}$,

   $$\sum_{k \in \mathbb{Z}^n} f(k) = \sum_{k \in \mathbb{Z}^n} \hat{f}(2\pi k).$$

   (e.g., by using the $\mathbb{Z}^n$-periodization $F(x) = \sum_{k \in \mathbb{Z}^n} f(x + k) \in \mathcal{S}^*$).

4. Complete the proof sketched in class of the following formula: For any compactly-support Borel measure $\mu$ on $\mathbb{R}^n$ and any $0 < t < n$,

   $$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x - y|^t} d\mu(x) d\mu(y) = c_{n,t} \int_{\mathbb{R}^n} \frac{1}{|\xi|^{n-t}} |\hat{\mu}(\xi)|^2 d\xi.$$

   (Carefully justify all passages. Recall that the Fourier transform of $|x|^{-t}$ is $c|\xi|^{t-n}$).

5. (Mattila-Orponen) Let $A, B \subset \mathbb{R}^2$ be two compacts, both of Hausdorff dimension greater than one. Prove that with positive probability of choosing the unit vector $\theta \in S^1$, the intersection $P_\theta(A) \cap P_\theta(B) \subseteq \mathbb{R}$ has a positive Lebesgue measure. Here $P_\theta(x) = \langle x, \theta \rangle$.

   (Hint: As in class, take Frostman measures $\mu$ and $\nu$, recall that $\mu_\theta = (P_\theta)_* \mu \in L^2(\mathbb{R})$ for almost any $\theta \in S^1$, and show that the $\theta$-average of $\langle \mu_\theta, \nu_\theta \rangle$ is positive).

6. Let $f \in \mathcal{S}$ satisfy $Supp(\hat{f}) \subseteq \{\xi \in \mathbb{R}^n \,;\, R/2 < |\xi| < 2R\}$ for some $R > 0$. Prove that $R\|f\|_\infty \leq C\|\nabla f\|_\infty$ for some $C > 1$ depending on $n$.