Harmonic Analysis, homework assignment no. 4

Please submit your solution in pdf format by Monday, July 20 at 2PM at the link:
`https://www.dropbox.com/request/MirW0BEcgJgPZsXuQWvX`

You are asked to solve at least 4 questions, at least one of them from questions 5–8.

1. Prove that for any $f \in C_c^\infty(\mathbb{R}^n)$, its Hilbert transform $g = Hf$ satisfies
$$\lim_{t\to\pm\infty} tg(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f.$$
When does $g \in L^1(\mathbb{R})$? and when does $g \in L^2(\mathbb{R})$?

2. Show that the Hilbert transform is *not* bounded from $L^1 \cap C^0$ to C^0, where C^0 is the space of continuous functions on $\mathbb{R}$ equipped with the L^∞-norm.
[Hint: φ a bump function on $[\varepsilon^2, \varepsilon]$).

3. Let $k \ge 0$ and let $m(\xi)$ be a k-homogeneous function which is smooth and non-vanishing in $\xi \in \mathbb{R}^n \setminus \{0\}$. Let $f, g \in L^1(\mathbb{R}^n)$ satisfy
$$\hat{g}(\xi) = m(\xi)\hat{f}(\xi) \qquad\qquad (\xi \neq 0).$$
Prove that for any $0 < s < 1$, if $g \in C^s$ then $f \in C^{k,s}_{loc}$.

4. Let $K(x, y)$ be a smooth function of $x, y \in \mathbb{R}^n$, such that for any multiindices α, β and $N > 0$ there exists a smooth, moderately growing function $A(x) = A_{\alpha,\beta,N}(x) > 0$ for which $|\partial_x^\alpha \partial_y^\beta K(x, y)| \le A(x)/(1 + |y|)^N$ throughout $\mathbb{R}^n \times \mathbb{R}^n$. Consider the integral operator
$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy$$
initially defined for $f \in \mathcal{S}$. Explain how to define T on $\mathcal{S}^*$, and prove that T maps tempered distributions to C^∞-functions.
[Hint: Recall the proof that $F * \varphi$ is C^∞ for $F \in \mathcal{S}^*$ and $\varphi \in \mathcal{S}$.]

5. Let $a_1, a_2, \ldots \in S^m$ be a sequence of symbols that converges pointwise to a function $a(x, \xi)$. Assume that $a_k \in S^m$ with uniform estimates in k, in the sense that for any multi-indices α, β there exists $A_{\alpha,\beta} > 0$ such that
$$|\partial_x^\alpha \partial_\xi^\beta a_k(x, \xi)| \le A_{\alpha,\beta}(1 + |\xi|)^{m-|\beta|}$$
for all $x, \xi \in \mathbb{R}^n$ and $k \ge 1$. Prove that $a \in S^m$ with the same bounds, and that for any $\varphi \in \mathcal{S}$ we have the convergence $Op[a_k]\varphi \longrightarrow Op[a]\varphi$ in the topology of $\mathcal{S}$.

6. Let $a_1, a_2, \ldots \in S^m$ be a sequence of symbols with uniform estimates such that $Op[a_k]\varphi \longrightarrow Op[a]\varphi$ pointwise for any $\varphi \in \mathcal{S}$. Prove that $a_k \longrightarrow a$ pointwise in $\mathbb{R}^n \times \mathbb{R}^n$. [Hint: $\hat{\varphi} \in \mathcal{S}$ a bump near ξ_0].

7. Let $a \in S^m, b \in S^\ell$ and $\gamma \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with $\gamma(0) = 1$. Set $a_\varepsilon(x,\xi) = a(x,\xi)\gamma(\varepsilon x, \varepsilon\xi)$ and similarly $b_\varepsilon(x,\xi) = b(x,\xi)\gamma(\varepsilon x, \varepsilon\xi)$. Set $c_\varepsilon = a_\varepsilon \sharp b_\varepsilon$.

   (a) Prove that there exists a subsequence $\varepsilon_k \searrow 0$ such that $c_{\varepsilon_k}(x,\xi)$ converges pointwise as $k \to \infty$ to some function $c(x,\xi)$. Prove that $c \in S^{m+\ell}$ with the same estimates and the same asymptotic expansion proven in class for c_ε.

   (b) Prove that $Op[c] = Op[a] \circ Op[b]$ as operators acting on $\mathcal{S}$.

8. Let $a \in S^m$ and $\gamma \in C_C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with $\gamma(0) = 1$. Set $a_\varepsilon(x,\xi) = a(x,\xi)\gamma(\varepsilon x, \varepsilon\xi)$. Define $Op[a]^*$ via $\langle Op[a]f, g\rangle = \langle f, Op[a]^* g\rangle$ for any $f, g \in \mathcal{S}$. Define

$$a_\varepsilon^*(x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\cdot(\eta-\xi)} \overline{a_\varepsilon(y,\eta)} \, dy d\eta.$$

   Prove that $Op[a_\varepsilon^*]\varphi \longrightarrow Op[a]^*\varphi$ pointwise for any $\varphi \in \mathcal{S}$ as $\varepsilon \to 0$.

   Bonus: Prove that $Op[a]^* \in S^m$ with an asymptotic expansion.