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Isometric embeddings into Riemannian surfaces with Hölder continuous curvature

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1 Introduction

Can a metric space (X, d_X) be embedded isometrically in a two-dimensional Riemannian manifold whose Gauss curvature is Lipschitz, or more generally, Hölder continuous? In this thesis we address what is probably the simplest nontrivial special case of this problem.

Let $\mathcal{X}$ denote the class of metric spaces (X, d_X) with the property that for some $x_0 \in X$, its complement $X \setminus \{x_0\}$ is isometric to a compact interval $I \subseteq \mathbb{R}$.

To each $X \in \mathcal{X}$ we can associate a function $\rho_X : I \to (0, \infty)$ defined by

$$\rho_X(t) = d_X(\varphi(t), x_0), \tag{1}$$

where $\varphi : I \to X \setminus \{x_0\}$ is an isometry. The function ρ_X determines the metric of X completely, and embedding X isometrically in a Riemannian manifold (M, g) is the same as finding a point $p \in M$ and a unit-speed minimizing geodesic $\gamma : I \to M$ such that

$$d_g(\gamma(t), p) = \rho_X(t) \qquad \text{for all } t \in I.$$

Given $X \in \mathcal{X}$ and $0 < \alpha \leq 1$, we would like to know:

- Does X embed isometrically in a two-dimensional Riemannian manifold whose Gauss curvature is bounded and α-Hölder?
- If so, then how small can the α-Hölder seminorm of the curvature be?

Recall that for $0 < \alpha \leq 1$, a real-valued function F on a metric space (X, d_X) is *α-Hölder* if there exists $L > 0$ such that

$$|F(p) - F(q)| \leq L \cdot d_X(p, q)^\alpha \qquad \text{for all } p, q \in X.$$

The smallest such L is denoted by $\|F\|_\alpha$ and called the α-Hölder seminorm of F. For $\alpha = 1$, Hölder continuity is the same as Lipschitz continuity.

A $C^{2,\alpha}$ *Riemannian surface* is a complete two-dimensional Riemannian manifold (M, g) with a C^2 metric, whose Gauss curvature K is bounded and α-Hölder (with respect to the Riemannian metric), and whose injectivity radius $\mathrm{inj}(M)$ is bounded from below. For simplicity, we will bound all three quantities $\|K\|_\infty$, $\|K\|_\alpha$ and $\mathrm{inj}(M)$ in terms of a single parameter $H > 0$, and say that (M, g) is a *$C^{2,\alpha}$ Riemannian surface with constant H.* See §2.3 for precise definitions.

The main result of this thesis is the following "finiteness principle":

Theorem 1. *There exist universal constants B, C such that the following holds. Let $H, R > 0$ satisfy $HR^2 \leq B$. Let $X \in \mathcal{X}$ and assume that the diameter of X is at most R. If every every subspace $S \subseteq X$ of cardinality at most 13 embeds isometrically in a $C^{2,\alpha}$ Riemmanian surface with constant H, then X itself embeds isometrically in a $C^{2,\alpha}$ Riemannian surface with constant CH.*

The assumption $HR^2 \leq B$ guarantees that all the embeddings involved are into strongly convex discs (a subset U of a Riemannian manifold is said to be *strongly convex* if every two points in U are joined by a unique minimizing geodesic lying entirely inside U; see §2.5). As a consequence, the function ρ_X associated with X by the formula (1) is convex. This assumption also guarantees that the Riemannian metrics on those discs is equivalent to the Euclidean metric up to universal constants.

We do not know whether the number 13 is sharp. A trivial lower bound is 4, since any 3-point metric space can be embedded in $\mathbb{R}^2$, and the lower bound 5 follows from the following fact, which is fundamental in Alexandrov geometry (see [4]). Let $\mathscr{S}_{K_0}$ denote the complete, simply connected surface of constant curvature $K_0 \in \mathbb{R}$. The number K_0 is said to be a *planar embedding curvature* of a metric space X if X embeds isometrically in $\mathscr{S}_{K_0}$.

Theorem (Wald-Berestovskii [16],[1])**.** *Any four-point metric space, of which exactly three points are collinear (i.e. embeds isometrically in $\mathbb{R}$), has a unique planar embedding curvature.*

The problem of embedding a metric space isometrically in a Riemannian manifold is, in some sense, the geometric counterpart of the classical Whitney problem, which asks under what conditions a real-valued function on a closed subset of $\mathbb{R}^n$ extends to a $C^{k,\alpha}$ function defined on all of $\mathbb{R}^n$ (recall that a $C^{k,\alpha}$ function is a function whose partial derivatives of order k are α-Hölder). For a comprehensive survey of the Whitney extension problem, see Fefferman and Israel [10].

We are not aware of much literature concerning isometric embeddings into manifolds of "moderately varying curvature". In a paper titled "the geometric Whitney problem", Fefferman, Ivanov, Kurylev, Lassas and Narayanan [11] have recently provided a characterization of metric spaces which are *quasi-isometric* to Riemannian manifolds of bounded geometry, i.e. metric spaces which can be mapped with small metric distortion into ε-nets of Riemannian manifolds with bounded sectional curvature and injectivity radius. By contrast, our embeddings are isometric rather than quasi-isometric, and the image does not have to be an ε-net.

Theorem 1 is inspired by finiteness principles arising both in the theory of isometric embeddings and in the theory of extensions of functions. A classical result (see Blumenthal [2], §39) states:

Theorem. *A metric space X embeds isometrically in $\mathscr{S}_{K_0}$ if every 5-point subspace of X does.*

The following finiteness principle is a well-known consequence of Whitney's extension theorem in dimension one, see [10].

Theorem. *Let $A \subseteq \mathbb{R}$ be a closed subset and let $f : A \to \mathbb{R}$. Suppose that for every subset $S \subseteq A$ consisting of at most $k+2$ points, the function $f|_S$ can be extended to a function $F_S \in C^{k,\alpha}(\mathbb{R})$ satisfying $\left\| F_S^{(k)} \right\|_\alpha \leq 1$. Then f can be extended to a function $F \in C^{k,\alpha}(\mathbb{R})$ with $\left\| F^{(k)} \right\|_\alpha \leq C$, where C is a universal constant.*

A generalization of this finiteness principle to $C^{k,\alpha}(\mathbb{R}^n)$ was proved by Fefferman [9], settling a conjecture by Brudnyi and Shvartsman.

Outline of the proof of Theorem 1

The thesis is devoted mainly to the proof of Theorem 1. The scheme of the proof is as follows. Let $X \in \mathcal{X}$ and let $\rho = \rho_X$. As noted above, embedding X isometrically in a $C^{2,\alpha}$ Riemannian surface (M, g) is the same as finding a point $p \in M$ and a unit-speed geodesic $\gamma : I \to M$ such that $\rho = d_g(\gamma(\cdot), p)$. The metric g will be constructed in polar normal coordinates r, θ centered at p (see §2.4). The advantages of these coordinates are twofold: first, the Riemannian distance of a point from the origin in normal coordinates is the same as the Euclidean distance, so we know that our curve γ has to be given in normal coordinates by $\gamma(t) = (\rho(t)\cos(\phi(t)), \rho(t)\sin(\phi(t)))$ for some function $\phi : I \to S^1$. Second, the metric in polar normal coordinates is always of the form $g = dr^2 + G^2 d\theta^2$ for some function G. Hence, the construction boils down to the determination of two functions: the angle function $\phi = \theta \circ \gamma$, and the metric coefficient G.

Two central difficulties which arise in the construction are:

- For $C^{2,\alpha}$ surfaces, the metric in polar normal coordinates is not necessarily C^2; it may even fail to be C^1 (see Hartman [13]). So it is possible that a metric with a rather "bad behavior" in polar normal coordinates would turn out to be $C^{2,\alpha}$. In §2.6 we settle the problem, interesting in its own right, of determining necessary and sufficient conditions on G for the metric $dr^2 + G^2 d\theta^2$ to be $C^{2,\alpha}$ in a sufficiently small disc around the origin.

- Both the metric g and the location of the curve γ are unknown, and each one of them determines the other. Our solution to this is to start with a "provisional" curve γ_0, then to construct the metric around γ_0, and finally to deform γ_0 into the final curve γ.

Below are the main steps of the construction process.

The curvature of the metric g has to be α-Hölder with a bounded seminorm, and the diameter of X is bounded, so, roughly speaking, there should be some number $K_0 \in \mathbb{R}$ such that the curvature is approximately K_0 on the region where X will be embedded. We use a simple formula to determine what K_0 is going to be (Proposition 3.8 and Corollary 3.9). Intuitively, we choose K_0 to be the planar embedding curvature of $\{x_0, x_1, x_2, x_3\}$, where $x_0 \in X$ is the point such that $X \setminus \{x_0\}$ is isometric to an interval, and $x_1, x_2, x_3 \in X \setminus \{x_0\}$ are points which are infinitesimally close to each other. Since x_1, x_2, x_3 are collinear, this embedding curvature is unique.

We now take a curve γ_0 on a disc $\Lambda \subseteq \mathbb{R}^2$, with the properties: (1) if the disc Λ is endowed with a metric of constant curvature K_0, then γ_0 is unit-speed; and (2) for all $t \in I$, $\gamma_0(t)$ has distance $\rho(t)$ from the origin. In polar normal coordinates, the metric of constant curvature K_0 is given by

$$dr^2 + (\sin_{K_0} r)^2 d\theta^2$$

(see §2.5 for the definition of $\sin_{K_0}$), and γ_0 is given by

$$\gamma_0(t) = (\rho(t)\cos(\phi_0(t)), \rho(t)\sin\phi_0(t))$$

for some $\phi_0 : I \to S^1$. We want to replace $\sin_{K_0} r$ and ϕ_0 by functions G and ϕ such that $\gamma := (\rho\cos\phi, \rho\sin\phi)$ is a unit-speed geodesic with respect to the metric $g = dr^2 + G^2 d\theta^2$.

The geodesic equations take a simple form in polar normal coordinates in two dimensions: It can be shown (Proposition 3.1) that a unit-speed curve γ, lying sufficiently close to origin, is a geodesic with respect to g if and only if

$$h(\gamma(t)) = \rho''(t)/(1 - \rho'(t)^2) \text{ for all } t \in I, \tag{2}$$

where $h := \partial_r G/G$. We prescribe these values of h on γ_0, extend h from γ_0 to the disc Λ using a carefully devised partition of unity, and then determine G from h on Λ by integration. Next, we apply a deformation in the θ coordinate, so that γ_0 becomes a unit-speed curve γ, which

satisfies (2) and whose distance from the origin at time t is $\rho(t)$. We then glue the disc Λ to a surface of constant curvature to obtain a complete surface.
Since the function h has a singularity at the origin, instead of extending it directly we work with the function $f := h - \cot_{K_0} r$, the difference between h and its counterpart in constant curvature ($\cot_{K_0}$ is also defined in §2.5). The extension of f from γ_0 to Λ takes more than a direct application of Whitney's extension theorem, because the desired extension has to be $C^{2,\alpha}$ in the r direction, while in the θ direction it is allowed to be just $C^{0,\alpha}$. A special set of conditions is needed to ensure that the curvature of the metric resulting from this extension has a bounded α-Hölder seminorm. In Proposition 3.11, this set of conditions is shown to follow from the finiteness condition of Theorem 1, and in Proposition 4.1, it is proven to be sufficient for the existence of the extension of f possessing all the desirable properties.

The thesis is organized as follows. Section §2 contains the background for the thesis. We begin with a review of relevant results from Whitney theory and Riemannian geometry. We then prove a stability lemma about the Ricatti equation, and use it to obtain estimates for the regularity of the metric of $C^{2,\alpha}$ Riemannian surfaces in polar normal coordinates. Finally, we prove Theorem 2, which characterizes $C^{2,\alpha}$ surfaces by their metrics in polar normal coordinates, a characterization which will become useful when constructing the embedding in the proof of Theorem 1.
In section §3, we study functions ρ_X corresponding to metric spaces $X \in \mathcal{X}$ satisfying the hypothesis of Theorem 1. The main result of the section is Proposition 3.11, which translates the finiteness condition of Theorem 1 into some technical estimates involving ρ and related functions. In Proposition 3.12 we prove that if the constant B in Theorem 1 is small enough, then γ_0 does not wind more than once around the origin, allowing us to construct the metric on the sector between γ_0 and the origin.
In section §4, we prove Theorem 1 in two steps: first, using Theorem 2, we reduce the proof to the problem of extending the domain of the function f mentioned above. Then, we show that the existence of this extension follows from the estimates of Proposition 3.11.

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2 Preliminaries

2.1 General notations

For two quantities A, B, we write $A \lesssim B$ or $A = O(B)$ if there is a positive constant C such that $A \leq CB$, and $A \sim B$ if there is a positive constant C such that $C^{-1}B \leq A \leq CB$. Unless stated otherwise, the implied constant C is universal.

For a point p in a metric space, and a positive number R, the open ball of radius R centered at p is denoted by $B_R(p)$.

The diameter of an subset S of a metric space is denoted by $\mathrm{diam} S$. The length of an interval $I \subseteq \mathbb{R}$ is denoted by $|I|$.

A *curve* is a path on a manifold, which has a well defined continuous tangent.

When working in coordinates, we will identify the coordinate domain on a Riemannian surface M with its image in $\mathbb{R}^2$, and the pullback metric of g on $\mathbb{R}^2$ with the metric on the manifold. We will often write (r, θ) or $(r\cos\theta, r\sin\theta)$ more compactly as $re^{i\theta}$. The coordinate θ will always vary between $-\pi$ and π.

For $K_0 \in \mathbb{R}$, we denote by $\mathscr{S}_{K_0}$ the complete simply-connected surface of constant curvature K_0.

2.2 $C^{k,\alpha}$ functions

Let $0 < \alpha \leq 1$, and let (X, d_X) be a metric space. A function $F : X \to \mathbb{R}$ is said to be α-Hölder if there exists $L > 0$ such that for all $p, q \in X$,

$$|F(p) - F(q)| \leq L \cdot d_X(p, q)^\alpha.$$

The smallest such L is called the α-Hölder seminorm of F and is denoted by $\|F\|_\alpha$.

We shall make extensive use of the (trivial) inequality

$$\|\varphi\psi\|_\alpha \leq \|\varphi\|_\alpha \|\psi\|_\infty + \|\varphi\|_\infty \|\psi\|_\alpha. \tag{3}$$

(here and throughout, $\|F\|_\infty := \sup|F|$).

It is also easy to verify that if X has finite diameter and $F : X \to \mathbb{R}$ is β-Hölder, then it is α-Hölder for all $0 < \alpha \leq \beta$ and

$$\|F\|_\alpha \leq \|F\|_\beta \cdot (\mathrm{diam} X)^{\beta - \alpha}. \tag{4}$$

Let $\Omega \subseteq \mathbb{R}^n$ be a domain. We denote by $C^{k,\alpha}(\Omega)$ the space of k-times differentiable functions $F : \Omega \to \mathbb{R}$ all of whose partial derivatives of order k are α-Hölder. We will write only $C^{k,\alpha}$ when the domain is clear from the context.
The space $C^{k,\alpha}(\Omega)$ can be endowed with the norm

$$\|F\|_{C^{k,\alpha}(\Omega)} := \max_{|J|\leq k} \left\|\partial^J F\right\|_\infty + \max_{|J|=k} \left\|\partial^J F\right\|_\alpha .$$

where $J = (j_1, \dots, j_k)$ is a multi-index and $|J| := \sum_i j_i$.
For a subset $A \subseteq \mathbb{R}$, a function $F : A \to \mathbb{R}$ and a finite subset $S \subseteq A$, we define the *divided difference of* F *with respect to* S, denoted by $F[S]$, by induction on the size of S:

$$F[\{x_0\}] := f(x_0)$$
$$F[\{x_0, \dots, x_k\}] := \frac{F[\{x_0, \dots, x_{k-1}\}] - F[\{x_1, \dots, x_k\}]}{x_0 - x_k}$$

One can check that the ordering of the points in S does not matter. By the mean-value theorem, if $|S| = k+1$ and F is C^k in an interval I containing S, then $F[S] = F^{(k)}(c)/k!$ for some point $c \in I$.

We shall use the following version of Whitney's Extension theorem in dimension one (Theorem 2.52 in [3]):

Theorem 2.1. *Let $k \in \mathbb{N} \cup \{0\}$ and $0 < \alpha \leq 1$, let A be a closed subset of $\mathbb{R}$ and let $f : A \to \mathbb{R}$. Suppose that there is a constant $L > 0$ such that for every subset $S \subseteq A$ consisting of at most $k+2$ points,*

$$|f[S]| \leq L \cdot (\mathrm{diam} S)^{\alpha-1}.$$

Then f can be extended a $C^{k,\alpha}$ function $F : \mathbb{R} \to \mathbb{R}$ with $\left\|F^{(k)}\right\|_\alpha \lesssim L$.

Theorem 2.1 implies the following corollaries:

Corollary 2.2. *Let F be defined on a closed interval $I \subseteq \mathbb{R}$, let $L > 0$ and suppose that for every subset $S \subseteq I$ consisting of at most $k+2$ points, there is a function $F_S \in C^{k,\alpha}$ which agrees with F on S and satisfies $\left\|F_S^{(k)}\right\|_\alpha \leq L$. Then $F \in C^{k,\alpha}$ and $\left\|F^{(k)}\right\|_\alpha \lesssim L$.*

Corollary 2.3. *Let $T_1, T_2, C > 0$ and $0 < \alpha \leq 1$, and let $I \subseteq \mathbb{R}$ be an interval with $|I| \lesssim (T_1/T_2)^{1/\alpha}$. Let f be defined on a closed subset $S \subseteq I$. Suppose for every distinct $x, y, z \in S$,*

$$|f(x) - f(y)| \leq T_1 |x - y|$$

and

$$\left| \frac{f(x) - f(y)}{x - y} - \frac{f(y) - f(z)}{y - z} \right| \le T_2 \cdot \operatorname{diam}\{x, y, z\}^\alpha.$$

Then f admits an extension $F : I \to \mathbb{R}$ with $\|F'\|_\infty \lesssim T_1$ and $\|F'\|_\alpha \lesssim T_2$.

Proof. Let $\lambda = (T_1/T_2)^{1/\alpha}$ and define $f_\lambda : \lambda^{-1}S \to \mathbb{R}$ by $f_\lambda(x) := f(\lambda x)$ (where for a set A and a scalar t we let $tA := \{ta \mid a \in A\}$). Our assumptions on f imply that, for $x < y < z \in \lambda^{-1}S$,

$$|f_\lambda(x) - f_\lambda(y)| \le T_1\lambda \cdot |x - y| \qquad \text{and} \tag{5}$$

$$\left| \frac{f_\lambda(x) - f_\lambda(y)}{x - y} - \frac{f_\lambda(y) - f_\lambda(z)}{y - z} \right| \le T_2\lambda^{1+\alpha} \cdot \operatorname{diam}\{x, y, z\}^\alpha = T_1\lambda \cdot \operatorname{diam}\{x, y, z\}^\alpha. \tag{6}$$

Dividing inequality (6) by $z - x$ we obtain

$$|f_\lambda[\{x, y, z\}]| \le T_1\lambda \cdot \operatorname{diam}\{x, y, z\}^{\alpha-1}.$$

By Theorem 2.1, f_λ admits an extension $F_\lambda : \lambda^{-1}I \to \mathbb{R}$ which is $C^{1,\alpha}$ with $\|F_\lambda'\|_\alpha \lesssim T_1\lambda$.
By the mean value theorem and (5), there is a point $w \in \lambda^{-1}I$ satisfying $|F_\lambda'(w)| \lesssim T_1\lambda$.
The function $F : I \to \mathbb{R}$ defined by $F(x) := F_\lambda(\lambda^{-1}x)$ extends f and satisfies $\|F'\|_\alpha \lesssim T_1\lambda^{-\alpha} = T_2$ and $|F'(\lambda w)| \lesssim T_1$. Since $|I| \lesssim \lambda$, it follows that $\|F'\|_\infty \lesssim T_1 + T_2 \cdot \lambda^\alpha \lesssim T_1$. □

2.3 $C^{2,\alpha}$ Riemmanian surfaces

A Riemannian manifold (M, g) is said to be C^2 if each point of M is contained in the domain of a coordinate chart in which the coefficients of the metric g are C^2 functions. This a rather minimal regularity assumption, which enables us to define the curvature tensor in the traditional way. It also guarantees the uniqueness of geodesics with prescribed initial conditions, as the coeffecients of the geodesic equations are locally Lipschitz.
For a Riemannian manifold (M, g), denote by $\operatorname{inj}(M)$ the injectivity radius of M, i.e, the largest number R such that for every $p \in M$, the exponential map $\exp_p : B_R(0) \to M$ is a diffeomorphism.

Definition 2.4. Let $H > 0$ and $0 < \alpha \le 1$. A $C^{2,\alpha}$ *Riemannian surface with constant H* is a complete two-dimensional C^2 Riemannian manifold satisfying

$$\|K\|_\infty \le H, \qquad \|K\|_\alpha \le H^{1+\alpha/2} \qquad \text{and} \qquad \operatorname{inj}(M) \ge \pi/\sqrt{H},$$

where K is the (Gaussian) curvature of (M, g), and its Hölder seminorm is taken with respect to the Riemannian metric.

We say that (M, g) is a $C^{2,\alpha}$ Riemannian surface, or simply a $C^{2,\alpha}$ surface, if there exists some $H > 0$ such that the requirements of Definition 2.4 are satisfied.

Example 2.5. The surface $\mathscr{S}_{K_0}$ for any $K_0 \in \mathbb{R}$ is a $C^{2,\alpha}$ Riemannian surface with constant $|K_0|$.

Example 2.6. The graph of a $C^{2,\alpha}$ function $F : \mathbb{R}^2 \to \mathbb{R}$, with the metric induced by $\mathbb{R}^3$, is a $C^{2,\alpha}$ surface.

We similarly define $C^{2,\alpha}$ Riemannian metrics on domains in $\mathbb{R}^2$:

Definition 2.7. A $C^{2,\alpha}$ *Riemannian metric (with constant H)* on a domain $\Lambda \subseteq \mathbb{R}^2$ is a Riemannian metric on Λ satisfying the conditions of Definition 2.4, excluding completeness and bounded injectivity radius.

Remark. If a surface $(M, \lambda^2 g)$ is obtained from a surface (M, g) by multiplication of the metric g by a scalar $\lambda^2 > 0$, and g is a $C^{2,\alpha}$ surface with constant H, then $(M, \lambda^2 g)$ is $C^{2,\alpha}$ with constant $\lambda^{-2} H$.

The following theorem appears in Kazdan and DeTurck [6]:

Theorem 2.8. *Let (M, g) be a $C^{2,\alpha}$ Riemannian surface. Then any point in M is contained in the domain of a coordinate chart in which the components of g are $C^{2,\alpha}$ functions.*

2.4 Polar normal coordinates

About any point p in a Riemannian surface (M, g) one can introduce normal coordinates, in which the coordinate map is the exponential map $\exp_p$, and lines through the origin correspond to geodesics emanating from p.

Using polar coordinates in $\mathbb{R}^2$, we obtain from the normal coordinates a new set of coordinates (r, θ), called *polar normal coordinates.* As a consequence of the Gauss lemma, the metric g can be written in polar normal coordinates as

$$g = dr^2 + G^2 d\theta^2 \tag{7}$$

for some function $G = G(r, \theta)$.

A *normal disc* on a Riemannian surface is a Riemannian disc on which normal coordinates may be defined (i.e. a disk $B_R(p)$ such that $\exp_p : B_R(0) \to B_R(p)$ is a diffeomorphism, where

$B_R(0) \subseteq T_pM$). When M is $C^{2,\alpha}$ with constant H, then by definition, any point is the center of a normal disc of radius $\pi/\sqrt{H}$. Whenever polar normal coordinates are used, the domain will be such a disc.

Polar normal coordinates do not posses the full regularity of the manifold. While Theorem 2.8 asserts the existence of coordinates in which the coefficients of g are $C^{2,\alpha}$, in polar normal coordinates the coefficient G is only C^{α} in general (an example appears in [13]). However, it is differentiable twice with respect to r and satisfies:

$$G(r,\theta) \xrightarrow{r\to 0} 0 \qquad \text{and} \qquad G(r,\theta)/r \xrightarrow{r\to 0} 1 \qquad \text{for all } \theta \in S^1, \tag{8}$$

as well as the Jabobi equation

$$\partial_r^2 G + KG = 0. \tag{9}$$

For smooth metrics, the Christoffel symbols for polar normal coordinates are

$$\begin{aligned} \Gamma^r_{\theta r} = \Gamma^r_{r\theta} = \Gamma^\theta_{rr} = \Gamma^r_{rr} = 0, \\ \Gamma^r_{\theta\theta} = -G\cdot\partial_r G, \quad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \partial_r G/G, \quad \Gamma^\theta_{\theta\theta} = \partial_\theta G/G. \end{aligned} \tag{10}$$

For $C^{2,\alpha}$ surfaces, however, the partial derivative $\partial_\theta G$ may not exist, thus $\Gamma^\theta_{\theta\theta}$ and subsequently the covariant derivative $\nabla_{\partial_\theta}\partial_\theta$ may not be defined. For this reason it is useful to introduce the oriented orthonormal frame $(\partial_r, \hat{\partial}_\theta)$, where $\hat{\partial}_\theta := \partial_\theta/G$ is the normalized version of the vector field ∂_θ. Fortunately, $\hat{\partial}_\theta$ can be covariantly differentiated with respect to both ∂_r and itself, even when the metric is merely C^2. To see this, let us first calculate the Christoffel symbols corresponding to the frame $(\partial_r, \hat{\partial}_\theta)$ for smooth metrics. As ∂_r and $\hat{\partial}_\theta$ are both parallel along radial geodesics, we have $\nabla_{\partial_r}\partial_r = \nabla_{\partial_r}\hat{\partial}_\theta = 0$, and using (10) and linearity we see that

$$\nabla_{\hat{\partial}_\theta}\partial_r = G^{-1}\nabla_{\partial_\theta}\partial_r = (\partial_r G/G^2)\partial_\theta = (\partial_r G/G)\hat{\partial}_\theta.$$

Differentiating the identities $\left\langle \hat{\partial}_\theta, \hat{\partial}_\theta \right\rangle = 1$ and $\left\langle \hat{\partial}_\theta, \partial_r \right\rangle = 0$ with respect to $\hat{\partial}_\theta$ then gives

$$\nabla_{\hat{\partial}_\theta}\hat{\partial}_\theta = -(\partial_r G/G)\partial_r.$$

Hence the Christoffel symbols for the frame $(\partial_r, \hat{\partial}_\theta)$ are

$$\begin{aligned} \Gamma^r_{rr} = \Gamma^{\hat\theta}_{rr} = 0, \qquad \Gamma^r_{r\hat\theta} = \Gamma^{\hat\theta}_{r\hat\theta} = 0, \\ \Gamma^r_{\hat\theta r} = 0, \quad \Gamma^{\hat\theta}_{\hat\theta r} = \partial_r G/G, \quad \Gamma^r_{\hat\theta\hat\theta} = -\partial_r G/G, \quad \Gamma^{\hat\theta}_{\hat\theta\hat\theta} = 0. \end{aligned} \tag{11}$$

Now let (M,g) be a $C^{2,\alpha}$ surface, let $\Lambda \subseteq M$ be a normal disc and consider the affine connection on Λ given by (11) with respect to $(\partial_r, \hat{\partial}_\theta)$. By the computations above, and the fact that

$$[\partial_r, \hat{\partial}_\theta] = -(\partial_r G/G)\hat{\partial}_\theta,$$

we see that this connection is symmetric and compatible with the metric g, hence coincides with the Levi-Civita connection on Λ. This proves that all the covariant derivatives involving ∂_r and $\hat{\partial}_\theta$ are well defined for any normal disc in a $C^{2,\alpha}$ surface.

The function $\partial_r G/G$, the logarithmic derivative of G with respect to r, which appears in the Christoffel symbols (11), is important for other reasons, and deserves its own letter:

$$h := \frac{\partial_r G}{G}. \tag{12}$$

With this notation we may summarize:

$$\nabla_{\partial_r}\partial_r = \nabla_{\partial_r}\hat{\partial}_\theta = 0 \qquad \nabla_{\hat{\partial}_\theta}\partial_r = h\hat{\partial}_\theta \qquad \nabla_{\hat{\partial}_\theta}\hat{\partial}_\theta = -h\partial_r. \tag{13}$$

Using the orthonormal frame $(\partial_r, \hat{\partial}_\theta)$ we can easily prove the following lemma.

Lemma 2.9. *Let (M, g) be a $C^{2,\alpha}$ Riemannian surface and let (r, θ) be polar normal coordinates centered at $p \in M$. If γ is a unit speed curve lying in a normal disc centered at p, and the function $\rho(t) := d_g(\gamma(t), p)$ is C^2 and satisfies $|\dot{\rho}| \neq 1$, then*

$$\nabla_{\dot\gamma}\dot\gamma = -\sqrt{1-\dot\rho^2}\left(\frac{\ddot\rho}{1-\dot\rho^2} - h\right)N, \tag{14}$$

where N is the inward-pointing unit normal *to γ, which is defined to be the unique unit vector field along γ such that $\langle\dot\gamma, N\rangle = 0$ and $\langle\partial_r, N\rangle < 0$.*

Remark. Since distance functions are 1-Lipschitz and γ is unit-speed, always $|\dot\rho| \leq 1$.

Proof. Since γ is unit-speed and $|\langle\partial_r, \dot\gamma\rangle| = |\dot\rho| \neq 1$, it follows that $\dot\theta = \pm\sqrt{1-\dot\rho^2}/G \neq 0$, hence $\theta \circ \gamma$ is monotone. Reversing the parametrization of γ does not alter either side of (14), hence we may assume without loss of generality that γ is oriented in the positive direction, i.e. that $\theta \circ \gamma$ is strictly increasing. The tangent to γ is then expressed in the orthonormal frame $(\partial_r, \hat{\partial}_\theta)$ by

$$\dot\gamma = \dot\rho\partial_r + \sqrt{1-\dot\rho^2}\hat{\partial}_\theta, \tag{15}$$

while the inward-pointing normal N is given by

$$N = -\sqrt{1-\dot\rho^2}\partial_r + \dot\rho\hat{\partial}_\theta. \tag{16}$$

Differentiating (15) covariantly along γ gives

$$\nabla_{\dot\gamma}\dot\gamma = \ddot\rho\partial_r + \dot\rho\nabla_{\dot\gamma}\partial_r + \frac{-\ddot\rho\dot\rho}{\sqrt{1-\dot\rho^2}}\hat{\partial}_\theta + \sqrt{1-\dot\rho^2}\nabla_{\dot\gamma}\hat{\partial}_\theta.$$

Using (13) and (15), linearity of the covariant derivative then yields

$$\nabla_{\dot\gamma}\dot\gamma = \big(\ddot\rho - h(1-\dot\rho^2)\big)\,\partial_r + \left(h\dot\rho\sqrt{1-\dot\rho^2} - \frac{\ddot\rho\dot\rho}{\sqrt{1-\dot\rho^2}}\right)\hat\partial_\theta. \tag{17}$$

Equation (14) then follows from (16) and (17). □

Remark. Consider the case where γ is circle centered at p, parametrized by unit speed. Then ρ is constant, and (14) reduces to $\nabla_{\dot\gamma}\dot\gamma = hN$. Thus the function h has the following geometric interpretation: it is the (signed) geodesic curvature of a unit-speed circle.
Another fact, which we shall not use, is that $h = \Delta r$, where Δ is the Laplace-Beltrami operator.

2.5 The Riccati equation and curvature comparison

Differentiating (12) with respect to r and using (9) gives the Riccati equation:

$$\partial_r h + h^2 + K = 0 \tag{18}$$

with the initial condition

$$h = 1/r + O(r) \text{ as } r \to 0. \tag{19}$$

In the case of constant K, the solution to (18),(19) is

$$\cot_K r := \begin{cases} \sqrt{-K}\coth(\sqrt{-K}r) & K<0,\ r>0 \\ 1/r & K=0,\ r>0 \\ \sqrt{K}\cot(\sqrt{K}r) & K>0,\ 0<r<\pi/\sqrt{K} \end{cases}$$

and the solution to the Jacobi equation (9) with the initial condition (8) is

$$\sin_K r := \begin{cases} \sinh(\sqrt{-K}r)/\sqrt{-K} & K<0 \\ r & K=0 \\ \sin(\sqrt{K}r)/\sqrt{K} & K>0 \end{cases}$$

It is also useful to introduce the functions

$$\Phi(x) := \begin{cases} \sqrt{-x}\coth(\sqrt{-x}) & x<0 \\ 1 & x=0 \\ \sqrt{x}\cot(\sqrt{x}) & 0<x<\pi^2 \end{cases} \tag{20}$$

and

$$\Psi(x) := \begin{cases} \sinh(\sqrt{-x})/\sqrt{-x} & x < 0 \\ 1 & x = 0 \\ \sin(\sqrt{x})/\sqrt{x} & 0 < x < \pi^2 \end{cases}$$

which are strictly decreasing and analytic. Note that $x \cot_K(x) = \Phi(Kx^2)$ and $\sin_K(x)/x = \Psi(Kx^2)$.

The Riccati equation satisfies the following well-known comparison principle ([15], Proposition 25):

Lemma 2.10. *Let $h : (0, R] \to \mathbb{R}$ be differentiable and satisfy*

$$h'(x) + h^2(x) + K(x) = 0$$
$$h(x) = 1/x + O(x) \text{ as } x \to 0$$

where $K : (0, R] \to \mathbb{R}$ safisfies $|K| \leq H$ for a constant $0 < H \leq \pi^2/R^2$. Then

$$\cot_H x \leq h(x) \leq \cot_{-H} x$$

for all $x \in (0, R]$.

Corollary 2.11. *Let G be a solution of* (8) *and* (9) *on some disc $B_R(0)$, and assume $|K| \leq H$ for some constant $H > 0$. Then*

$$\sin_H r \leq G(r, \theta) \leq \sin_{-H} r \qquad \text{for all } 0 < r \leq R \text{ and } \theta \in S^1.$$

In particular, if $R \leq \pi/2\sqrt{H}$ then

$$G(r, \theta)/r \sim 1 \qquad \text{for all } 0 < r \leq R \text{ and } \theta \in S^1.$$

Proof. Fix $\theta \in S^1$ and let

$$Y(r) = Y_\theta(r) := G(r, \theta)/r.$$

By (8), $Y \xrightarrow{r \to 0} 1$, and if we let $h := \partial_r G/G$ then

$$\partial_r \log Y = \partial_r G/G - 1/r = h - 1/r.$$

We know that h satisfies the Riccati equation (18), (19), hence by Lemma 2.10,

$$\partial_r \log \left(\frac{\sin_H r}{r} \right) = \cot_H r - 1/r \leq \partial_r \log Y \leq \cot_{-H} r - 1/r = \partial_r \log \left(\frac{\sin_{-H} r}{r} \right).$$

Integrating this inequality and using the fact that $Y, r^{-1}\sin_H r$ and $r^{-1}\sin_{-H} r$ all tend to 1 as $r \to 0$, we obtain

$$\Psi(\pi^2/4) \le \Psi(Hr^2) \le \frac{\sin_H r}{r} \le Y(r,\theta) \le \frac{\sin_{-H} r}{r} = \Psi(-Hr^2) \le \Psi(-\pi^2/4),$$

and as $\Psi(\pi^2/4) = 2/\pi > 0$, the corollary follows. $\square$

A subset U of a Riemannian manifold is said to be *strongly convex* if every $p, q \in U$ are joined by a unique minimizing geodesic in M, which lies entirely in U.

Theorem 2.12 (Whitehead). *Let (M,g) be a complete Riemannian manifold with sectional curvature $\le K_{max}$ and injectivity radius $\ge Z$. For every $R \le \frac{1}{2}\min\{\pi/\sqrt{K_{max}}, Z\}$ and every $p \in M$, the disc $B_R(p)$ is strongly convex.*

Corollary 2.13. *In $C^{2,\alpha}$ Riemannian surfaces with constant H, open discs of radius $\le \pi/(2\sqrt{H})$ are strongly convex.*

Combining this with Corollary 2.11, one obtains

Corollary 2.14. *Let (M,g) be a $C^{2,\alpha}$ Riemannian surface with constant H and let $p \in M$. For every $v, w \in T_pM$ with $\max\{|v|,|w|\} \le \pi/2\sqrt{H}$,*

$$|v-w|_g \sim d_g(\exp_p v, \exp_p w)$$

where $|\cdot|_g := \sqrt{\langle\cdot,\cdot\rangle_g}$ is the Riemannian norm on T_pM.

Having recalled some classical results about bounded curvature, we now prove a stability lemma for the Riccati equation, which will be useful in our setting, where the curvature is not merely bounded, but also satisfies a Hölder condition.

Lemma 2.15. *Let $h_i : (0,R] \to \mathbb{R}$ $(i = 1,2)$ be solutions to the Riccati equation:*

$$h_i' + h_i^2 + K_i = 0 \tag{21}$$

$$h_i(x) = 1/x + O(x) \quad \text{as } x \to 0 \qquad i = 1,2 \tag{22}$$

where $K_i : (0,R] \to \mathbb{R}$ are functions satisfying:

1. $\|K_i\|_\infty \le H$
2. $\|K_i\|_\alpha \le L$ *for some* $0 < \alpha \le 1$

3. $\|K_1 - K_2\|_\infty \leq T$

for constants $H, L, R, T > 0$*. Assume* $HR^2 \leq \pi^2/4$*. Then, for every* $r > 0$*, the function* $f := h_1 - h_2$ *has the following properties on the interval* $[r, R]$*:*

(*a*) $\|f\|_\infty \lesssim TR$

(*b*) $\|f'\|_\infty \lesssim T$

(*c*) $\|f\|_\alpha \lesssim TR^{1-\alpha}$

(*d*) $\|f'\|_\alpha \lesssim L + TR^{-\alpha}(1 + R/r)$

Proof. By (21), we have:

$$\begin{aligned} f'(x) = h_1'(x) - h_2'(x) &= h_2(x)^2 - h_1(x)^2 + K_2(x) - K_1(x) = \\ &= (h_2(x) - h_1(x))(h_2(x) + h_1(x)) + K_2(x) - K_1(x) = \\ &= -f(x)\left(h_2(x) + h_1(x) - \frac{2}{x}\right) - \frac{2f(x)}{x} + K_2(x) - K_1(x). \end{aligned} \tag{23}$$

Set $g(x) := x^2 f(x)$ for $0 < x \leq R$ and $g(0) := 0$; then g is C^1 on $[0, R]$ by (22), and (23) implies:

$$\begin{aligned} g'(x) &= 2xf(x) + x^2 f'(x) \\ &= -x^2 f(x)\left(h_2(x) + h_1(x) - \frac{2}{x}\right) + x^2(K_2(x) - K_1(x)) \end{aligned} \tag{24}$$

By the Riccati comparison principle (Lemma 2.10 above), for all $0 < x \leq R$,

$$\cot_H x \leq h_i(x) \leq \cot_{-H} x \qquad i = 1, 2 \tag{25}$$

whence

$$\Phi(Hx^2) = x \cot_H x \leq x h_i(x) \leq x \cot_{-H} x = \Phi(-Hx^2). \tag{26}$$

By assumption, $HR^2 \leq \pi^2/4$, and its not hard to show that $|\Phi(t) - 1| \leq 4|t|/\pi^2$ for $|t| \leq \pi^2/4$ (e.g. by convexity of the nonnegative functions $1/x - \cot x$ and $\coth x - 1/x$ on the intervals $[0, \pi/2]$ and $[-\pi/2, 0]$, respectively). Thus

$$|xh_i(x) - 1| \leq 4Hx^2/\pi^2 \qquad x \in [0, R].$$

This, together with (24) and assumption 3, imply that

$$g'(x) \leq 8Hx|g(x)|/\pi^2 + Tx^2. \tag{27}$$

Fix $r \in [0, R]$ and $x \in [r, R]$, and let

$$x_0 = \sup\{t \in [0, x] \mid g(t) = 0\}.$$

Since $g(0) = 0$, necessarily $g(x_0) = 0$. Exchanging h_1 and h_2 if necessary, we may assume that g is positive on (x_0, x). Hence (27) becomes

$$g'(t) \leq 8Htg(t)/\pi^2 + Tt^2 \qquad t \in [x_0, x]. \tag{28}$$

Setting $\sigma = 4/\pi^2$, this last equation can be written as:

$$\frac{d}{dt}\left(e^{-\sigma H t^2} g(t)\right) \leq T e^{-\sigma H t^2} t^2 \qquad t \in [x_0, x].$$

Integrating this we get

$$\begin{aligned} e^{-\sigma H x^2} g(x) &\leq T \int_{x_0}^{x} e^{-\sigma H t^2} t^2 dt \\ &\leq Tx \int_0^x e^{-\sigma H t^2} t dt \\ &= \frac{Tx}{2\sigma H}\left(1 - e^{-\sigma H x^2}\right) \leq \frac{T}{2} x^3 \end{aligned}$$

where the identity $1 - e^a < -a$ was applied in the last passage. It follows that:

$$g(x) \leq \frac{T}{2} e^{\sigma H x^2} x^3 \leq \frac{T}{2} e^{\sigma H R^2} x^3 \lesssim T x^3 \tag{29}$$

because $HR^2 \leq \pi^2/4$. Conclusion (a) follows, as

$$f(x) = g(x)/x^2 \lesssim Tx \leq TR. \tag{30}$$

Now, by (28) and (29),

$$\begin{aligned} |f'(x)| &= \left| \frac{g'(x)}{x^2} - \frac{2g(x)}{x^3} \right| \\ &\leq \frac{2\sigma H |g(x)|}{x} + T + \frac{2|g(x)|}{x^3} \\ &\lesssim THx^2 + T \leq T(HR^2 + 1) \lesssim T \end{aligned}$$

hence (b) holds; it also follows that

$$\begin{aligned} |f(x_1) - f(x_2)| &\lesssim T|x_1 - x_2| \\ &\lesssim TR^{1-\alpha}|x_1 - x_2|^\alpha \end{aligned}$$

which proves (c).

To obtain (d), we shall bound the C^α seminorm of each term in (23) separately. Since $|K_i| \leq H$, and $1/x$ is the solution to the Riccati equation with $K \equiv 0$, we can apply the estimates (a) and (c), which we have already proved, to the pairs $h_1, 1/x$ and $h_2, 1/x$, with T replaced by H, and get:

$$\begin{aligned} \left\| h_i - \frac{1}{x} \right\|_\infty &\lesssim HR \\ \left\| h_i - \frac{1}{x} \right\|_\alpha &\lesssim HR^{1-\alpha} \end{aligned}$$

This last estimate, together with (a),(c), and the inequality (3) gives

$$\left\| f(x)\left(h_1(x)-\frac{1}{x}+h_2(x)-\frac{1}{x}\right)\right\|_\alpha \lesssim TR\cdot HR^{1-\alpha}+TR^{1-\alpha}\cdot HR \lesssim (HR^2)\cdot TR^{-\alpha}\lesssim TR^{-\alpha}.$$

As for the second term in (23), we have by (a) and (b) and the fact that $\|1/x\|_\alpha \lesssim r^{-1-\alpha}$ on $[r,R]$:

$$\|f/x\|_\alpha \lesssim TR\cdot r^{-1-\alpha}+TR^{1-\alpha}\cdot r^{-1}\lesssim TR^{1-\alpha}r^{-1}.$$

Finally, the last two terms have, by assumption, C^α seminorms bounded by L. This finishes the proof of (d). $\square$

It follows from the proof of Lemma 2.15 and from Lemma 2.10 that:

Corollary 2.16. *Let $H>0$. For any continuous function $K:[0,\pi/(2\sqrt{H}))\to[-H,H]$ there is a unique solution to the Riccati equation* (18), (19) *and to the Jacobi equation* (9), (8) *on the interval* $(0,\pi/(2\sqrt{H}))$.

We now apply Lemma 2.15 to the study of $C^{2,\alpha}$ Riemannian surfaces.

Corollary 2.17. *Let $H,R>0$, $0<\alpha\le 1$, and suppose that $HR^2\le\pi^2/4$. Let g be a $C^{2,\alpha}$ metric with constant H on $B_R(0)\subseteq\mathbb{R}^2$ given by $g=dr^2+G^2d\theta^2$, and let $h=\partial_rG/G$. Suppose that there is some $K_0\in[-H,H]$ such that the curvature K satisfies*

$$|K(q)-K_0|\le T \textit{ for all } q\in B_R(0).$$

Then the function $f:=h-\cot_{K_0}r$ satisfies for all $R_0\in(0,R]$:

(A) $\|f\|_\infty\lesssim TR_0$ *on* $B_{R_0}(0)$

(B) $\|\partial_r f\|_\infty\lesssim T$ *on* $B_{R_0}(0)$

(C) $\|f\|_\alpha\lesssim LR_0$ *on* $B_{R_0}(0)$

(D) $\|\partial_r f\|_\alpha\lesssim L\lambda^{-1}$ *on* $B_{R_0}(0)\setminus B_{\lambda R_0}(0)$ *for all* $\lambda<1$

where the C^α norms are with respect to the Euclidean metric on $B_{R_0}(0)$, and $L:=H^{1+\alpha/2}$.

Proof. Let $R_0\in(0,R]$. By Corollary 2.14, the Riemannian metric and the Euclidean metric on $B_{R_0}(0)$ are comparable on the disc $B_R(0)$ up to a universal factor, and so $\|K\|_\alpha\lesssim L$ with respect to the Euclidean metric on this disc. Thus

$$|K(z)-K(z')|\lesssim\min\{T,Lr^\alpha\}\text{ for all } r\in(0,R]\text{ and } z,z'\in B_r(0).$$

Fix $\theta\in(-\pi,\pi]$. The function $h(re^{i\theta})$ is a solution to the Riccati equation (18),(19), whereas $\cot_{K_0}r$ is the solution to the same equation for $K\equiv K_0$. The hypotheses of Lemma 2.15

thus hold, with $h_1(x) = h(xe^{i\theta})$ and $h_2(x) = \cot_{K_0}(x)$ (recall that by Definition 2.4, $\|K\|_\alpha \leq H^{1+\alpha/2} = L$).

From conclusions (a) and (b) we obtain on $B_{R_0}(0)$

$$\|f\|_\infty \lesssim TR_0 \qquad \text{and} \qquad \|\partial_r f\|_\infty \lesssim T.$$

thus (A) and (B) hold. Moreover, for all $\theta \in (-\pi, \pi]$ and $0 \leq r_1 \leq r_2 \leq R_0$, it follows from conclusions (c) and (d) of Lemma 2.15 with $r = r_1$, $R = r_2$ and $T = r_2^\alpha$, that

$$\begin{aligned} |f(r_1 e^{i\theta}) - f(r_2 e^{i\theta})| &\lesssim L r_2 |r_1 - r_2|^\alpha \quad \text{and} \\ |\partial_r f(r_1 e^{i\theta}) - \partial_r f(r_2 e^{i\theta})| &\lesssim L(1 + r_2/r_1))|r_1 - r_2|^\alpha. \end{aligned} \tag{31}$$

Now fix $0 \leq r \leq R_0$, let $\theta_1, \theta_2 \in S^1$, and note that for $s \in [0, r]$,

$$|K(se^{i\theta_1}) - K(se^{i\theta_2})| \leq L d_g(se^{i\theta_1}, se^{i\theta_2})^\alpha \sim L|se^{i\theta_1} - se^{i\theta_2}|^\alpha \lesssim L r^\alpha |\theta_1 - \theta_2|^\alpha.$$

This time, we apply Lemma 2.15 to the pair $h_j(x) := h(xe^{i\theta_j})$. By conclusions (a) and (b) thereof:

$$\begin{aligned} |f(re^{i\theta_1}) - f(re^{i\theta_2})| &= |h(re^{i\theta_1}) - h(re^{i\theta_2})| \lesssim L r^{\alpha+1} |\theta_1 - \theta_2|^\alpha, \\ |\partial_r f(re^{i\theta_1}) - \partial_r f(re^{i\theta_2})| &= |\partial_r \left(h(re^{i\theta_1}) - h(re^{i\theta_2}) \right)| \lesssim L r^\alpha |\theta_1 - \theta_2|^\alpha. \end{aligned} \tag{32}$$

Putting together (31) and (32), we get that for $z_j = r_j e^{i\theta_j}$, $0 \leq r_1 \leq r_2 \leq R_0$:

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq |f(r_1 e^{i\theta_1}) - f(r_1 e^{i\theta_2})| + |f(r_1 e^{i\theta_2}) - f(r_2 e^{i\theta_2})| \\ &\lesssim L r_1^{\alpha+1} |\theta_1 - \theta_2|^\alpha + L r_2 |r_1 - r_2|^\alpha \\ &\lesssim L r_2 (r_1^\alpha |\theta_1 - \theta_2|^\alpha + |r_1 - r_2|^\alpha) \\ &\lesssim L R_0 |z_1 - z_2|^\alpha \end{aligned}$$

and if $\lambda R_0 \leq r_1 \leq r_2 \leq R_0$, then

$$\begin{aligned} |\partial_r f(z_1) - \partial_r f(z_2)| &\leq |\partial_r f(r_1 e^{i\theta_1}) - \partial_r f(r_1 e^{i\theta_2})| + |\partial_r f(r_1 e^{i\theta_2}) - \partial_r f(r_2 e^{i\theta_2})| \\ &\lesssim L r_1^\alpha |\theta_1 - \theta_2|^\alpha + L(1 + r_2/r_1)|r_1 - r_2|^\alpha \\ &\lesssim (L r_2/r_1)(r_1^\alpha |\theta_1 - \theta_2|^\alpha + |r_1 - r_2|^\alpha) \\ &\lesssim \lambda^{-1} L |z_1 - z_2|^\alpha. \end{aligned}$$

completing the proof of (C) and (D) and of the corollary. $\square$

2.6 Metric coefficients of $C^{2,\alpha}$ surfaces in polar normal coordinates

As was mentioned in §2.4, the full regularity of a $C^{2,\alpha}$ surface is not revealed in polar normal coordinates: the coefficient G in (7) can fail to be anything more than $C^{0,\alpha}$. In this section we prove the following theorem, which provides necessary and sufficient conditions for a metric of the form (7), defined on a sufficiently small disc, to extend to a $C^{2,\alpha}$ surface.

Theorem 2. *Let $0 < \alpha \leq 1$, let $R > 0$ and let $G : B_R(0) \setminus \{0\} \to (0, \infty)$. The two statements below are equivalent in the following sense: if (1) holds for some $\tilde{H} < \pi^2/4R^2$, then (2) holds with $H = C\tilde{H}$, and if (2) holds for some $H < \pi^2/64R^2$, then (1) holds with $\tilde{H} = CH$, where $C > 0$ is a universal constant.*

(1) *There exists a $C^{2,\alpha}$ Riemannian surface (M, g) with constant $\tilde{H}$ and a point $p \in M$ such that $B_R(p)$ is strongly convex, and g is given in polar normal coordinates on $B_R(p)$ by*

$$g = dr^2 + G^2 d\theta^2. \tag{33}$$

(2) *The following conditions hold:*

 (*a*) *The function G is C^2 in r.*

 (*b*) *$G \to 0$ and $G/r \to 1$ as $r \to 0$.*

 (*c*) *The function $K := -\partial_r^2 G/G$ satisfies $\|K\|_\infty \leq H$ and $\|K\|_\alpha \leq H^{1+\alpha/2}$.*

The implication (1) $\implies$ (2) follows immediately from the facts stated in §2.4, together with Corollary 2.14. We thus set out to prove the reverse implication.
Suppose that G satisfies $(a) - (c)$. By property (b), the metric g extends continuously to the origin, and is thus a well defined metric on $B_R(0)$. Moreover, condition (c) implies that K extends continuously to the origin; let

$$K_0 := K(0).$$

We shall construct a $C^{2,\alpha}$ surface by gluing the disc $(B_R(0), g)$ to the surface of constant curvature $\mathscr{S}_{K_0}$ with a disc removed.
To this end, let us first extend G to all of $\mathbb{R}^2$ in such a way that $(a) - (c)$ are still satisfied, and $K \equiv K_0$ outside the disc

$$\Lambda := B_{2R}(0).$$

This can be done, e.g., by defining $K(re^{i\theta}) := K((2R-r)e^{i\theta})$ for $r \in [R, 2R]$ and $K(re^{i\theta}) := K_0$ for $r \geq 2R$, and then letting G be the unique solution to the Jacobi equation (9),(8) with respect to the extended K.

If $K_0 \leq 0$, then G is positive on $\mathbb{R}^2 \setminus \{0\}$ by Corollary 2.11, and therefore formula (33) defines a Riemannian metric g on $\mathbb{R}^2$, which is locally isometric to $\mathscr{S}_{K_0}$ outside Λ. If $K_0 > 0$ then (33) defines a metric on $B_{\pi/\sqrt{K_0}}(0)$, which can be extended to a metric on the quotient $\overline{B_{\pi/\sqrt{K_0}}(0)}/\partial B_{\pi/\sqrt{K_0}}(0)$, locally isometric outside Λ to $\mathscr{S}_{K_0}$, a sphere of radius $1/\sqrt{K_0}$. In both cases let us denote the resulting Riemannian surface by (M, g).

We must first prove that (M, g) is a C^2 surface. We do so by introducing new coordinates on the disc Λ -namely, *isothermal coordinates*- in which the metric g is C^2.

Proposition 2.18. *There is a C^2 function $u : \Lambda \to \mathbb{R}$ such that (Λ, g) is isometric to $(\Lambda, \tilde{g})$, where $\tilde{g}$ is a metric on Λ given by*

$$\tilde{g} := e^u dz\overline{dz} = e^u(dx^2 + dy^2).$$

Moreover, $\|u\|_\infty \lesssim 1$.

Proof. Let $K_n : \mathbb{R}^2 \to \mathbb{R}$ be a sequence of smooth functions satisfying

- $\|K_n\|_\infty \leq H$
- $\|K_n - K\|_\infty \xrightarrow{n\to\infty} 0$ on Λ
- $K_n|_{B_{1/n}(0)} \equiv K(0)$.

(to get a sequence satisfying the first two conditions one can take $K_n = K * \varphi_n$, where φ_n is an appropriate sequence of mollifiers- see e.g. [8], pp.713-716. The third property can be achieved using a partition of unity).

Let

$$\Lambda' = \sqrt{2}\Lambda = B_{\sqrt{8}R}(0)$$

(the specific radii here are not very important). Let $G_n : \Lambda' \setminus \{0\} \to \mathbb{R}$ be the solution to the Jacobi equation (9), (8) with K replaced by K_n. Then G_n is smooth (by smoothness of solutions to ordinary differential equations, and smooth dependence of parameters- see e.g. [14]) and positive by Corollary 2.11. Thus

$$g_n := dr^2 + G_n^2 d\theta^2$$

is a well-defined smooth metric on Λ', whose curvature is K_n and is constant near the origin. For each n we now introduce an isothermal coordinate chart on (Λ', g_n), consisting of a diffeomorphism $\varphi_n : \Lambda' \to \Lambda'$ and a smooth function $u_n : \Lambda' \to \mathbb{R}$ such that $\varphi_n(0) = 0$ and

$$\tilde{g}_n := \varphi_n^* g_n = e^{u_n} dz\overline{dz}$$

(the asterisk denotes pullback). Such a coordinate transformation always exists (see e.g. [6]).

Lemma 2.19. *The functions u_n satisfy*

$$\sup_{\Lambda'} |u_n| \lesssim 1 \qquad \text{and} \tag{34}$$

$$\|u_n\|_{C^{2,\beta}(\Lambda)} \le C(H, R, \beta) \quad \text{for all } \beta < \alpha. \tag{35}$$

Proof. The estimate (34) is an immediate consequence of the following theorem by Eilat [7]:

Theorem 2.20 (Eilat). *Let $H, R > 0$ satisfy $HR^2 < \pi^2/8$. Let (M, g) be a smooth Riemannian surface with injectivity radius $\mathrm{inj}(M) > 2R$ and curvature $|K| \le H$, and let $p \in M$. Suppose that the metric g has the expression $e^u dz\overline{dz}$ in an isothermal coordinate chart $z : \overline{B_R(p)} \to \overline{B_R(0)}$ satisfying $z(p) = 0$ and $z(\partial B_R(p)) = \partial B_R(0)$. Then*

$$\|u\|_\infty \lesssim 1. \tag{36}$$

Indeed, the discs (Λ', g_n) can be extended to complete surfaces (M_n, g_n) by gluing them to $\mathscr{S}_{K_0}$, as we did with (Λ, g). See below for the proof of the injectivity radius bound for (M, g)- the same argument implies that the injectivity radii of such M_n's are bounded from below by $\pi/4\sqrt{H} > 2R$. Thus Theorem 2.20 gives us (34).

To obtain (35), we recall Liouville's equation:

$$\Delta u_n = -2 \cdot (K_n \circ \varphi_n) \cdot e^{u_n}. \tag{37}$$

and make use of the following elliptic regularity estimates (see [12], Theorems 3.9 and 4.8):

Proposition 2.21. *Let $0 < \beta < 1$ and $0 < R_1 < R_2$ and let $\Omega_j = B_{R_j}(0)$. Let $u \in C^2(\Omega_2)$. Then*

$$\sup_{\Omega_1} |\nabla u| \lesssim \sup_{\Omega_2} |u| + \sup_{\Omega_2} |\Delta u| \tag{38}$$

$$\|u\|_{C^{2,\beta}(\Omega_1)} \lesssim \sup_{\Omega_2} |u| + \|\Delta u\|_{C^{0,\beta}(\Omega_2)} \tag{39}$$

with the implied constants depending on R_1, R_2 and β.

The implied constants in the rest of this proof may depend on β, H and R. From (37), (34) and the fact that $|K_n| \le H$, it follows that $|\Delta u_n| \lesssim 1$. Let $\Lambda'' := B_{\sqrt{7}R}(0)$. The estimate (38) then implies that $\sup_{\Lambda''} |\nabla u_n| \lesssim 1$, whence $\|u_n\|_{C^{0,\beta}(\Lambda'')} \lesssim 1$. Now (37), (34) and the fact that $\|K_n\|_\alpha \lesssim H^{1+\alpha/2}$ imply that $\|\Delta u_n\|_{C^{0,\beta}(\Lambda'')} \lesssim 1$, so (35) follows from (39). Lemma 2.19 is proved. $\square$

By Lemma 2.19 and the Arzelà-Ascoli theorem, we may assume that $u_n \to u$ uniformly on Λ for some $u \in C^2$. Set

$$\tilde{g} := e^u dz\overline{dz}.$$

We finish the proof of Proposition 2.18 by proving that the sequence φ_n converges to an isometry

$$\varphi : (\Lambda, \tilde{g}) \to (\Lambda, g).$$

Claim. *The distance functions d_{g_n} and $d_{\tilde{g}_n}$, corresponding to the metrics g_n and $\tilde{g}_n$, converge uniformly on $\Lambda \times \Lambda$ to d_g and $d_{\tilde{g}}$, respectively.*

Proof. Let $\varepsilon > 0$ and let $n \in \mathbb{N}$ be so large that $\sup_\Lambda |K - K_n| < \varepsilon$. By Lemma 2.15, the functions $h := \partial_r G/G$ and $h_n := \partial_r G_n/G$ satisfy:

$$\sup_\Lambda |h - h_n| \lesssim \varepsilon r \qquad \text{on } B_r(0) \text{ for all } 0 < r < 2R. \tag{40}$$

Therefore, for all $\theta \in S^1$ and $0 < r < 2R$,

$$\frac{G(re^{i\theta})}{G_n(re^{i\theta})} = \exp\left(\int_0^r (h - h_n)(se^{i\theta})ds\right) = \exp(O(\varepsilon r^2)) = 1 + O(\varepsilon r^2). \tag{41}$$

By Corollary 2.11, if n is large enough then $G_n \sim r$ whence

$$G^2 - G_n^2 = G_n^2((G/G_n)^2 - 1) \lesssim \varepsilon r^4. \tag{42}$$

It follows that for every tangent vector $v = v^\theta \partial_\theta + v^r \partial_r \in T(\Lambda \setminus \{0\})$,

$$|v|_g^2 - |v|_{g_n}^2 = (v^r)^2 + G^2(v^\theta)^2 - (v^r)^2 - G_n^2(v^\theta)^2 \lesssim \varepsilon r^4 (v^\theta)^2 \le \varepsilon r^2 |v|^2, \tag{43}$$

where $|v|_g, |v|_{g_n}$ and $|v|$ denote the norm of v with respect to g, g_n and the Euclidean metric, respectively. Thus

$$|v|_g - |v|_{g_n} \lesssim \varepsilon r |v|.$$

For any $\gamma : [0, 1] \to \Lambda$, we then have that

$$\mathrm{Len}_g(\gamma) - \mathrm{Len}_{g_n}(\gamma) \le \max_{[0,1]}(|\dot{\gamma}|_g - |\dot{\gamma}|_{g_n}) \lesssim \varepsilon R \cdot \max_{[0,1]} |\dot{\gamma}|. \tag{44}$$

Now, since $G \sim r \sim G_n$, there is a universal constant $Z > 0$ such that for every $p, q \in \Lambda$,

$$d_g(p,q) = \inf\{\mathrm{Len}_g(\gamma) \mid \gamma : [0,1] \to \Lambda, \gamma(0) = p, \gamma(1) = q, |\dot\gamma| \le Z\} \text{ and}$$
$$d_{g_n}(p,q) = \inf\{\mathrm{Len}_{g_n}(\gamma) \mid \gamma : [0,1] \to \Lambda, \gamma(0) = p, \gamma(1) = q, |\dot\gamma| \le Z\}$$

(in words: to measure the distances in g and g_n between p and q, it suffices to consider curves whose tangents are no larger than Z in Euclidean norm).
By (44), these two infima differ by at most $CRZ\varepsilon$, where C is a universal constant. This proves uniform convergence of d_{g_n} to d_g. Convergence of $d_{\tilde g_n}$ to $d_{\tilde g}$ is similar: by uniform convergence of u_n to u, we have $|v|_{\tilde g_n} \to |v|_{\tilde g}$ uniformly in v, where v is a tangent vector with Euclidean norm bounded by some universal constant; the metrics $\tilde g_n$ and $\tilde g$ are all comparable to the Euclidean metric by a universal factor because of (34); and the rest of the argument is analogous. □

The metrics $d_g, d_{g_n}, d_{\tilde g}$ and $d_{\tilde g_n}$ are all comparable to the Euclidean metric by a universal factor, and therefore the functions φ_n, which are isometries from $(\Lambda, \tilde g_n)$ to (Λ, g_n), are uniformly bi-Lipschitz with respect to the Euclidean metric. By the Arzelà-Ascoli theorem, (a subsequence of) φ_n converges uniformly to a function $\varphi : \Lambda \to \Lambda$, and by the claim, for all $p, q \in \Lambda$,

$$d_g(p,q) = \lim_n d_{g_n}(p,q) = \lim_n d_{\tilde g_n}(\varphi_n(p), \varphi_n(q)) = d_{\tilde g}(\varphi(p), \varphi(q)). \tag{45}$$

This proves that $\varphi : (\Lambda, \tilde g) \to (\Lambda, g)$ is an isometry. Proposition 2.18 is proved. □

Proposition 2.18 shows that (M, g) is a C^2 Riemannian surface, because in isothermal coordinates on Λ the coefficients of g are C^2, and outside Λ the metric has constant curvature. By the Hopf-Rinow theorem (see [5], Theorem 10.1) M is complete, because $\exp_0$ is defined on all of T_0M. Since the rays $t \mapsto tv$ for $v \in \mathbb{R}^2$ are unit-speed geodesics and the metric is Euclidean at the origin, formula (33) expresses g in polar normal coordinates.
It follows from (9) that the curvature of M is K, and is constant coutside Λ. By (c), we have that $\|K\|_\infty \le H$, and $\|K\|_\alpha \le H^{1+\alpha/2}$ with respect to the Euclidean metric on Λ. By Corollary 2.11, $G \sim r$ on Λ, so the Euclidean metric on Λ is equivalent, up to a universal factor, to the metric d_g induced by g, whence $\|K\|_\alpha \lesssim H^{1+\alpha/2}$ with respect to the intrinsic metric[1] of Λ. We will soon show that Λ is strongly convex, from which it follows that its intrinsic metric coincides with the metric d_g inherited from (M, g), so $\|K\|_\alpha \lesssim \tilde H^{1+\alpha/2}$ with respect to the metric d_g as desired.

[1]The intrinsic metric of a subset U of a Riemannian manifold (M, g) is defined by $d_U(p,q) = \inf \mathrm{Len}_g(\gamma)$, where the infimum is over curves joining p and q and lying inside U.

We first prove that $\mathrm{inj}(M) \gtrsim \pi/\sqrt{\tilde{H}}$.

By Klingenberg's Lemma ([5], Corollary 5.7), $\mathrm{inj}(M) \geq \min\{\pi/\sqrt{\tilde{H}}, l/2\}$, where l is the length of the shortest closed geodesic in M. Thus it suffices to show that the length of any closed geodesic in M is $\gtrsim \pi/\sqrt{\tilde{H}}$. In fact, we shall prove that a closed geodesic in M can only exist if $K_0 > 0$, in which case its length must be $> \pi/2\sqrt{K_0} \geq \pi/2\sqrt{\tilde{H}}$.

A closed geodesic on M cannot lie entirely inside Λ. Indeed, Proposition 3.1 implies that if γ is a unit-speed geodesic lying in Λ, and ρ is defined by $\rho(t) := d_g(\gamma(t), 0)$, then $\rho \in C^2$ and $\rho'' > 0$. But if γ is closed, then ρ must attain a maximum, a contradiction.

We now argue that a closed geodesic intersecting $M \setminus \Lambda$ can exist only if $K_0 > 0$, and must be of length $\geq \pi/2\sqrt{K_0}$ in this case. By our construction, $M \setminus \Lambda$ is isometric to the complement of an open disc of radius $2R$ in $\mathscr{S}_{K_0}$. If $K_0 \leq 0$, then no closed geodesic lies entirely outside such a disc, nor can a geodesic return to it once it has left it. If $K_0 > 0$, then $\mathscr{S}_{K_0}$ is a sphere of radius $1/\sqrt{K_0}$, and a Riemannian disc of radius $2R$ on such a sphere is contained in a hemisphere. A closed geodesic lying entirely outside this disc has length $\pi/\sqrt{K_0}$, and a geodesic which leaves it, only returns after traversing a distance strictly larger than $\pi/2\sqrt{K_0}$. This proves that any closed geodesic on M has length $> \pi/2\sqrt{K_0}$ in this case. This finishes the proof that $\mathrm{inj}(M) \gtrsim \pi/\sqrt{\tilde{H}}$.

Finally, we prove that Λ is strongly convex, i.e. that any $z, z' \in \Lambda$ are joined by a unique minimizing geodesic in M, lying entirely within Λ. Such a geodesic exists by completeness, and lies in Λ because as was argued above, any geodesic joining $z, z' \in \Lambda$ which leaves Λ has length $> \pi/2\sqrt{K_0}$ (K_0 must be positive for such a geodesic to exist), while

$$\mathrm{diam}\Lambda = 4R \leq \pi/2\sqrt{\tilde{H}} \leq \pi/2\sqrt{K_0}.$$

No two points of Λ are conjugate, because $\tilde{H}R^2 \leq \pi^2$; therefore, by Klingenberg's Lemma, non-uniquness of minimizing geodesics between some pair of points in Λ would imply the existence of a closed geodesic lying in Λ, and we have already proved that this is not possible. This finishes the proof of the strong convexity of Λ.

We have proved that (M, g) is a $C^{2,\alpha}$ Riemmanian surface, and that Λ is a strongly convex disc on this surface where g is expressed in polar normal coordinates by (7). The proof of Theorem 2 is complete.

3 On distance functions in $C^{2,\alpha}$ Riemannian surfaces

Recall that $\mathcal{X}$ is the class of metric spaces (X, d_X) such that $X \setminus \{x_0\}$ is isometric to a compact interval $I \subseteq \mathbb{R}$ for some $x_0 \in X$. If $\varphi : I \to X \setminus \{x_0\}$ is such an isometry, then X is completely determined by the function $\rho_X : I \to (0, \infty)$ given by $\rho_X = d_X(\varphi(\cdot), x_0)$. By the triangle inequality,

$$\rho_X(t) - \rho_X(t') \leq |t - t'| \leq \rho_X(t) + \rho_X(t') \qquad \text{for all } t, t' \in I. \tag{46}$$

Conversely, a function $\rho : I \to (0, \infty)$ satisfying (46) determines a metric space $X_\rho \in \mathcal{X}$, whose underlying set is $I \sqcup \{x_0\}$, where x_0 is an additional point and $\sqcup$ indicates disjoint union, and whose metric d_{X_ρ} is given by:

$$d_{X_\rho}(t, t') = |t - t'| \qquad \text{for } t, t' \in I$$
$$d_{X_\rho}(t, x_0) = \rho(t) \qquad \text{for } t \in I.$$

We define X_ρ similarly for any function ρ on a set $S \subseteq \mathbb{R}$ and satisfying (46). Note however that by definition, $X_\rho \in \mathcal{X}$ only if S is a compact interval.

Let

$$\rho_{\mathcal{X}} := \{\rho : I \to (0, \infty) \mid \rho \text{ satisfies (46) }, I \subseteq \mathbb{R} \text{ is a compact interval}\}.$$

If we agree that each $X \in \mathcal{X}$ comes equipped with an interval $I_X \subseteq \mathbb{R}$ and an isometry $\varphi_X : I_X \to X \setminus \{x_0\}$, so that ρ_X is uniquely defined by the formula $\rho_X = d_X(\varphi_X(\cdot), x_0)$, then by the above,

$$\rho_{\mathcal{X}} = \{\rho_X \mid X \in \mathcal{X}\} \qquad \text{and} \qquad \mathcal{X} = \{X_\rho \mid \rho \in \rho_{\mathcal{X}}\}.$$

We now restate Theorem 1 with this new notation.

Theorem 1′. *There exist universal constants B, C such that the following holds. Let $H, R > 0$ satisfy $HR^2 \leq B$. Let $I \subseteq \mathbb{R}$ be a compact interval and let $\rho \in \rho_{\mathcal{X}}$, $\rho : I \to (0, R]$. If for every subset $S \subseteq I$ of cardinality at most 12, the metric space $X_{\rho|_S}$ embeds isometrically in a $C^{2,\alpha}$ Riemmanian surface with constant H, then X_ρ embeds isometrically in a $C^{2,\alpha}$ Riemannian surface with constant CH.*

Remark. The condition $\mathrm{diam} X \leq R$ implies $\rho_X \leq R$, while the condition $\rho \leq R$ implies $\mathrm{diam} X_\rho \leq 2R$ by (46). So the hypotheses of the two formulations are equivalent up to a constant factor.

In this section we study functions $\rho \in \rho_{\mathcal{X}}$ satisfying the hypothesis of Theorem 1:

($\star$) For every subset $S \subseteq I$ of consisting of at most 12 points, the metric space $X_{\rho|_S}$ embeds isometrically in a $C^{2,\alpha}$ Riemannian surface with constant H.

Our main goal is to prove Proposition 3.11, in which the finiteness condition ($\star$) is shown to imply certain estimates involving ρ and some functions associated with it.

Remark. For $\rho \in \rho_{\mathcal{X}}$ and $\lambda > 0$, if we define ρ^λ by

$$\rho^\lambda(t) = \lambda\rho(\lambda^{-1}t),$$

then $\rho^\lambda \in \rho_{\mathcal{X}}$, and X_ρ embeds isometrically in (M, g) if and only if X_{ρ^λ} embeds isometrically in $(M, \lambda^2 g)$. As noted in §2.3, if (M, g) is $C^{2,\alpha}$ with constant H, then $(M, \lambda^2 g)$ is $C^{2,\alpha}$ with constant $\lambda^{-2}H$. So it suffices to prove Theorem 1 with $H = 1$. Nevertheless we choose to keep track of the parameter H, and one can set $H = 1$ (and later also $L = 1$) at any point to simplify reading.

When working in polar normal coordinates, the functions G, h are always defined as in the previous section, i.e. $g = dr^2 + G^2 d\theta^2$ is the metric, and $h = \partial_r G/G$.

Proposition 3.1. *Let $H, R, m > 0$ and assume that $HR^2 < \pi^2/4$. Let $g = dr^2 + G^2 d\theta^2$ be a $C^{2,\alpha}$ Riemannian metric with constant H on $B_R(0)$, let $\gamma : I \to B_R(0) \setminus B_m(0)$ be a unit-speed curve and set $\rho(t) := d_g(\gamma(t), 0)$. Then γ is a minimizing geodesic if and only if $\rho \in C^{2,\alpha}$ and*

$$\rho''(t) = h(\gamma(t))(1 - \rho'(t)^2) \quad \text{for all } t \in I. \tag{47}$$

Furthermore, if γ is a unit speed geodesic then

$$\|\ddot{\rho}\|_\alpha \lesssim m^{-1-\alpha}(1 + R/m) \qquad \text{and} \qquad \|\ddot{\rho}\|_\infty \lesssim m^{-1}. \tag{48}$$

Proof. Suppose $\rho \in C^{2,\alpha}$ and (47) holds. By Lemma 2.9, the curve γ is a geodesic provided that $\dot{\rho}$ is never 1 or -1. If $\dot{\rho} \equiv \pm 1$ then γ is a geodesic by the definition of normal coordinates. Those are the only possibilities: Suppose $|\rho'(t)| < 1$ for some $t \in I$ and let J be the maximal relatively open subinterval containing t such that $|\dot{\rho}| < 1$ on J. Then by the above $\gamma|_J$ is a geodesic. If $J \neq I$ then $\rho'(a) = \pm 1$ for some $a \in \partial J$. By Gauss's Lemma, $\langle \dot{\gamma}(a), \partial_r \rangle = \pm 1$ and $\langle \dot{\gamma}(a), \partial_\theta \rangle = 0$ (the curve γ has no corners because $\rho \in C^2$). It follows by uniqueness of geodesics that $\gamma|_J$ is a radial geodesic so $\dot{\rho} \equiv \pm 1$ on J, a contradiction. Thus any curve satisfying (47) is a geodesic, and is minimizing by Corollary 2.13.

It remains to prove the other implication and the estimates (48). To this end, we first suppose that the metric g is smooth.

Let $\gamma : I \to B_R(p)\setminus B_m(p)$ be a unit-speed minimizing geodesic. We assume that γ is non-radial (for radial geodesics (47) and (48) hold trivially). Then ρ is smooth because g is, and it satisfies (47) because of Lemma 2.9.

By Corollary 2.17 with $K_0 = 0$, we have that $h = 1/r + f$ where f satisfies on $B_R(0)$:

$$\begin{aligned}\|f\|_\alpha &\lesssim H^{1+\alpha/2}R = (HR^2)^{1+\alpha/2}R^{-1-\alpha} \lesssim R^{-1-\alpha}\\ \|f\|_\infty &\lesssim HR = (HR^2)R^{-1} \lesssim R^{-1}.\end{aligned}$$

Since $\|1/r\|_\alpha \lesssim m^{-1-\alpha}$ on $B_R(0) \setminus B_m(0)$ we get that

$$\|h\circ\gamma\|_\alpha \lesssim m^{-1-\alpha} \text{ and } \|h\circ\gamma\|_\infty \lesssim m^{-1}.$$

From the triangle inequality and the fact that γ is a minimizing geodesic, it follows that $|I| \le 2R$. Thus by (47) and the above estimates,

$$\begin{aligned}\|\ddot\rho\|_\infty &\lesssim \|h\circ\gamma\|_\infty \left\|1-\dot\rho^2\right\|_\infty \lesssim m^{-1} \qquad \text{and}\\ \|\ddot\rho\|_\alpha &\le \|h\circ\gamma\|_\alpha \left\|1-\dot\rho^2\right\|_\infty + \|h\circ\gamma\|_\infty \left\|1-\dot\rho^2\right\|_\alpha \lesssim m^{-1-\alpha} + m^{-1}\|\ddot\rho\|_\infty |I|^{1-\alpha}\\ &\lesssim m^{-1-\alpha} + m^{-2}R^{1-\alpha} \lesssim m^{-1-\alpha}(1+R/m),\end{aligned}$$

and (48) is proved in the smooth case.

Now we relax the assumption of smoothness. As in the proof of Proposition 2.18, there is a sequence of smooth metrics $g_n = dr^2 + G_n^2 d\theta^2$, such that the functions $h_n := \partial_r G_n/G_n$ converge uniformly to h on $B_R(0)$, and the distance functions d_{g_n} converge uniformly on $B_R(0)\times B_R(0)$ to d_g.

Let $\gamma : I \to B_R(0)\setminus B_m(0)$ be a unit-speed geodesic with respect to g, and let $\gamma_n : I \to \mathbb{R}^2$ be a geodesic in the metric g_n with the same initial conditions as γ (it might not be unit speed). Set

$$\rho_n(t) = d_{g_n}(\gamma_n(t), 0).$$

For sufficiently large n, the curve γ_n lies in $B_R(0)\setminus B_{m/2}(0)$, and g_n is a $C^{2,\alpha}$ metric with constant $< \pi^2/4R^2$. Since the proposition holds for smooth metrics, $\|\ddot\rho_n\|_\infty$ and $\|\ddot\rho_n\|_\alpha$ are bounded by constants independent of n ($|\dot\rho_n|$ is bounded by 1). By passing to a subsequence and applying the Arzelà-Ascoli theorem, we can assume that $\gamma_n, \rho_n, \dot\rho_n, \ddot\rho_n$ all converge uniformly.

Since $d_{g_n} \to d_g$ uniformly, the limit of the sequence γ_n is a unit speed geodesic, which by uniqueness of geodesics must be γ, because they share the same initial conditions. If follows that $\rho_n \to \rho$, and since $\dot\rho_n$ and $\ddot\rho_n$ converge uniformly, ρ is differentiable twice and $\ddot\rho_n \to \ddot\rho$. A pointwise limit of α-Hölder functions with a uniformly bounded seminorm is again α-Hölder with the same bound; hence $\rho \in C^{2,\alpha}$ and satisfies (48). Finally, equation (47) holds because $h_n \circ \gamma_n \to h \circ \gamma$. $\square$

Corollary 3.2. *Let $m, H, R > 0$ satisfy $HR^2 < \pi^2/4$. Let $\rho \in \rho_{\mathcal{X}}$, $\rho : I \to [m, R]$, and assume that for all $S \subseteq I$ with $|S| \leq 3$, $X_{\rho|S}$ embeds isometrically in a $C^{2,\alpha}$ Riemmanian surface with constant H. Then*

$$\rho \in C^{2,\alpha}, \quad |\dot\rho| \leq 1, \quad \|\ddot\rho\|_\alpha \lesssim m^{-1-\alpha}(1 + R/m) \quad \textit{and} \quad \|\ddot\rho\|_\infty \lesssim m^{-1}. \tag{49}$$

Proof. By our assumptions and the previous proposition, for every subset $S \subseteq I$ of cardinality at most 3, we can find a function ρ_S which satisfies (49) and agrees with ρ on S, i.e. $\rho_S(s) = \rho(s)$ for all $s \in S$. By Corollary 2.2 it follows that $\rho \in C^{2,\alpha}$ and $\|\ddot\rho\|_\alpha \lesssim m^{-1-\alpha}(1 + R/m)$.
By (46), ρ must be 1-Lipschitz, so $|\dot\rho| \leq 1$.
Given $t \in I$ and $\varepsilon > 0$, let $t', t'' \in (t - \varepsilon, t + \varepsilon)$, let $S := \{t, t', t''\}$, and let ρ_S be as above. Recall the definition of the divided differences $\rho[\cdot]$ from §2.2. By the mean value theorem and fact that $\|\ddot\rho_S\|_\alpha \lesssim m^{-1}$,

$$\rho''(t) = 2\rho[S] + O(\varepsilon^\alpha) = 2\rho_S[S] + O(\varepsilon^\alpha) = \rho_S''(t) + O(\varepsilon^\alpha)$$

with the implied constants depending only on m, R. Since $|\rho_S| \lesssim m^{-1}$ by (48), and ε is arbitrary, it follows that $|\ddot\rho| \lesssim m^{-1}$. $\square$

Proposition 3.3. *Let $H, R > 0$ satisfy $HR^2 < \pi^2/16$. Let (M, g) be a $C^{2,\alpha}$ Riemmanian surface with constant H, let $p \in M$, and let $\gamma : I \to B_R(p)$ be a non-radial unit-speed minimizing geodesic. Set $\rho(t) := d_g(\gamma(t), p)$. Then*

$$\frac{\rho(t)\rho''(t)}{1 - \rho'(t)^2} \sim 1 \qquad \textit{for all } t \in I. \tag{50}$$

Proof. By Proposition 3.1 (applied to polar normal coordinates centered at p),

$$\frac{\rho''(t)}{1 - \rho'(t)^2} = h(\gamma(t)) \qquad t \in I.$$

Since h satisfies the Riccati equation (18),(19) on radial geodesics, Lemma 2.10 implies that

$$\Phi(H\rho(t)^2) = \rho(t)\cot_H(\rho(t)) \leq \rho(t)h(\gamma(t)) \leq \rho(t)\cot_{-H}(\rho(t)) = \Phi(-H\rho(t)^2)$$

so by monotonicity of Φ,

$$1/2 \leq \Phi(\pi^2/16) \leq \rho(t)\rho''(t)/(1 - \rho'(t)^2) \leq \Phi(-\pi^2/16) \leq 2.$$

and (50) follows. $\square$

Corollary 3.4. *Let $H, R > 0$ satisfy $HR^2 < \pi^2/16$. Let $\rho \in \rho_{\mathcal{X}}$, $\rho : I \to (0, R]$ and assume that for all $S \subseteq I$ with $|S| \le 4$, $X_{\rho|_S}$ embeds isometrically in a $C^{2,\alpha}$ Riemannian surface with constant H. Then ρ satisfies* (50) *for every t such that $\rho'(t) \ne \pm 1$. In particular, $\rho''(t) > 0$ for any such t.*

Proof. Let $t \in I$ and let $\varepsilon > 0$. let t', t'' be points in an ε-neighbourhood of t (all three points should be distinct), and let $T := \{t', t''\}$ and $S := \{t, t', t''\}$.
By Corollary 3.2 and the mean-value theorem,

$$\rho[T] = \rho'(t) + O(\varepsilon^\alpha)$$

and

$$\rho[S] = \rho''(t)/2 + O(\varepsilon^\alpha).$$

where the implied constants depend only on R and $m := \min \rho$. Thus

$$\frac{\rho(t)\rho''(t)}{1 - \rho'(t)^2} = \frac{2\rho(t)\rho[S]}{1 - \rho[T]^2} + O(\varepsilon^\alpha) \quad \text{as } \varepsilon \to 0.$$

By assumption, $X_{\rho|_S}$ embeds isometrically in a $C^{2,\alpha}$ Riemannian surface with constant H. Hence there is a unit speed geodesic $\gamma : \mathbb{R} \to M$ which satisfies

$$\tilde{\rho}(s) = \rho(s) \qquad \text{for all } s \in S$$

where $\tilde{\rho}(s) := d_g(\gamma(s), p)$ for some $p \in M$.
Thus, using Proposition 3.1 and the same argument as above,

$$\frac{\tilde{\rho}(t)\tilde{\rho}''(t)}{1 - \tilde{\rho}'(t)^2} = \frac{2\tilde{\rho}(t)\tilde{\rho}[S]}{1 - \tilde{\rho}[T]^2} + O(\varepsilon^\alpha) = \frac{2\rho(t)\rho[S]}{1 - \rho[T]^2} + O(\varepsilon^\alpha) \qquad \text{as } \varepsilon \to 0,$$

whence

$$\frac{\rho(t)\rho''(t)}{1 - \rho'(t)^2} = \frac{\tilde{\rho}(t)\tilde{\rho}''(t)}{1 - \tilde{\rho}(t)^2} + O(\varepsilon^\alpha) \qquad \text{as } \varepsilon \to 0.$$

Since ε can be chosen arbitrarily small, and $\tilde{\rho}$ satisfies (50) for all ε, we are done. ☐

Corollary 3.5. *Let $H, R > 0$ satisfy $HR^2 < \pi^2/16$. Let $\rho \in \rho_{\mathcal{X}}$, $\rho : I \to (0, R]$ and assume that for all $S \subseteq I$ with $|S| \le 4$, $X_{\rho|_S}$ embeds isometrically in a $C^{2,\alpha}$ Riemannian surface with constant H.*
Then for any $t \in I$ such that $\rho'(t) \ne \pm 1$,

$$-\frac{d}{dt}\log(1 - \rho'(t)^2) \sim \frac{d}{dt}\log \rho(t). \tag{51}$$

Proof. Using Corollary 3.4,

$$-\frac{d}{dt}\log(1-\rho'(t)^2) = \frac{2\rho'(t)\rho''(t)}{1-\rho'(t)^2} \sim \frac{\rho'(t)}{\rho(t)} = \frac{d}{dt}\log\rho(t).$$

$\square$

Corollary 3.6. *Let $H, R > 0$ satisfy $HR^2 < \pi^2/16$. Let $\rho \in \rho_{\mathcal{X}}$, $\rho : I \to (0, R]$ and assume that for all $S \subseteq I$ with $|S| \le 4$, $X_{\rho|S}$ embeds isometrically in a $C^{2,\alpha}$ Riemannian surface with constant H. Then exactly one of the following holds:* (1) $\ddot\rho \equiv \pm 1$, *or* (2) $|\dot\rho| < 1$ *and* $\ddot\rho > 0$.

Proof. Suppose $\dot\rho \not\equiv \pm 1$. Hence $1 - \rho'(t_0)^2 = a > 0$ for some $t_0 \in I$. Let $t \in I$. Then by Proposition 3.5, if $\dot\rho \ne \pm 1$ on the interval $[t_0, t]$ then for some universal constant C,

$$\begin{aligned}
1-\rho'(t)^2 &= a\cdot \exp\int_{t_0}^{t}\frac{d}{ds}\log(1-\rho'(s)^2)ds\\
&\ge a\cdot\exp\int_{t_0}^{t} -C\frac{d}{ds}\log\rho(s)ds\\
&= a\rho(t_0)^C/\rho(t)^C\\
&\ge am^C/R^C > 0
\end{aligned}$$

so $|\rho'(t)| < 1$. By Corollary 3.4, $\ddot\rho > 0$ for all $t \in I$. $\square$

We will henceforth assume that case (2) in Corollary 3.6 holds; case (1) is trivial.

Let $\mu > 0$. We say that a function $\varphi : I \to (0, \infty)$ *varies by at most a factor of μ on the interval I* if it satisfies $\mu^{-1} \le \varphi(t)/\varphi(t') \le \mu$ for all $t, t' \in I$.

Corollary 3.7. *Let $H, R > 0$ satisfy $HR^2 < \pi^2/16$. Let (M, g) be a $C^{2,\alpha}$ Riemannian surface with constant H, let $p \in M$ and let $\gamma : I \to B_R(p)$ be a non-radial unit-speed geodesic. Suppose that γ is given in polar normal coordinates at p by $\gamma = \rho e^{i\phi}$.*

Then, if ρ varies by at most a factor of μ on some subinterval $J \subseteq I$, then $\dot\phi$ varies by at most a factor of $C\mu^{C'}$ on J, where $C, C' > 0$ are universal constants.

Proof. Since γ is unit speed, $\dot\phi = \sqrt{1-\dot\rho^2}/G$ (we may assume that $\dot\phi > 0$). Hence by Corollary 2.11,

$$\dot\phi \sim \sqrt{1-\dot\rho^2}/\rho =: \omega.$$

It thus suffices to show that ω varies by at most a factor of $\mu^{C'}$ on J, for a universal constant C'. It follows from Corollary 3.5 that

$$\left|\frac{d}{dt}\log\omega(t)\right| = \left|\frac{1}{2}\frac{d}{dt}\log(1-\rho'(t)^2) - \frac{d}{dt}\log\rho(t)\right| \lesssim \left|\frac{d}{dt}\log\rho(t)\right| \qquad t\in I. \tag{52}$$

Thus for any $t, t' \in I$, and a universal constant C',

$$\left|\log \frac{\omega(t)}{\omega(t')}\right| = \left|\int_t^{t'} \frac{d}{ds} \log \omega(s) ds\right| \leq C' \cdot \left|\int_t^{t'} \frac{d}{ds} \log \rho(s) ds\right| = C' \cdot \left|\log \frac{\rho(t)}{\rho(t')}\right| \leq C' \log \mu. \quad (53)$$

and we are done. ☐

In the next proposition, we use the distance function of a geodesic from a point to determine the curvature somewhere in between. This is useful because when we are given some function $\rho \in \rho_{\mathcal{X}}$ and we wish to embed X_ρ in a nice Riemannian surface, it is good to have an idea of what the curvature should roughly be in the region where X_ρ embeds.
For two points p, q in a strongly convex subset of a Riemannian manifold we denote by $[p, q]$ the minimal geodesic joining p and q, parametrized by unit speed from p to q.
Recall the function Φ from §2.5, which is strictly decreasing and onto $\mathbb{R}$.

Proposition 3.8. *Let $H, R > 0$ satisfy $HR^2 < \pi^2/16$. Let (M, g) be a $C^{2,\alpha}$ Riemannian surface with constant H, let $p \in M$ let $\gamma : I \to B_R(p)$ be a non-radial unit speed geodesic and let $\rho(t) = d_g(\gamma(t), p)$. Set*

$$\kappa(t) := \frac{1}{\rho(t)^2}\Phi^{-1}\left(\frac{\rho(t)\rho''(t)}{1 - \rho'(t)^2}\right) \quad (54)$$

Then, for each $t \in I$ there is a point q_t on the geodesic segment $[p, \gamma(t)]$ such that $K(q_t) = \kappa(t)$.

Remark. We know that a metric space consisting of four points, three of which are collinear, embeds uniquely in a surface of constant curvature (see introduction). We can think of $\kappa(t)$ as the embedding curvature of a quadruple of points, three of which lie on the geodesic γ infinitesimally close to $\gamma(t)$, and the fourth of which is p.

Proof. We will show that K must be $\leq \kappa(t)$ somewhere on the segment $[p, \gamma(t)]$, and $\geq \kappa(t)$ somewhere else. The proposition then follows by the intermediate value theorem.
By (47), $\rho''(t)/(1 - \rho'(t)^2) = h(\gamma(t))$ for all t. Thus (54) is equivalent to

$$\Phi(\kappa(t)\rho(t)^2) = \rho(t)h(\gamma(t))$$

which by the definitions of Φ and $\cot_K$ is equivalent to

$$\cot_{\kappa(t)} \rho(t) = h(\gamma(t))$$

Since h satisfies the Riccati equation (18),(19) on the geodesic segment $[p, \gamma(t)]$, it follows from Lemma 2.10 that K cannot be strictly smaller nor strictly larger than $\kappa(t)$ on the entire segment. ☐

Corollary 3.9. *Let $H, R > 0$ satisfy $HR^2 < \pi^2/16$. Let $\rho \in \rho_{\mathcal{X}}$, $\rho : I \to (0, R]$, and suppose that for every $S \subseteq I$ with $|S| \leq 6$, the metric space X_ρ embeds isometrically in a $C^{2,\alpha}$ Riemannian surface with constant H. Then for every $t_1, t_2 \in I$,*

$$|\kappa(t_1)| \leq H \qquad \text{and} \qquad |\kappa(t_1) - \kappa(t_2)| \lesssim H^{1+\alpha/2} \cdot \max\{\rho(t_1), \rho(t_2)\}^\alpha.$$

Proof. Let $t_1, t_2 \in I$. Let $\varepsilon > 0$, and choose $t_i', t_i'' \in (t_i - \varepsilon, t_i + \varepsilon)$ for $i = 1, 2$, such that all six points are distinct. Set $T_i := \{t_i', t_i''\}$ and $S_i := \{t_i, t_i', t_i''\}$, and let $S = S_1 \cup S_2$. By assumption, the metric space $X_{\rho|_S}$ embeds isometrically in a $C^{2,\alpha}$ Riemannian surface (M, g) with constant H. As in the proof of Corollary 3.4, we let $\gamma : \mathbb{R} \to M$ and $p \in M$ satisfy $d_g(\gamma(s), p) = \rho(s)$ for all $s \in S$, set $\tilde{\rho} = d_g(\gamma(\cdot), p)$, and deduce from Proposition 3.1 and Corollary 3.2 that $\rho'(t_i) = \tilde{\rho}'(t_i) + O(\varepsilon^\alpha)$ and $\rho''(t_i) = \tilde{\rho}''(t_i) + O(\varepsilon^\alpha)$ for $i = 1, 2$.

Let $\tilde{\kappa}$ be defined as in (54), with ρ replaced by $\tilde{\rho}$. The function Φ^{-1} is locally Lipschitz, and its argument in the definitions of κ and $\tilde{\kappa}$ is bounded by a universal constant by virtue of Proposition 3.3 and Corollary 3.4. By the previous paragraph, we thus have that

$$\kappa(t_i) = \tilde{\kappa}(t_i) + O(\varepsilon^\alpha), \ i = 1, 2.$$

Assume without loss of generality that $\rho(t_2) \geq \rho(t_1)$. By Proposition 3.8, there are $q_1, q_2 \in B_{\rho(t_2)}(p)$ with $K(q_i) = \tilde{\kappa}(t_i)$. $i = 1, 2$ (with K denoting the curvature of M as usual). Since $\|K\|_\infty \leq H$ and $\|K\|_\alpha \leq H^{1+\alpha/2}$, it follows that

$$|\tilde{\kappa}(t_1)| \leq H \qquad \text{and} \qquad |\tilde{\kappa}(t_1) - \tilde{\kappa}(t_2)| \leq 2H^{1+\alpha/2} \cdot \rho(t_2)^\alpha.$$

Since ε can be arbitrarily small, the last two formulas imply the corollary. $\square$

Using Corollary 3.9, we can present an example demonstrating the fact, that the optimal constant H of a $C^{2,\alpha}$ surface in which X_ρ can be embedded, is very sensitive to small changes in ρ.

Define a two-parameter family of functions $\rho_{\varepsilon,\alpha} : [-1, 1] \to (0, \infty)$ for $0 < \alpha \leq 1$ and small $\varepsilon > 0$ by

$$\rho_{\varepsilon,\alpha}(t) := \sqrt{t^2 + \varepsilon^2} + \varepsilon^{3+\alpha}. \tag{55}$$

The function $\tilde{\rho}_\varepsilon(t) := \sqrt{t^2 + \varepsilon^2}$ measures the Euclidean distance to the origin in $\mathbb{R}^2$ along the line $t \mapsto (t, \varepsilon)$; in our terminology, $X_{\tilde{\rho}_\varepsilon}$ admits an isometric embedding into the Euclidean plane. However, an embedding of $X_{\rho_{\varepsilon,\alpha}}$ into a $C^{2,\alpha}$ Riemannian surface with constant H is possible only if $H > c$ for some positive constant c independent of ε. To see this, define $\kappa_{\varepsilon,\alpha}$ as in (54), with ρ replaced by $\rho_{\varepsilon,\alpha}$. Then

$$\kappa_{\varepsilon,\alpha}(t) = \frac{1}{(\sqrt{t^2+\varepsilon^2}+\varepsilon^{3+\alpha})^2}\Phi^{-1}\left(1+\frac{\varepsilon^{3+\alpha}}{\sqrt{t^2+\varepsilon^2}}\right).$$

Since $\Phi(0) = 1$ and $\Phi'(0) = -1/3$, it follows that

$$\kappa_{\varepsilon,\alpha}(0) = \frac{\Phi^{-1}(1+\varepsilon^{2+\alpha})}{(\varepsilon+\varepsilon^{3+\alpha})^2} = \varepsilon^{\alpha}\cdot(-3+O(\varepsilon))$$

and

$$\kappa_{\varepsilon,\alpha}(\varepsilon) = \frac{\Phi^{-1}(1+2^{-1/2}\varepsilon^{2+\alpha})}{(2^{1/2}\varepsilon+\varepsilon^{3+\alpha})^2} = 2^{-3/2}\cdot\varepsilon^{\alpha}\cdot(-3+O(\varepsilon)),$$

whence

$$\frac{\kappa_{\varepsilon,\alpha}(\varepsilon)-\kappa_{\varepsilon,\alpha}(0)}{\varepsilon^{\alpha}} \sim 1. \tag{56}$$

Suppose that $X_{\rho_{\varepsilon,\alpha}}$ embeds isometrically in a $C^{2,\alpha}$ surface (M,g) with constant H. By Corollary 3.9, the values $\kappa_{\varepsilon,\alpha}(0)$ and $\kappa_{\varepsilon,\alpha}(\varepsilon)$ are both attained by the curvature of M on a disc of radius $\rho_{\varepsilon,\alpha}(\varepsilon) \lesssim \varepsilon$. It then follows from (56) that $\|K\|_{\alpha} \gtrsim 1$ and therefore $H \gtrsim 1$.
Replacing $\varepsilon^{3+\alpha}$ by ε^3 in (55) forces $\|K\|_{\alpha}$ to tend to infinity as $\varepsilon \to 0$ for all $0 < \alpha \leq 1$, and replacing it by ε^{λ} for $\lambda < 3$ forces $\|K\|_{\infty} \to \infty$.
On the other hand, we shall now construct an isometric embedding of $X_{\rho_{\varepsilon,\alpha}}$ into a $C^{2,\alpha}$ surface with constant $\lesssim 1$. In fact, it will be a $C^{2,\beta}$ surface with constant $\lesssim \varepsilon^{\alpha-\beta}$ for any $0 < \beta \leq 1$.
Define $h_{\varepsilon,\alpha} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ by

$$h_{\varepsilon,\alpha} = h_{\varepsilon,\alpha}(r) := \frac{\xi(r)}{r} + \frac{1-\xi(r)}{r-\varepsilon^{3+\alpha}},$$

where $\xi : (0,\infty) \to [0,1]$ is a smooth function supported on $[0,\varepsilon]$, identically 1 on $[0,\varepsilon/2]$, and satisfying $|\xi'| \lesssim \varepsilon^{-1}$ and $|\xi''| \lesssim \varepsilon^{-2}$.
We claim that for sufficiently small ε, the curve $\gamma_{\varepsilon,\alpha} : [-1,1] \to \mathbb{R}^2$ defined by

$$\gamma_{\varepsilon,\alpha}(t) := \rho_{\varepsilon,\alpha}(t)e^{i\phi_{\varepsilon,\alpha}(t)}$$

is a unit-speed geodesic with respect to the metric

$$g_{\varepsilon,\alpha} = dr^2 + G_{\varepsilon,\alpha}^2 d\theta^2, \tag{57}$$

where

$$G_{\varepsilon,\alpha} = G_{\varepsilon,\alpha}(r) := r\exp\int_{\varepsilon/2}^{r}\left(h_{\varepsilon,\alpha}(s)-\frac{1}{s}\right)ds \qquad \text{and} \qquad \phi_{\varepsilon,\alpha}(t) := \int_0^t \frac{\sqrt{1-\rho'_{\varepsilon,\alpha}(s)^2}}{G_{\varepsilon,\alpha}(\rho_{\varepsilon,\alpha}(s))}ds.$$

Set $K_{\varepsilon,\alpha} := h_{\varepsilon,\alpha}^2 + \partial_r h_{\varepsilon,\alpha}$. Then $K_{\varepsilon,\alpha} \equiv 0$ unless $\varepsilon/2 < r < \varepsilon$. For $r \in (\varepsilon/2, \varepsilon)$, a straightforward calculation shows that

$$K_{\varepsilon,\alpha} = \frac{\varepsilon^{6+2\alpha}\xi(\xi-1)}{r^2(r-\varepsilon^{3+\alpha})^2} - \frac{\varepsilon^{3+\alpha}\xi'}{r(r-\varepsilon^{3+\alpha})}$$

which, by the properties of ξ and the fact that $r \sim \varepsilon$, implies that $\|K\|_\infty \lesssim \varepsilon^\alpha$, and

$$\|K_{\varepsilon,\alpha}\|_\beta = O(\varepsilon^{\alpha-\beta}) \quad \text{for all} \quad 0 < \beta \leq 1.$$

We have that $\|K_\varepsilon, \alpha\|_\infty \xrightarrow{\varepsilon} 0$, so by similar arguments as in the proof of Theorem 2, we get that $(\mathbb{R}^2, g_{\varepsilon,\alpha})$ is a $C^{2,\beta}$ surface with constant $\lesssim \varepsilon^{\alpha-\beta}$, which is flat outside $B_\varepsilon(0) \setminus B_{\varepsilon/2}(0)$, and whose metric takes the form (57) in polar normal coordinates centered at the origin. The curvature of this metric is $K_{\varepsilon,\alpha}$, and by definition $h_{\varepsilon,\alpha} = \partial_r G_{\varepsilon,\alpha}/G_{\varepsilon,\alpha}$.
It follows from the definitions that $\gamma_{\varepsilon,\alpha}$ is unit-speed, and since $\rho_{\varepsilon,\alpha} \geq \varepsilon$, a simple calculation shows

$$\rho''_{\varepsilon,\alpha}(t) = h_{\varepsilon,\alpha}(\rho_{\varepsilon,\alpha}(t))(1 - \rho'_{\varepsilon,\alpha}(t)^2), \tag{58}$$

which by Proposition 3.1 implies that $\gamma_{\varepsilon,\alpha}$ is a unit-speed minimizing geodesic with respect to $g_{\varepsilon,\alpha}$. Hence $X_{\rho_{\varepsilon,\alpha}}$ embeds isometrically in $(\mathbb{R}^2, g_{\varepsilon,\alpha})$, which is a $C^{2,\alpha}$ surface with constant $\lesssim 1$.
The process described above can give an idea of our general strategy for constructing embeddings. Here matters where quite simple, because the metric we proposed was rotationally symmetric.

Polar coordinates are convenient for our problem, because we seek a geodesic γ satisfying $r \circ \gamma = \rho$. Eventually, we will have to determine $\theta \circ \gamma$ as well. What is a reasonable first guess? If, by chance, X_ρ embeds in a surface of *constant* curvature, that is, if $\rho = d_g(\gamma(\cdot), p)$ for a point p and a geodesic γ on $\mathscr{S}_{K_0}$, then the θ-coordinate of γ is uniquely determined (up to translation) by the r-coordinate, simply by demanding that the curve be unit-speed; in polar normal coordinates this means that we demand $\dot{r}^2 + (\sin_{K_0} r)^2\dot{\theta}^2 = 1$ on γ, and this is achieved by letting $\gamma = \rho e^{i\phi_0}$ where

$$\phi_0(t) := \int_{t_0}^t \frac{\sqrt{1-\rho'(s)^2}}{\sin_{K_0}\rho(s)} ds \tag{59}$$

for some $t_0 \in I$. With this motivation in mind, given an arbitrary function ρ, we take ϕ_0 to be our "approximate θ-coordinate", where we naturally choose t_0 to be the argument at which ρ is smallest (recall that ρ must be strictly convex by Corollary 3.6), and K_0 to be $\kappa(t_0)$. Immediately from this definition we get an analogue of Corollary 3.7.

Proposition 3.10. *Let $H, R > 0$ satisfy $HR^2 < \pi^2/16$. Let $\rho : I \to (0, R]$, and suppose that for every $S \subseteq I$ with $|S| \leq 4$, the metric space $X_{\rho|S}$ embeds isometrically in a $C^{2,\alpha}$ Riemannian surface with constant H. Let $t_0 = \text{argmin}\rho$, let $K_0 = \kappa(t_0)$, and define $\phi_0 : I \to \mathbb{R}$ as in (59). Then if ρ varies by at most a factor of μ on some subsinterval J, then $\dot{\phi}_0$ varies by at most a factor of μ^C on J, where C is a universal constant.*

Proof. By definition, $\dot{\phi}_0 = \sqrt{1 - \dot{\rho}^2}/\sin_{K_0}\rho$. By Corollary 3.5, the logarithmic derivative of the numerator is within a universal factor of the logarithmic derivative of ρ (both in absolute value). The same holds for the denominator, as

$$\left|\frac{d}{dt}\log\sin_{K_0}\rho(t)\right| = |\rho'(t)\cot_{K_0}\rho(t)| = \left|\frac{\rho'(t)}{\rho(t)}\right| |\Phi(K_0\rho(t)^2)| \sim \left|\frac{\rho'(t)}{\rho(t)}\right|$$

because $|K_0\rho^2| \leq \pi^2/16$. So altogether,

$$\left|\frac{d}{dt}\log\phi_0'(t)\right| \sim \left|\frac{d}{dt}\log\rho(t)\right| \qquad \text{for all } t \in J,$$

and we can proceed as in the proof of Corollary 3.7. $\square$

Remark. We haven't really used the fact that ρ varies by at most a factor of μ *on the entire interval*, so actually, the following stronger statement is true: for every $t, t' \in I$, if $\mu^{-1} \leq \rho(t)/\rho(t') \leq \mu$, then $\mu^{-C} \leq \phi_0(t)/\phi_0(t') \leq \mu^C$.

The following proposition is the main conclusion of this section. It provides some estimates on the function f_0 defined below. These estimates allow the extension of f_0 described in Proposition 4.1, which in turn determines the metric into which X_ρ will be embedded.

Proposition 3.11. *Let $H, R > 0$ satisfy $HR^2 \leq \pi/16$. Let $\rho : I \to (0, R]$ and suppose that for every $S \subseteq I$ with $|S| \leq 12$, the space $X_{\rho|S}$ embeds isometrically in a $C^{2,\alpha}$ Riemannian surface with constant H. Let $t_0 = \text{argmin}\rho$ and $K_0 = \kappa(t_0)$, and define ϕ_0 by (59) and f_0 by*

$$f_0(t) := \frac{\rho''(t)}{1 - \rho'(t)^2} - \cot_{K_0}\rho(t).$$

Let $t_1, t_2, t_3, t_4 \in I$, let $r_j := \rho(t_j)$ and assume that ρ varies by at most a factor of 4 on each of the intervals $\text{conv}\{t_1, t_2\}$ and $\text{conv}\{t_3, t_4\}$, where conv denotes convex hull. Then the following *hold:*

$$|f_0(t_1)| \lesssim Lr_1^{1+\alpha}; \tag{60}$$

$$|f_0(t_1) - f_0(t_2)| \lesssim Lr_1^{\alpha}|r_1 - r_2| + Lr_1^{1+\alpha}|\phi_0'(t_1)|^{\alpha}|t_1 - t_2|^{\alpha} \tag{61}$$

$$\begin{aligned}\left|\frac{f_0(t_1) - f_0(t_2)}{r_1 - r_2}\right. &+2f_0(t_1)\cot_{K_0} r_1 - \frac{f_0(t_3) - f_0(t_4)}{r_3 - r_4} - 2f_0(t_3)\cot_{K_0} r_3\Bigg| \\ &\lesssim L \cdot \operatorname{diam}\{t_1, t_2, t_3, t_4\}^{\alpha} \\ &+ Lr_1^{1+\alpha}|\phi_0'(t_1)|^{\alpha}|t_1 - t_2|^{\alpha}/|r_1 - r_2| + Lr_3^{1+\alpha}|\phi_0'(t_3)|^{\alpha}|t_3 - t_4|^{\alpha}/|r_3 - r_4|.\end{aligned} \tag{62}$$

where $L := H^{1+\alpha/2}$.

If ρ *varies by at most a factor of* 4 *on the entire interval* $\operatorname{conv}\{t_1, t_2, t_3, t_4\}$, *then also*

$$\begin{aligned}\left|\frac{f_0(t_1) - f_0(t_2)}{r_1 - r_2} - \frac{f_0(t_3) - f_0(t_4)}{r_3 - r_4}\right| &\lesssim L \cdot \operatorname{diam}\{t_1, t_2, t_3, t_4\}^{\alpha} \\ + Lr_1^{1+\alpha}|\phi_0'(t_1)|^{\alpha}\Big(|t_1 - t_2|^{\alpha}/|r_1 - r_2| &+ |t_3 - t_4|^{\alpha}/|r_3 - r_4|\Big).\end{aligned} \tag{63}$$

Proof. Let $\varepsilon > 0$, and for $1 \leq j \leq 4$ choose points t_j', t_j'' with $\operatorname{diam}\{t_j, t_j', t_j''\} \leq \varepsilon$. Let $S_j = \{t_j', t_j'', t_j\}$ and $T_j = \{t_j', t_j''\}$ and let $S = \bigcup_{j=1}^4 S_j$.

By assumption there is a $C^{2,\alpha}$ Riemannian surface (M, g) with constant H, a unit-speed minimizing geodesic $\gamma : \mathbb{R} \to M$ and a point $p \in M$ such that

$$d_g(\gamma(t), p) = \rho(t) \text{ for all } t, t' \in S$$

As usual we work in polar normal coordinates centered at p, with the usual identification between $B_{\pi/\sqrt{H}}(0) \subseteq \mathbb{R}^2$ and $B_{\pi/\sqrt{H}}(p) \subseteq M$. Let $\gamma = \tilde{\rho}e^{i\tilde{\phi}}$ be the expression for γ in these coordinates. Let

$$q_j = \gamma(t_j) \quad \text{ and } \quad r_j = \rho(t_j) = \tilde{\rho}(t_j) = d_g(q_j, p).$$

By Proposition 3.1,

$$h(\gamma(t))(1 - \tilde{\rho}'(t)^2) = \tilde{\rho}''(t) \qquad \text{for all } t \in I.$$

Proceeding as in the proof of Corollary 3.4, one sees that

$$\frac{\rho''(t_j)}{1 - \rho'(t_j)^2} = \frac{2\rho[S_j]}{1 - \rho[T_j]} + O(\varepsilon^{\alpha}) = \frac{\tilde{\rho}''(t_j)}{1 - \tilde{\rho}'(t_j)^2} + O(\varepsilon^{\alpha}) = h(q_j) + O(\varepsilon^{\alpha}).$$

Here and in the rest of this proof, the constant in $O(\varepsilon^{\alpha})$ may depend on ρ and on the t_j's, but not e.g. on the choice of t_j', t_j'' and the resulting ambient surface (M, g).

Thus

$$f_0(t_j) = \frac{\rho''(t_j)}{1 - \rho'(t_j)^2} - \cot_{K_0}(r_j) = h(q_j) - \cot_{K_0}(r_j) + O(\varepsilon^{\alpha}).$$

Let $f : B_R(0) \to \mathbb{R}$ be defined by $f := h - \cot_{K_0} r$. By the above,

$$f_0(t_j) = f(q_j) + O(\varepsilon^\alpha). \qquad j = 1, 2, 3, 4. \tag{64}$$

Let $\tilde{\kappa}$ be defined as in (54), with ρ replaced by $\tilde{\rho}$. Arguing as in the proof of Corollary 3.9, we have that $\kappa(t_i) = \tilde{\kappa}(t_j) + O(\varepsilon^\alpha)$. Corollary 3.9 itself then implies that

$$K_0 := \kappa(t_0) = \kappa(t_j) + O(Lr_j^\alpha) = \tilde{\kappa}(t_j) + O(Lr_j^\alpha + \varepsilon^\alpha),$$

where $L := H^{1+\alpha/2}$. Since ε is arbitrarily small we can write $K_0 = \tilde{\kappa}(t_j) + O(Lr_j^\alpha)$.
By Proposition 3.8, for each $1 \le j \le 4$, the value $\tilde{\kappa}(t_j)$ is attained by K on $B_{r_j}(p)$, so since $\|K\|_\alpha \le L$,

$$|K(w) - K_0| \lesssim Lr_j^\alpha \qquad \text{for all } 1 \le j \le 4 \text{ and } w \in B_{r_j}(p).$$

Applying Corollary 2.17 with $T = O(Lr_1^\alpha)$ and recalling that $r_1 \sim r_2$, we have the estimates:

1. $\|f\|_\infty \lesssim Lr_1^{1+\alpha}$ on $B_{r_1 \vee r_2}(0)$
2. $\|\partial_r f\|_\infty \lesssim Lr_1^\alpha$ on $B_{r_1 \vee r_2}(0)$
3. $\|f\|_\alpha \lesssim Lr_1$ on $B_{r_1 \vee r_2}(0)$
4. $\|\partial_r f\|_\alpha \lesssim L$ on $B_{r_1 \vee r_2}(0) \setminus B_{r_1 \wedge r_2}(0)$,

and similarly with r_1 and r_2 replaced by r_3 and r_4. Here $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$.

We are ready to derive the estimates of the proposition. We always take ε small enough to absorb its error term into the others. The bound on $\|f\|_\infty$ together with (64) give us

$$|f_0(t_1)| = |f(q_1)| + O(\varepsilon^\alpha) \lesssim Lr_1^{1+\alpha},$$

proving (60). The estimates on $\|\partial_r f\|_\infty$ and $\|f\|_\alpha$ give

$$\begin{aligned}
|f_0(t_1) - f_0(t_2)| &= |f(q_1) - f(q_2)| + O(\varepsilon^\alpha) \\
&\le \left| f(r_1 e^{i\tilde{\phi}(t_1)})) - f(r_2 e^{i\tilde{\phi}(t_1)}) \right| + \left| f(r_2 e^{i\tilde{\phi}(t_1)}) - f(r_2 e^{i\tilde{\phi}(t_2)})) \right| + O(\varepsilon^\alpha) \\
&\lesssim Lr_1^\alpha |r_1 - r_2| + Lr_2 |\rho(t_2) e^{i\tilde{\phi}(t_1)} - \rho(t_2) e^{i\tilde{\phi}(t_2)}|^\alpha \\
&\lesssim Lr_1^\alpha |r_1 - r_2| + Lr_2^{1+\alpha} |\tilde{\phi}(t_1) - \tilde{\phi}(t_2)|^\alpha \\
&\lesssim Lr_1^\alpha |r_1 - r_2| + Lr_1^{1+\alpha} \max_{[t_1, t_2]} |\tilde{\phi}'|^\alpha |t_1 - t_2|^\alpha
\end{aligned}$$

(in the last inequality we used $r_1 \sim r_2$). Because of Corollaries 3.7 and 2.11,

$$\max_{[t_1, t_2]} |\tilde{\phi}'| \sim |\tilde{\phi}'(t_1)| = \left| \frac{\sqrt{1 - \tilde{\rho}'(t_1)^2}}{G(\gamma(t_1))} \right| \sim \left| \frac{\sqrt{1 - \tilde{\rho}'(t_1)^2}}{\tilde{\rho}(t_1)} \right| \sim \left| \frac{\sqrt{1 - \rho'(t_1)^2}}{\rho(t_1)} \right| \sim |\phi_0'(t_1)| \tag{65}$$

(in the penultimate passage we used $\rho'(t_1) = \tilde{\rho}'(t_1) + O(\varepsilon)$ and $\rho(t_1) = \tilde{\rho}(t_1)$). Thus

$$|f_0(t_1) - f_0(t_2)| \lesssim Lr_1^\alpha |r_1 - r_2| + Lr_1^{1+\alpha} |\phi_0'(t_1)|^\alpha |t_1 - t_2|^\alpha.$$

which is (61). Now for the proof of (62). We have

$$\begin{aligned}
&\left| \frac{f_0(t_1) - f_0(t_2)}{r_1 - r_2} + 2f_0(t_1) \cot_{K_0}(r_1) - \frac{f_0(t_3) - f_0(t_4)}{r_3 - r_4} - 2f_0(t_3) \cot_{K_0}(r_3) \right| \\
&= \left| \frac{f(q_1) - f(q_2)}{r_1 - r_2} + 2f(q_1) \cot_{K_0}(r_1) - \frac{f(q_3) - f(q_4)}{r_3 - r_4} - 2f(q_3) \cot_{K_0}(r_3) \right| + O(\varepsilon^\alpha) \\
&\lesssim \left| \frac{f(r_1 e^{i\tilde{\phi}(t_1)}) - f(r_2 e^{i\tilde{\phi}(t_1)})}{r_1 - r_2} + 2f(q_1) \cot_{K_0}(r_1) - \frac{f(r_3 e^{i\tilde{\phi}(t_3)}) - f(r_4 e^{i\tilde{\phi}(t_3)})}{r_3 - r_4} - 2f(q_3) \cot_{K_0}(r_3) \right| \\
&+ Lr_1 \cdot \frac{r_1^\alpha |\tilde{\phi}(t_1) - \tilde{\phi}(t_2)|^\alpha}{|r_1 - r_2|} + Lr_3 \cdot \frac{r_3^\alpha |\tilde{\phi}(t_3) - \tilde{\phi}(t_4)|^\alpha}{|r_3 - r_4|}
\end{aligned}$$

where in the last passage we used the triangle inequality, the fact that $r_1 \sim r_2$ and $r_3 \sim r_4$, and our local bounds on $\|f\|_\alpha$.

Since $\|\partial_r f\|_\alpha \lesssim L$ on the annuli $\{r_1 \wedge r_2 \leq r \leq r_1 \vee r_2\}$ and $\{r_3 \wedge r_4 \leq r \leq r_3 \vee r_4\}$, this last expression is

$$\begin{aligned}
&\lesssim |\partial_r f(q_1) + 2f(q_1) \cot_{K_0}(r_1) - \partial_r f(q_3) - 2f(q_3) \cot_{K_0}(r_3)| + L|r_1 - r_2|^\alpha + L|r_3 - r_4|^\alpha \\
&+ Lr_1^{1+\alpha} \max_{[t_1,t_2]} |\tilde{\phi}'|^\alpha |t_1 - t_2|^\alpha / |r_1 - r_2| + Lr_3^{1+\alpha} \max_{[t_3,t_4]} |\tilde{\phi}'|^\alpha |t_3 - t_4|^\alpha / |r_3 - r_4|
\end{aligned}$$

which by (65) is

$$\begin{aligned}
&\lesssim |\partial_r f(q_1) + 2f(q_1) \cot_{K_0}(r_1) - \partial_r f(q_3) - 2f(q_3) \cot_{K_0}(r_3)| + L|r_1 - r_2|^\alpha + L|r_3 - r_4|^\alpha \\
&+ Lr_1^{1+\alpha} |\phi_0'(t_1)|^\alpha |t_1 - t_2|^\alpha / |r_1 - r_2| + Lr_3^{1+\alpha} |\phi_0'(t_3)|^\alpha |t_3 - t_4|^\alpha / |r_3 - r_4|.
\end{aligned} \tag{66}$$

We would like to use the identity

$$f^2 + \partial_r f + 2f \cot_{K_0} r = K_0 - K \tag{67}$$

which follows by a short calculation from the Riccati equation for h and $\cot_{K_0} r$. In order to do so we need to introduce the term $f(q_1)^2 - f(q_3)^2$ into the first summand in (66).

To this end we apply our estimates on $\|f\|_\alpha$ and $\|f\|_\infty$ again to obtain (using (3)):

$$|f(q_1)^2 - f(q_3)^2| \lesssim Lr_3^{1+\alpha} \cdot Lr_3 |q_1 - q_3|^\alpha \lesssim L^2 R^{2+\alpha} |t_1 - t_3|^\alpha \lesssim L|t_1 - t_3|^\alpha$$

(the last passage holds since $L = H^{1+\alpha/2}$ and $HR^2 \lesssim 1$). Thus the expression in (66) is

$$\begin{aligned}
&\lesssim |f(q_1)^2 + \partial_r f(q_1) + 2f(q_1) \cot_{K_0}(r_1) - f(q_3)^2 - \partial_r f(q_3) - 2f(q_3) \cot_{K_0}(r_3)| \\
&+ Lr_1^{1+\alpha} |\phi_0'(t_1)|^\alpha |t_1 - t_2|^\alpha / |r_1 - r_2| + Lr_3^{1+\alpha} |\phi_0'(t_3)|^\alpha |t_3 - t_4|^\alpha / |r_3 - r_4| \\
&+ L \cdot \mathrm{diam}\{t_1, t_2, t_3, t_4\}^\alpha.
\end{aligned}$$

We now make use of (67) and the fact that $\|K\|_\alpha \lesssim L$ with respect to the Euclidean metric:

$$
\begin{aligned}
&= |K(q_1) - K(q_3)| + L \cdot \operatorname{diam}\{t_1, t_2, t_3, t_4\}^\alpha \\
&+ Lr_1^{1+\alpha}|\phi_0'(t_1)|^\alpha |t_1 - t_2|^\alpha / |r_1 - r_2| + Lr_3^{1+\alpha}|\phi_0'(t_3)|^\alpha |t_3 - t_4|^\alpha / |r_3 - r_4| \\
&\lesssim L|q_1 - q_3|^\alpha + L \cdot \operatorname{diam}\{t_1, t_2, t_3, t_4\}^\alpha \\
&+ Lr_1^{1+\alpha}|\phi_0'(t_1)|^\alpha |t_1 - t_2|^\alpha / |r_1 - r_2| + Lr_3^{1+\alpha}|\phi_0'(t_3)|^\alpha |t_3 - t_4|^\alpha / |r_3 - r_4| \\
&\lesssim L \cdot \operatorname{diam}\{t_1, t_2, t_3, t_4\}^\alpha \\
&+ Lr_1^{1+\alpha}|\phi_0'(t_1)|^\alpha |t_1 - t_2|^\alpha / |r_1 - r_2| + Lr_3^{1+\alpha}|\phi_0'(t_3)|^\alpha |t_3 - t_4|^\alpha / |r_3 - r_4|
\end{aligned}
$$

using in the last passage the fact that γ is a unit-speed minimizing geodesic to absorb the first term into the second. Inequalities (60)-(62) are proved.

Now assume that ρ varies by at most a factor of 4 on $\operatorname{conv}(t_1, t_2, t_3, t_4)$. Then, applying (60) and (61), and noting that $|\cot_{K_0} x| \lesssim x^{-1}$ and $|\cot'_{K_0} x| \lesssim x^{-2}$ when $|K_0 x^2| \leq \pi^2/16$, we have

$$
\begin{aligned}
&|f_0(t_1)\cot_{K_0}(r_1) - f_0(t_3)\cot_{K_0}(r_3)| \\
&\leq |f_0(t_1)||\cot_{K_0}(r_1) - \cot_{K_0}(r_3)| + |f_0(t_1) - f_0(t_3)||\cot_{K_0}(r_3)| \\
&\lesssim Lr_1^{1+\alpha} \cdot r_1^{-2}|r_1 - r_3| + \Big(Lr_1^\alpha |r_1 - r_2| + Lr_1^{1+\alpha}|\phi_0'(t_1)|^\alpha |t_1 - t_2|^\alpha\Big) r_3^{-1} \\
&\lesssim Lr_1^{\alpha-1}|r_1 - r_3| + Lr_1^\alpha |\phi_0'(t_1)|^\alpha |t_1 - t_2|^\alpha \\
&\lesssim L|r_1 - r_3|^\alpha + Lr_1^\alpha |\phi_0'(t_1)|^\alpha |t_1 - t_2|^\alpha.
\end{aligned}
$$

hence, by (62),

$$
\begin{aligned}
&\left| \frac{f_0(t_1) - f_0(t_2)}{r_1 - r_2} - \frac{f_0(t_3) - f_0(t_4)}{r_3 - r_4} \right| \lesssim L \cdot \operatorname{diam}\{t_1, t_2, t_3, t_4\}^\alpha \\
&+ Lr_1^{1+\alpha}|\phi_0'(t_1)|^\alpha |t_1 - t_2|^\alpha / |r_1 - r_2| + Lr_3^{1+\alpha}|\phi_0'(t_3)|^\alpha |t_3 - t_4|^\alpha / |r_3 - r_4|.
\end{aligned}
$$

By assumption, $r_1 \sim r_3$, and by Proposition 3.10, $\phi_0'(t_1) \sim \phi_0'(t_3)$. So (63) follows. $\square$

Our study of the functions ρ, κ, ϕ_0 and f_0 is almost finished. There is one last fact we need to prove. We intend to use ϕ_0 as our "provisional" θ coordinate for a geodesic-to-be γ. We can't afford to have γ wind more that once around the origin, because our plan is to construct the metric on the sector between γ and the origin. The following proposition asserts that if the constant B in the hypothesis of Theorem 1 is taken small enough, then this problem will not arise.

Proposition 3.12. *With the assumptions and notations of Proposition 3.11, there are universal constants $B_1, c_1 > 0$ such that if $HR^2 \leq B_1$ then $|\phi_0(t)| \leq (1 - c_1) \cdot \pi$ for all $t \in I$.*

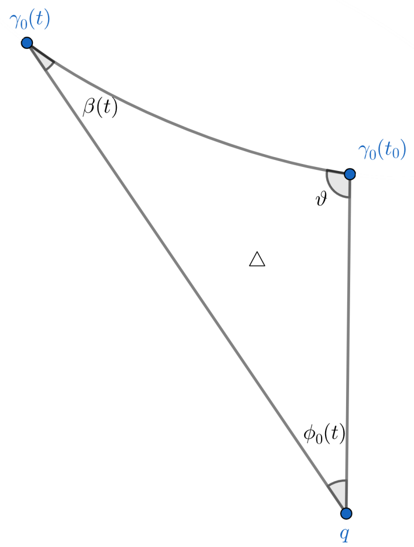

Figure 1: Proof of Proposition 3.12

Proof. Recall that $\mathscr{S}_{K_0}$ denotes the complete, simply connected surface of constant curvature K_0. Fix $q \in \mathscr{S}_{K_0}$ and let γ_0 be a curve in $\mathscr{S}_{K_0}$ given in polar normal coordinates at q by $\gamma_0 = \rho e^{i\phi_0}$. Then, since the metric of $\mathscr{S}_{K_0}$ in polar normal coordinates is given by $dr^2 + (\sin^2_{K_0} r)d\theta^2$, the curve γ_0 is a unit-speed curve. By Lemma 2.9, if N is the inward-pointing unit normal to γ_0, then

$$\nabla_{\dot{\gamma}_0}\dot{\gamma}_0 = -\sqrt{1-\dot{\rho}^2}\left(\frac{\ddot{\rho}}{1-\dot{\rho}^2} - \cot_{K_0}\rho\right)N = -\sqrt{1-\dot{\rho}^2}\cdot f_0 \cdot N. \tag{68}$$

Fix $t \in I$. For definiteness we assume $t > t_0$, and hence $\phi_0(t) > 0$. The case $t < t_0$ is similar. Consider the curve η which is the concatenation of the geodesic $[q, \gamma_0(t_0)]$, the curve $\gamma_0|_{[t_0,t]}$ and the geodesic $[\gamma_0(t), q]$ (see Figure 1). If $t - t_0$ is small enough, the curve η is the boundary of a triangle, two sides of which are geodesics, hence by the Gauss-Bonnet formula:

$$\phi_0(t) = \pi - \beta(t) - \vartheta + K_0 \cdot \mathrm{Area}(\triangle) + \int_{t_0}^{t} \langle \nabla_{\dot{\gamma}_0}\dot{\gamma}_0, \nu\rangle \tag{69}$$

where $\triangle$ is the triangle, Area indicates Riemannian area, ν is the oriented unit normal[2] to γ, and ϑ and $\beta(t)$ are the interior angles of $\triangle$ at $\gamma_0(t_0)$ and $\gamma_0(t)$ respectively (see Figure 1).
Since ρ attains a minimum at t_0, the angle ϑ is at least $\pi/2$ (if $t_0 \in I^\circ$ then $\vartheta = \pi/2$).
The term $K_0 \cdot \mathrm{Area}(\triangle)$ is nonpositive when $K_0 \leq 0$, and if $K_0 > 0$ then it is bounded by K_0 times the Riemannian area of the sector $\{0 \leq r \leq R,\ 0 \leq \theta \leq \phi_0(t)\}$, which is $K_0^{-1} \cdot \phi_0(t)(1 - \cos(\sqrt{K_0}R))$. Suppose $HR^2 \leq B_1$, where B_1 is a universal constant to be determined soon. By

[2]Given an orientation on $B_R(q)$, The *oriented unit normal* to a curve γ is the unique vector field ν along γ such that $(\dot{\gamma}, \nu)$ form an oriented orthonormal frame along γ. The choice of orientation is not important here.

Corollary 3.9, $|K_0| \leq H$. Thus by the above argument, $K_0 \cdot \text{Area}(\triangle) \leq \phi_0(t)(1 - \cos\sqrt{B_1})$.
The normals N and ν differ at most by a sign. Thus by (68), the last term of (69) is

$$\pm \int_{t_0}^{t} \sqrt{1 - \rho'(s)^2} \cdot f_0(s) ds,$$

which by (60) is

$$\lesssim LR^{1+\alpha} \int_{t_0}^{t} \sqrt{1 - \rho'(s)^2} ds \lesssim LR^{2+\alpha} \leq B_1^{1+\alpha}.$$

where the second inequality holds because $|t - t_0| \leq \rho(t) + \rho(t_0) \leq 2R$ by (46).
Putting together (69) and the estimates above, we conclude that

$$\phi_0(t) \leq \frac{\pi/2 + O(B_1^{1+\alpha})}{\cos(\sqrt{B_1})}.$$

If follows that if the universal constant B_1 is chosen small enough, then $\phi_0(t) \leq (1 - c_1)\pi$ for some universal constant $0 < c_1 < 1$. In particular, the curve η bounds a triangle for all $t > t_0$, so our analysis applies to all $t > t_0$. $\square$

4 Proof of Theorem 1

From now on we fix $H, R > 0$, an interval I, and a function $\rho \in \rho_{\mathcal{X}}$, $\rho : I \to (0, R]$ with the property

($\star$) For every subset $S \subseteq I$ containing at most 12 points, the 13-point metric space $X_{\rho|_S}$ embeds isometrically in a $C^{2,\alpha}$ Riemannian surface with constant H.

We will also assume that $HR^2 \leq B$, where $B > 0$ is a universal constant to be determined later. For the moment, we assume it is at most $\min\{\pi^2/16, B_1\}$, where B_1 is the constant from Proposition 3.12, so that all the conclusions of Section 3 hold.
By Corollary 3.2 we know that $\rho \in C^{2,\alpha}$, and by Corollary 3.6, $\ddot{\rho} > 0$ (unless $\rho \equiv \pm 1$, but we exclude this trivial case). Translating if neccesary, we may assume that the unique minimum of ρ is attained at $t = 0$,

$$m := \min_I \rho = \rho(0).$$

We set $K_0 = \kappa(0)$, where κ is defined in (54), and define ϕ_0 and f_0 as before:

$$\phi_0(t) = \int_0^t \frac{\sqrt{1 - \rho'(s)^2}}{\sin_{K_0} \rho(s)} ds \qquad\qquad f_0(t) = \frac{\rho''(t)}{1 - \rho'(t)^2} - \cot_{K_0} \rho(t). \tag{70}$$

Our goal is to show that X_ρ embeds isometrically in some $C^{2,\alpha}$ Riemannian surface (M,g) with constant $\lesssim H$, that is, to find a point $p \in M$ and a unit speed geodesic $\gamma : I \to M$ such that

$$d_g(\gamma(t), p) = \rho(t) \text{ for all } t \in I. \tag{71}$$

It suffices to find a function $G : B_R(0) \to (0,\infty)$ satisfying conditions $(a)-(c)$ of Theorem 2, and a unit-speed curve $\gamma = \rho e^{i\phi} : I \to B_R(0)$ satisfying $\ddot\rho = (h\circ\gamma)(1-\dot\rho^2)$, where $h = \partial_r G/G$. Indeed, by Theorem 2, conditions $(a)-(c)$ imply the existence of a $C^{2,\alpha}$ Riemannian surface (M,g) with constant $\lesssim H$, whose expression in polar normal coordinates centered at some point $p \in M$ is $dr^2 + G^2 d\theta^2$. The curve γ is then a unit-speed minimizing geodesic on $B_R(p)$ by Proposition 3.1, and satisfies (71).

As a first step towards finding G and γ, we define a "provisional" curve $\gamma_0 : I \to B_R(0)$ by

$$\gamma_0(t) = \rho(t)e^{i\phi_0(t)}.$$

Recall the discussion preceding Proposition 3.10: if the disc $B_R(0)$ is endowed with the metric $dr^2 + (\sin_{K_0} r)^2 d\theta^2$, which is a metric of constant curvature K_0, then γ_0 is a unit speed curve, whose distance from the origin (both Euclidean and Riemannian) at time t is $\rho(t)$; however, it is not necessarily a geodesic.

As was mentioned above, by demanding that $h = \ddot\rho/(1-\dot\rho^2)$ on our curve, we can guarantee that it will be a geodesic in the new metric. Hence we know what we would like h to be on the final curve γ, and G can be determined from h by integration. The function f_0, which we have defined in (70), is the difference, on γ_0, between this desired h and $\cot_{K_0} r$, the counterpart of h in constant curvature K_0. If we could extend the domain of $f_0 \circ \gamma_0^{-1}$ from γ_0 to all of $B_R(0)$, maintaining some appropriate regularity, then this extension would be the additive correction needed to turn $\cot_{K_0} r$ into h, or, after integration with respect to r and exponentiation, the multiplicative correction needed to turn $\sin_{K_0} r$ into G. The following proposition asserts the existence of this extension.

Proposition 4.1. *There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is C^1 with respect to r, and which extends f_0 in the sense that*

$$f \circ \gamma_0 = f_0. \tag{72}$$

Moreover f satisfies:

$$f|_{B_{m/2}(0)} \equiv 0 \tag{73}$$

$$\|f/r\|_\infty \lesssim H, \tag{74}$$

$$\left\| f^2 + 2f\cot_{K_0} r + \partial_r f \right\|_\alpha \lesssim H^{1+\alpha/2}. \tag{75}$$

We first finish the proof of Theorem 1 given Proposition 4.1.

Let us emphasize again what the role of f is: we wish to construct a metric of the form $g = dr^2 + G^2 d\theta^2$ on the disc $B_R(0)$ with $f = h - \cot_{K_0} r$, where $h = \partial_r G/G$. Thus f determines G by the formula

$$G(re^{i\theta}) = (\sin_{K_0} r) \exp \int_0^r f(ue^{i\theta}) du.$$

If we were to define G this way, we would get for any $t \in I$:

$$h(\gamma_0(t)) = \frac{\partial_r G(\gamma_0(t))}{G(\gamma_0(t))} = \cot_{K_0} \rho(t) + f(\gamma_0(t)) = \cot_{K_0} \rho(t) + f_0(t) = \frac{\rho''(t)}{1 - \rho'(t)^2}$$

and γ_0 would satisfy equation (47). Since we are in polar normal coordinates, it would also satisfy $d_g(\gamma_0(t), 0) = \rho(t)$. But there is a problem: satisfying (47) is equivalent to being a geodesic only for unit speed curves, and γ_0 is not unit-speed with respect to g.

Our remedy to this is to anticipate what G it going to be on the curve γ_0, and first alter γ_0 so as to make it unit-speed with respect to the future metric g. To this end, define $G_0 : I \to \mathbb{R}$ by

$$G_0(t) := (\sin_{K_0} \rho(t)) \exp \int_0^{\rho(t)} f(ue^{\phi_0(t)}) du. \tag{76}$$

Note that G_0 is positive, because if $K_0 > 0$ then $\rho(t) \leq R \leq \pi/(2\sqrt{H}) \leq \pi/(2\sqrt{K_0})$ (it follows from Corollary 3.9 that $K_0 = \kappa(0) \leq H$). Now let

$$\phi(t) := \int_0^t \frac{\sqrt{1 - \rho'(s)^2}}{G_0(s)} ds \tag{77}$$

and define a new curve γ by

$$\gamma(t) := \rho(t) e^{i\phi(t)}. \tag{78}$$

The curve γ differs from γ_0 only by some deformation in the θ direction; this is good, because we want $\gamma(t)$ to be at distance $\rho(t)$ from the origin for all $t \in I$. This θ-deformation is not too dramatic, indeed we now prove that it is bi-Lipschitz with universal constants, provided that the constant B is small enough. By definition,

$$(\phi \circ \phi_0^{-1})'(\phi_0(t)) = \frac{\sqrt{1 - \rho'(t)^2}}{G_0(t)} \cdot \left(\frac{\sqrt{1 - \rho'(t)^2}}{\sin_{K_0} \rho(t)} \right)^{-1} = \frac{\sin_{K_0} \rho(t)}{G_0(t)} = \exp \left(- \int_{m/2}^{\rho(t)} f(ue^{i\phi_0(t)}) du \right).$$

By (74), for all $\theta \in S^1$,

$$\left| \int_{m/2}^{\rho(t)} f(ue^{i\theta}) du \right| \leq R \int_{m/2}^{\rho(t)} \left| \frac{f(ue^{i\theta})}{u} \right| du \leq C_1 H R^2 \leq C_1 B \tag{79}$$

where $C_1 > 0$ is a universal constant, so

$$e^{-C_1 B} \leq (\phi \circ \phi_0^{-1})' \leq e^{C_1 B}.$$

Let c_1 be the constant from Proposition 3.10. Then since $B \le B_1$, we have that $|\phi_0(t)| \le \pi \cdot (1 - c_1)$ for all $t \in I$. If B is at most $B_2 := C_1^{-1} \log((1 - c_1/2)/(1 - c_1))$, then $\phi \circ \phi_0^{-1}$ is Lipschitz with constant at most $(1 - c_1/2)/(1 - c_1)$, and so is its inverse. It follows that $|\phi(t)| \le \pi \cdot (1 - c_1/2)$ for all $t \in I$. There is a homeomorphism $\psi : S^1 \to S^1$ which is bi-Lipschitz with universal constants[3] and extends $\phi \circ \phi_0^{-1}$: for example, we can take ψ to be the unique extension of $\phi \circ \phi_0^{-1}$ which is continuous on $[-\pi, \pi]$, linear outside $\phi_0(I)$, and satisfies $\psi(\pm\pi) = \pm\pi$.

Having transformed γ_0 into γ, we are ready to define the function G. Let

$$G(re^{i\theta}) := (\sin_{K_0} r) \exp \int_0^r f(ue^{i\psi^{-1}(\theta)}) du. \tag{80}$$

Then G is C^2 with respect to r, and by virtue of (74), $G \to 0$ and $G/r \to 1$ as $r \to 0$. Differentiating (80) logarithmically with respect to r we get that

$$h(re^{i\theta}) = \partial_r G(re^{i\theta})/G(re^{i\theta}) = \cot_{K_0} r + f(re^{i\psi^{-1}(\theta)}), \tag{81}$$

which, after differentiation with respect to r and some rearrangement, becomes

$$-\partial_r^2 G(re^{i\theta})/G(re^{i\theta}) = K_0 - f(re^{i\psi^{-1}(\theta)})^2 - 2f(re^{i\psi^{-1}(\theta)}) \cot_{K_0} r - \partial_r f(re^{i\psi^{-1}(\theta)}). \tag{82}$$

Set $K := -\partial_r^2 G/G$.

By (75),(82) and the fact that ψ^{-1} is Lipschitz with a universal constant, we have $\|K\|_\alpha \lesssim H^{1+\alpha/2}$ with respect to the Euclidean metric on $B_R(0)$. Moreover, $K(0) = K_0 \le H$ by (82) and (73), so

$$\|K\|_\infty \lesssim H + H^{1+\alpha/2} R^\alpha = H(1 + (HR^2)^{\alpha/2}) \lesssim H.$$

We would like to apply Theorem 2. To do so, we need to add the following extra assumption on the constant B:

$$B \le B_3 := \pi^2/(64 C_2), \tag{83}$$

where C_2 is a universal constant such that $\|K\|_\infty \le C_2 H$ and $\|K\|_\alpha \le (C_2 H)^{1+\alpha/2}$. This guarantees

$$C_2 H R^2 \le C_2 B \le \pi^2/64,$$

allowing us to apply Theorem 2.

We obtain a $C^{2,\alpha}$ Riemannian surface (M, g) and a strongly convex disc $B_R(p) \subseteq M$, such that the metric g is given in polar normal coordinates on $B_R(p)$ by $g = dr^2 + G^2 d\theta^2$. Hence we can

[3] Here S^1 is considered as $\mathbb{R}/2\pi\mathbb{Z}$ with the quotient metric.

regard γ as a curve on M, whose expression in polar normal coordinates centered at p is (78). Since $B_R(p)$ is strongly convex, it follows that

$$d_g(\gamma(t), p) = \rho(t) \qquad \text{for all } t \in I.$$

To finish the proof of the theorem we need to show that γ is a unit-speed geodesic. By our definitions,

$$\begin{aligned}
\rho'(t)^2 + G(\gamma(t))^2\phi'(t)^2 &= \rho'(t)^2 + \phi'(t)^2(\sin_{K_0}\rho(t))^2 \exp\left(2\int_0^{\rho(t)} f(ue^{i\psi^{-1}(\phi(t))})du\right) \\
&= \rho'(t)^2 + \phi'(t)^2(\sin_{K_0}\rho(t))^2 \exp\left(2\int_0^{\rho(t)} f(ue^{i\phi_0(t)})du\right) \\
&= \rho'(t)^2 + \phi'(t)^2 G_0(t)^2 \\
&= 1
\end{aligned}$$

thus γ is unit speed, and so by Proposition 3.1, it suffices to check that $\ddot{\rho} = h(1-\dot{\rho}^2)$. Indeed,

$$\begin{aligned}
\rho''(t) &= (f_0(t) + \cot_{K_0}(\rho(t)))(1-\rho'(t)^2) \\
&= (f(\gamma_0(t)) + \cot_{K_0}(\rho(t)))(1-\rho'(t)^2) \\
&\overset{(81)}{=} h(\rho(t)e^{i\psi(\phi_0(t))})(1-\rho'(t)^2) \\
&= h(\rho(t)e^{i\phi(t)})(1-\rho'(t)^2) \\
&= h(\gamma(t))(1-\rho'(t)^2).
\end{aligned}$$

Theorem 1 is proved, with $B = \min\{B_1, B_2, B_3\}$. □

Proof of Proposition 4.1. The proof consists of the following main ingredients. First, we construct a partition of unity, subordinated to a cover of a sector $\Omega^0 \subseteq \mathbb{R}^2$ in which γ_0 lies. Next, we define local extensions of f_0 (in the sense of (72)), satisfying some estimates due to the results of Section 3. Then we use the partition of unity to glue the local extensions together, and show that the resulting global extension satisfies (74) and (75).

By Proposition 3.12, $|\phi_0| \leq \theta_0$ for a universal constant $\pi/2 < \theta_0 < \pi$, so γ_0 is contained in the sector

$$\Omega^0 := \{re^{i\theta} \mid -\theta_0 \leq \theta \leq \theta_0\}.$$

Recall that $m = \min\rho = \rho(0)$. Let

$$R_k := 2^k m \qquad k \in \mathbb{Z}.$$

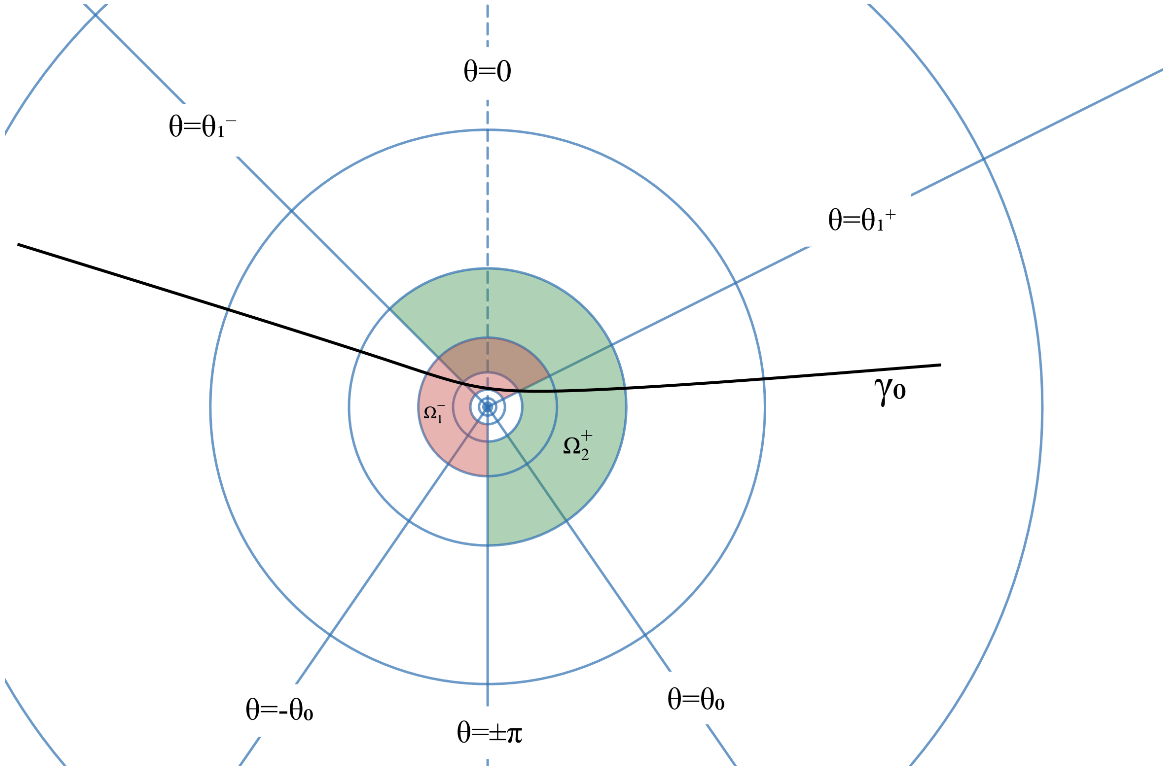

Figure 2: The cover of Ω^0, with the sets Ω_2^+ and Ω_1^- highlighted.

If there exists $t_1^+ > 0$ such that $\rho(t_1^+) = R_1$ then let $\theta_1^+ := \phi_0(t_1^+)$, otherwise let $\theta_1^+ = \theta_0$. Similarly let $\theta_1^- := \phi_0(t_1^-)$ where $t_1^- < 0$ satisfies $\rho(t_1^-) = R_1$, or $\theta_1^- = -\theta_0$ if no such t_1^- exists. Set

$$\Omega_k^- := \{re^{i\theta} \mid R_{k-1} < r < R_{k+1},\ -\pi < \theta < \theta_1^+\}$$
$$\Omega_k^+ := \{re^{i\theta} \mid R_{k-1} < r < R_{k+1},\ \theta_1^- < \theta < \pi\}.$$

(see Figure 2). The sets Ω_k^ℓ cover Ω^0 (in fact, they cover $\mathbb{R}^2 \setminus \{\theta = \pi\}$).

Claim 1. $\theta_1^+ - \theta_1^- \gtrsim 1$.

Proof. If 0 is an endpoint of I, then either $\theta_1^- = -\theta_0$ or $\theta_1^+ = \theta_0$; in both cases the difference $\theta_1^+ - \theta_1^-$ is at least θ_0, because always $\theta_1^- \le 0 \le \theta_1^+$. So we may assume that 0 is an interior point of I.

We can also assume that t_1^+ exists, because otherwise $\theta_1^+ = \theta_0$ so again $\theta_1^+ - \theta_1^- \ge \theta_0$. Since ρ attains its minimum at $0 \in I^\circ$, $\rho'(0) = 0$. By Corollary 3.5, since ρ increases on $[0, t_1^+]$ by a factor of 2, the quantity $1 - \dot\rho^2$ decreases by at most some constant factor on this interval. It follows that $\dot\rho \le 1 - c$, whence $d \log \rho / dt \le (1-c)/m$, on the interval $[0, t_1^+]$, for some universal $0 < c < 1$. Thus

$$\log 2 = \int_0^{t_1^+} (\log \rho)'(s) ds \le t_1^+(1-c)/m.$$

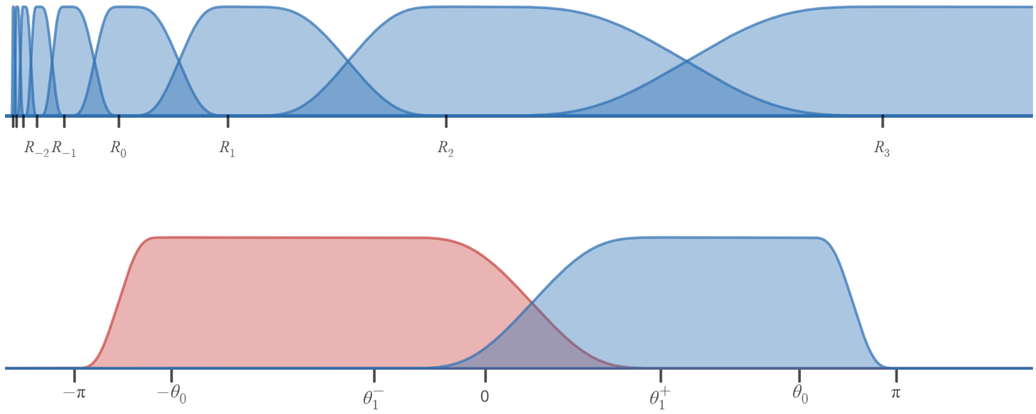

Figure 3: The functions ξ_k (top) and the functions ζ^- and ζ^+ (bottom, left and right respectively).

Since $\phi_0'(0) = \sqrt{1-\rho'(0)^2}/\sin_{K_0} m = 1/\sin_{K_0} m \sim 1/m$ by Corollary 2.11, Proposition 3.10 implies that $\dot{\phi}_0 \sim 1/m$ on $[0, t_1^+]$. Therefore

$$\theta_1^+ - \theta_1^- \geq \theta_1^+ = \phi_0(t_1^+) = \int_0^{t_1^+} \phi_0'(s)ds \sim t_1^+/m \geq \log 2/(1-c).$$

and the claim is proved. □

Claim 2. *There exist functions* $\{\varphi_k^\ell\}_{k\in\mathbb{Z},\ell\in\{+,-\}}$ *with the following properties:*

$$\sum_{k,\ell} \varphi_k^\ell \equiv 1 \ \text{on } \Omega^0$$

$$\mathrm{supp}\varphi_k^\ell \subseteq \Omega_k^\ell, \qquad |\nabla\varphi_k^\ell| \lesssim R_k^{-1} \qquad \text{and} \qquad |\mathrm{Hess}(\varphi_k^\ell)| \lesssim R_k^{-2} \qquad \begin{matrix} k\in\mathbb{Z} \\ \ell\in\{+,-\} \end{matrix}, \tag{84}$$

where Hess *is the Hessian.*

Proof. Let $\xi : \mathbb{R} \to [0,1]$ be a smooth function supported on $(-\infty, 1]$ and identically 1 on $(-\infty, 0]$. For $k \in \mathbb{Z}$ let

$$\xi_k(x) := \begin{cases} 1 - \xi(x/R_{k-1} - 1) & x \leq R_k \\ \xi(x/R_k - 1) & x \geq R_k. \end{cases}$$

(see figure 3). Then ξ_k is smooth, positive and supported on $[R_{k-1}, R_{k+1}]$ and satisfies $|\xi_k'| \lesssim R_k^{-1}$ and $|\xi_k''| \lesssim R_k^{-2}$. One also sees easily that $\sum_k \xi_k \equiv 1$ on $(0, \infty)$. Next, set

$$\zeta^-(x) := \begin{cases} \xi(-(x+\theta_0)/(-\theta_0+\pi)) & x \leq -\theta_0 \\ 1 & -\theta_0 \leq x \leq \theta_1^- \\ \xi((x-\theta_1^-)/(\theta_1^+-\theta_1^-)) & x \geq \theta_1^- \end{cases} \quad \zeta^+(x) = \begin{cases} 1 - \xi((x-\theta_1^-)/(\theta_1^+-\theta_1^-)) & x \leq \theta_1^+ \\ 1 & \theta_1^+ \leq x \leq \theta_0 \\ \xi((x-\theta_0)/(\pi-\theta_0)) & x \geq \theta_0. \end{cases}$$

(see Figure 3). Then ζ^- and ζ^+ are supported on $[-\pi, \theta_1^+]$ and $[\theta_1^-, \pi]$ respectively, and their sum is identically 1 on $[-\theta_0, \theta_0]$. Moreover, all the denominators in their definition are bounded below by a positive constant: the number θ_0 is a universal constant strictly between 0 and $\pi/2$, and $\theta_1^+ - \theta_1^-$ is bounded below by a universal constant by Claim 1. Hence we have $|(\zeta^\pm)'| \lesssim 1$ and $|(\zeta^\pm)''| \lesssim 1$.

For $k \in \mathbb{Z}$ and $\ell \in \{+, -\}$, define $\varphi_k^\ell : \mathbb{R}^2 \to \mathbb{R}$ by

$$\varphi_k^\ell(re^{i\theta}) := \xi_k(r)\zeta^\ell(\theta).$$

Then

$$\sum_{k,\ell} \varphi_k^\ell(re^{i\theta}) = \sum_k \xi_k(r)(\zeta^+(\theta) + \zeta^-(\theta)) = \sum_k \xi_k(r) = 1$$

whenever $\theta \in [-\theta_0, \theta_0]$, i.e. whenever $re^{i\theta} \in \Omega^0$, and $\mathrm{supp}\varphi_k^\ell \subseteq \Omega_k^\ell$ by construction. It is also elementary to verify that $|\nabla \varphi_k^\ell| \lesssim R_k^{-1}$ and $|\mathrm{Hess}(\varphi_k^\ell)| \lesssim R_k^{-2}$. □

Since each point $q \in \mathbb{R}^2$ is contained in at most 4 of the domains Ω_k^ℓ, it follows that at most 4 of the functions φ_k^ℓ are nonzero at q (namely, $\varphi_k^+, \varphi_k^-, \varphi_{k+1}^+$ and φ_{k+1}^- for some $k \in \mathbb{Z}$). Observe also that by the inequality (4),

$$\begin{aligned} \left\|\varphi_k^\ell\right\|_\alpha &\lesssim \sup|\nabla\varphi_k^\ell| \cdot \left(\mathrm{diam}\Omega_k^\ell\right)^{1-\alpha} \lesssim R_k^{-1} \cdot R_k^{1-\alpha} = R_k^{-\alpha} \quad \text{and} \\ \left\|\partial_r\varphi_k^\ell\right\|_\alpha &\lesssim \sup|\mathrm{Hess}(\varphi_k^\ell)| \cdot \left(\mathrm{diam}\Omega_k^\ell\right)^{1-\alpha} \lesssim R_k^{-2} \cdot R_k^{1-\alpha} = R_k^{-1-\alpha}. \end{aligned} \tag{85}$$

Claim 3. *For every k, ℓ, the set*

$$I_k^\ell := \gamma_0^{-1}(\Omega_k^\ell)$$

is either empty or a relatively open subinterval of I.

Proof. We can write

$$I_k^- = \rho^{-1}([R_{k-1}, R_{k+1}]) \cap \phi_0^{-1}((-\pi, \theta_1^+)) \quad \text{and} \quad I_k^+ = \rho^{-1}([R_{k-1}, R_{k+1}]) \cap \phi_0^{-1}((\theta_1^-, \pi)). \tag{86}$$

By convexity of ρ, the set $\rho^{-1}([R_{k-1}, R_{k+1}])$ consists of at most two connected components, while by monotonicity of ϕ_0, the sets $\phi_0^{-1}((-\pi, \theta_1^+))$ and $\phi_0^{-1}((\theta_1^-, \pi))$ are connected.

If $\rho^{-1}([R_{k-1}, R_{k+1}])$ is connected then we are done. If it consists of two connected components J_1, J_2, then it must be the case that $k \geq 2$, and we have $J_1 \subseteq (-\infty, t_1^-)$ and $J_2 \subseteq (t_1^+, \infty)$. But

$$\phi_0^{-1}((-\pi, \theta_1^+)) \subseteq (-\infty, t_1^+) \quad \text{and} \quad \phi_0^{-1}((\theta_1^-, \pi)) \subseteq (-t_1^+, \infty),$$

so

$$I_k^- = J_1 \cap \phi_0^{-1}((-\pi, \theta_1^+)) \quad \text{and} \quad I_k^+ = J_2 \cap \phi_0^{-1}((\theta_1^-, \pi)),$$

thus I_k^+, I_k^- are connected, being the intersection of intervals, and relatively open by continuity.

$\square$

Our partition of unity is ready. We shall now construct a local extension $f_k^\ell : \Omega_k^\ell \to \mathbb{R}$ for each k, ℓ.

First we set

$$f_k^\ell \equiv 0 \quad \text{whenever} \quad I_k^\ell = \varnothing. \tag{87}$$

This includes in particular all f_k^ℓ with $k \leq -1$.

The nontrivial local extensions are described in the following claim. As usual, denote

$$L := H^{1+\alpha/2}.$$

Claim 4. *For each $(k, \ell) \in \mathbb{Z} \times \{+, -\}$ such that $I_k^\ell \neq \varnothing$, there is a function $f_k^\ell : \Omega_k^\ell \to \mathbb{R}$ satisfying $f_k^\ell(\gamma_0(t)) = f_0(t)$ for all $t \in I_k^\ell$, as well as:*

$$\left\|f_k^\ell\right\|_\infty \lesssim LR_k^{1+\alpha}, \quad \left\|f_k^\ell\right\|_\alpha \lessapprox LR_k, \quad \left\|\partial_r f_k^\ell\right\|_\infty \lessapprox LR_k^\alpha, \quad \textit{and} \quad \left\|\partial_r f_k^\ell\right\|_\alpha \lessapprox L. \tag{88}$$

Proof. By definition, $\rho \sim R_k$ on $I_k^- \cup I_k^+$. By the remark following Proposition 3.12, if $I_k^\ell \neq \varnothing$ then there exists some $\lambda_k > 0$ such that

$$\phi_0'(t) \sim \lambda_k \text{ for all } t \in I_k^\ell.$$

Recall that ρ attains its unique minimum m at $t = 0$. If $|\rho'(0)| \leq 1/2$, in particular if $0 \in I^\circ$, then the extension in the case $k \in \{0, 1\}$ is different from when $k \geq 2$ and is described in the next three paragraphs. If $|\rho'(0)| > 1/2$ then the construction is the same for all $k \geq 0$ (as noted above, $I_k^\ell = \varnothing$ for $k < 0$). Suppose $|\rho'(0)| \leq 1/2$ and $k \in \{0, 1\}$. In this case we let f_k^ℓ be the unique continuous function which is independent of r and constant outside the annular sector

$$S_k^\ell := \theta^{-1}(\phi_0(I_k^\ell)) \cap \Omega_k^\ell = \{re^{i\theta} \mid r \in [R_{k-1}, R_{k+1}],\ \theta \in \phi_0(I_k^\ell)\},$$

(see Figure 4) and satisfies $f_k^\ell(\gamma_0(t)) = f_0(t)$ for all $t \in I_k^\ell$. Explicitly: first extend the function f_0 to be constant along radial lines, i.e. $f_k^\ell(re^{i\phi_0(t)}) := f_0(t)$ for all $t \in I_k^\ell$ and $R_{k-1} \leq r \leq R_{k+1}$, and then extend f_k^ℓ to be continuous on Ω_k^ℓ and constant on $\Omega_k^\ell \setminus S_k^\ell$.

Since $\phi_0'(0) = \sqrt{1 - \rho'(0)^2}/\sin_{K_0} m \sim 1/\sin_{K_0} m \sim 1/m$ by Corollary 2.11, we have $\lambda_k R_k \sim 1$ (recall that $k \in \{0, 1\}$ so $R_k \sim m$ and $\lambda_k \sim \lambda_0$). For every $t_1, t_2, t_3, t_4 \in I_k^\ell$, the conditions of

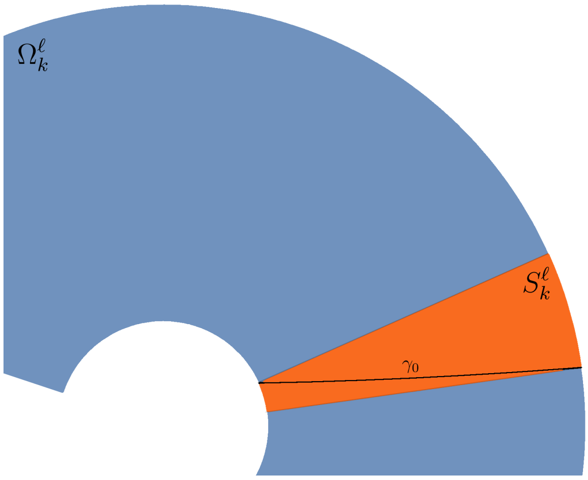

Figure 4: The sector S_k^ℓ

Proposition 3.11 are satisfied, because ρ varies by at most a factor of 4 on I_k^ℓ. It then follows directly from (60) that $\|f_0\|_\infty \lesssim LR_k^{1+\alpha}$ on I_k^ℓ; and (61) implies that $\|f_0\|_\alpha \lesssim LR_k$ on I_k^ℓ, because for every $t_1, t_2 \in I_k^\ell$,

$$
\begin{aligned}
|f_0(t_1) - f_0(t_2)| &\overset{(61)}{\lesssim} LR_k^\alpha|\rho(t_1) - \rho(t_2)| + LR_k^{1+\alpha}\lambda_k^\alpha|t_1 - t_2|^\alpha \\
(|\rho(t_1) - \rho(t_2)| \lesssim R_k \text{ and } R_k\lambda_k \sim 1) \qquad &\lesssim LR_k|\rho(t_1) - \rho(t_2)|^\alpha + LR_k|t_1 - t_2|^\alpha \\
(|\dot\rho| < 1) \qquad &\lesssim LR_k|t_1 - t_2|^\alpha.
\end{aligned}
$$

From our construction and the fact that $\phi_0' \sim \lambda_k$ on I_k^ℓ it follows that

$$\left\|f_k^\ell\right\|_\alpha = \left\|f_0 \circ \phi^{-1} \circ \theta\right\|_\alpha \sim \lambda_k^{-\alpha} R_k^{-\alpha} \left\|f_0\right\|_\alpha \sim \left\|f_0\right\|_\alpha .$$

Thus $\left\|f_k^\ell\right\|_\infty \lesssim LR_k^{1+\alpha}$ and $\left\|f_k^\ell\right\|_\alpha \lesssim LR_k$ on S_k^ℓ, while $\partial_r f_k^\ell \equiv 0$. Since f_k^ℓ is continuous on Ω_k^ℓ and constant on $\Omega_k^\ell \setminus S_k^\ell$, these estimates hold on all of Ω_k^ℓ. This completes the proof in the exceptional case where $|\rho'(0)| \le 1/2$ and $k \in \{0, 1\}$.

We now describe the construction when $k \ge 2$ or when $|\rho'(0)| \ge 1/2$. The idea is to first find a $C^{1,\alpha}$ function depending only on r, and which agrees with f_0 on a discrete net of "sample points", and then to add to it a correction which is purely a function of θ so that the resulting function agrees with f_0 on all of $\gamma_0 \cap \Omega_k^\ell$. The correction vanishes on all of the sample points and so does not cause too much "noise", as we shall see.

First let us argue that in this case, $|\dot\rho| \sim 1$ on I_k^ℓ. By convexity, $|\dot\rho|$ attains its minimum at 0, so if $|\rho'(0)| \ge 1/2$ then $1/2 \le |\dot\rho| < 1$ everywhere. So we may assume that $k \ge 2$.

Let $t \in I_k^\ell$. By Corollary 3.5, since $\rho(t) \ge 2\rho(0)$, there is a universal constant $0 < c < 1$ such

that $1-\rho'(t)^2 \le c\cdot(1-\rho'(0)^2) \le c$ whence $\sqrt{1-c} \le |\rho'(t)| < 1$. Thus $|\dot\rho| \sim 1$ on I_k^ℓ.
Set

$$\delta_k := \lambda_k^\alpha R_k^{1+\alpha}$$

and let $\{s_j\}_{j\in\mathcal{J}}$ be a δ_k-net in I_k^ℓ. By δ_k-net we mean that $|s_j - s_{j'}| \ge \delta_k$ for every $j, j' \in \mathcal{J}$ and that for every $t \in I_k^\ell$ there is some $j \in \mathcal{J}$ such that $|s_j - t| \le \delta_k$. Let

$$x_j := \rho(s_j)$$

and define a function $y = y_k^\ell : \{x_j\}_{j\in\mathcal{J}} \to \mathbb{R}$ by

$$y(x_j) := f_0(s_j), \qquad j \in \mathcal{J}.$$

We now evoke Proposition 3.11 again, with the intention of using Corollary 2.3 to extend y to the interval $[R_{k-1}, R_{k+1}] \supseteq \{x_j\}_{j\in\mathcal{J}}$ (one should think of y as a function of r, while f_0 is a function of t).
By (60) and the fact that $\rho \sim R_k$ on I_k^ℓ, we have for all $j \in \mathcal{J}$,

$$|y(x_j)| \lesssim LR_k^{1+\alpha}.$$

By (61), for $j \ne j' \in \mathcal{J}$,

$$\begin{aligned}
|y(x_j) - y(x_{j'})| &\lesssim LR_k^\alpha |x_j - x_{j'}| + LR_k^{1+\alpha}\lambda_k^\alpha |s_j - s_{j'}|^\alpha \\
&= LR_k^\alpha |x_j - x_{j'}| + L\delta_k |s_j - s_{j'}|^\alpha \\
(\{s_j\} \text{ is a } \delta_k\text{-net}) \qquad &\le LR_k^\alpha |x_j - x_{j'}| + L|s_j - s_{j'}|^{1+\alpha} \\
(|\dot\rho| \sim 1 \text{ on } I_k^\ell) \qquad &\lesssim LR_k^\alpha |x_j - x_{j'}| + L|x_j - x_{j'}|^{1+\alpha} \\
(x_j, x_{j'} \in \rho(I_k^\ell) \subseteq [R_{k-1}, R_{k+1}]) \qquad &\lesssim LR_k^\alpha |x_j - x_{j'}|.
\end{aligned}$$

By (63) and the fact that $|\dot\rho| \sim 1$, for distinct $j, j', j'' \in \mathcal{J}$ we have

$$\begin{aligned}
\left| \frac{y(x_j) - y(x_{j'})}{x_j - x_{j'}} - \frac{y(x_{j'}) - y(x_{j''})}{x_{j'} - x_{j''}} \right| &\lesssim L \cdot \operatorname{diam}\{s_j, s_{j'}, s_{j''}\}^\alpha \\
&\quad + LR_k^{1+\alpha}\lambda_k^\alpha (|s_j - s_{j'}|^{\alpha-1} + |s_{j'} - s_{j''}|^{\alpha-1}) \\
(\{s_j\} \text{ is a } \delta_k\text{-net}) \qquad &\lesssim L \cdot \operatorname{diam}\{s_j, s_{j'}, s_{j''}\}^\alpha + LR_k^{1+\alpha}\lambda_k^\alpha \delta_k^{\alpha-1} \\
&= L \cdot \operatorname{diam}\{s_j, s_{j'}, s_{j''}\}^\alpha + L\delta_k^\alpha \\
(\{s_j\} \text{ is a } \delta_k\text{-net and } |\dot\rho| \sim 1) \qquad &\lesssim L \cdot \operatorname{diam}\{x_j, x_{j'}, x_{j''}\}^\alpha.
\end{aligned}$$

Since $|\dot\rho| \sim 1$, we also have that $|I_k^\ell| \lesssim R_k = (LR_k^\alpha/L)^{1/\alpha}$. Corollary 2.3 now implies that we can extend y from the set $\{x_j\}_{j\in\mathcal{J}}$ to the interval $[R_{k-1}, R_{k+1}]$ with the estimates $\|y'\|_\infty \lesssim LR_k^\alpha$

and $\|y'\|_\alpha \lesssim L$. Since $y(x_j) \lesssim LR_k^{1+\alpha}$ it also follows that $\|y\|_\infty \lesssim LR_k^{1+\alpha}$.

By construction, the function $y \circ r \circ \gamma_0$ agrees with f_0 on the net $\{s_j\}_{j\in\mathcal{J}}$:

$$(y \circ r)(\gamma_0(s_j)) = y(\rho(s_j)) = y(x_j) = f_0(s_j) \quad \text{for all } j \in \mathcal{J}.$$

We now define $f_k^\ell : S_k^\ell \to \mathbb{R}$ by

$$f_k^\ell(re^{i\phi_0(t)}) := y(r) + f_0(t) - y(\rho(t))$$

and extend f_k^ℓ to all of Ω_k^ℓ by letting it be continuous and independent of θ outside S_k^ℓ.

We have

$$f_k^\ell(\gamma_0(t)) = y(\rho(t)) - f_0(t) - y(\rho(t)) = f_0(t) \quad \text{for all} \quad t \in I_k^\ell,$$

and f_k^ℓ is $C^{1,\alpha}$ in r with the same estimates as y; that is, $\left\|\partial_r f_k^\ell\right\|_\infty \lesssim LR_k^\alpha$ and $\left\|\partial_r f_k^\ell\right\|_\alpha \lesssim L$.

To see that $\left\|f_k^\ell\right\|_\infty \lesssim LR_k^{1+\alpha}$, note that it suffices to prove this on the sector S_k^ℓ. Each point on S_k^ℓ is joined to a point on γ_0 by a radial line segment of length $\lesssim R_k$, and $|f_0| \lesssim LR_k^{1+\alpha}$ on I_k^ℓ by (60), so

$$\left\|f_k^\ell\right\|_\infty \lesssim \left\|f_0|_{I_k^\ell}\right\|_\infty + R_k \cdot \left\|\partial_r f_k^\ell\right\|_\infty \lesssim LR_k^{1+\alpha} + R_k \cdot LR_k^\alpha \lesssim LR_k^{1+\alpha}.$$

It remains to estimate $\left\|f_k^\ell\right\|_\alpha$. Here, again, we may restrict our attention to S_k^ℓ.

Fix $r \in [R_{k-1}, R_{k+1}]$; we now prove that

$$\left\|f_k^\ell(re^{i\phi_0(\cdot)})\right\|_\alpha \lesssim L\delta_k.$$

Let $t, t' \in I_k^\ell$, and assume first that $|t - t'| \le \delta_k$. Take $s_j \ne s_{j'}$ with $\operatorname{diam}\{t, t', s_j, s_{j'}\} \le 2\delta_k$. Then:

$$\begin{aligned}
|f_k^\ell(re^{i\phi_0(t)}) - f_k^\ell(re^{i\phi_0(t')})| &= |y(r) - f_0(t) - y(\rho(t)) - y(r) + f_0(t') + y(\rho(t'))| \\
&= |f_0(t) - f_0(t') - y(\rho(t)) + y(\rho(t'))| \\
&= |\rho(t) - \rho(t')| \left| \frac{f_0(t) - f_0(t')}{\rho(t) - \rho(t')} - \frac{y(\rho(t)) - y(\rho(t'))}{\rho(t) - \rho(t')} \right| \\
\begin{matrix}\text{(because } \|y'\|_\alpha \lesssim L \text{ and} \\ \operatorname{diam}\{\rho(t), \rho(t'), x_j, x_{j'}\} \lesssim \delta_k)\end{matrix} \quad &\lesssim |\rho(t) - \rho(t')| \left(\left| \frac{f_0(t) - f_0(t')}{\rho(t) - \rho(t')} - \frac{y(x_j) - y(x_{j'})}{x_j - x_{j'}} \right| + L\delta_k^\alpha \right) \\
&= |\rho(t) - \rho(t')| \left(\left| \frac{f_0(t) - f_0(t')}{\rho(t) - \rho(t')} - \frac{f_0(s_j) - f_0(s_{j'}))}{\rho(s_j) - \rho(s_{j'})} \right| + L\delta_k^\alpha \right) \\
\text{(by (63)), since } |\dot\rho| \sim 1) \quad &\lesssim |\rho(t) - \rho(t')| \left(L\delta_k^\alpha + LR_k^{1+\alpha}\lambda_k^\alpha(|t - t'|^{\alpha-1} + |s_j - s_{j'}|^{\alpha-1}) \right) \\
(|\dot\rho| \sim 1) \quad &\lesssim L\delta_k^\alpha |t - t'| + L\delta_k |t - t'|^\alpha \\
(|t - t'| \le \delta_k) \quad &\lesssim L\delta_k |t - t'|^\alpha.
\end{aligned}$$

(there is a slight technicality here. If $|I_k^\ell| \le \delta_k$, then the net $\{s_j\}$ consists of a single point, and the above analysis fails. To avoid this situation, in such a case we define the extension as follows: let s_1 and s_2 be the endpoints of I_k^ℓ, and consider two cases: $|I_k^\ell| \le b_k$ or $|I_k^\ell| > b_k$, where $b_k := (R_k \lambda_k^\alpha)^{\frac{1}{1-\alpha}}$. In the former case we let y be the linear interpolant

$$y(x) = f_0(s_1) + (f_0(s_2) - f_0(s_1))(x - \rho(s_1))/(\rho(s_2) - \rho(s_1)), \quad x \in \rho(I_k^\ell)$$

and proceed as above. In the latter case we define the extension as in the case where $k = \{0, 1\}$ and $|\rho'(0)| \le 1/2$. The calculations are similar to the ones above, and are omitted. The main point is that each of the two cases corresponds to another term in the right hand side of (61) being dominant).

If $|t - t'| \ge \delta_k$, then let $s_j \ne s_{j'}$ satisfy $|t - s_j|, |t' - s_{j'}| \le \delta_k$.

Since $y(\rho(s_j)) - f_0(s_j) = 0$ for all $j \in \mathcal{J}$,

$$\begin{aligned}
|f_k^\ell(re^{i\phi_0(t)}) - f_k^\ell(re^{i\phi_0(t')})| &= |f_0(t) - y(\rho(t)) - f_0(t') + y(\rho(t'))| \\
&\le |f_0(t) - f_0(s_j) + y(\rho(s_j)) - y(\rho(t))| \\
&+ |f_0(t') - f_0(s_{j'}) + y(\rho(s_{j'})) - y(\rho(t'))| \\
&\lesssim L\delta_k(|t - s_j|^\alpha + |t' - s_{j'}|^\alpha) \\
&\lesssim L\delta_k^{1+\alpha} \\
&\lesssim L\delta_k |t - t'|^\alpha.
\end{aligned}$$

where in the third passage we have applied the previous calculation to the pairs t, s_j and $t', s_{j'}$.

Thus $\left\|f_k^\ell(re^{i\phi_0(\cdot)})\right\|_\alpha \lesssim L\delta_k$ on I_k^ℓ for all $r \in [R_{k-1}, R_{k+1}]$.

Let $z_1, z_2 \in S_k^\ell$, and write $z_j = r_j e^{i\phi_0(t_j)}$ for some $r_j \in [R_{k-1}, R_{k+1}]$ and $t_j \in I_k^\ell$. Then

$$\begin{aligned}
|f_k^\ell(z_1) - f_k^\ell(z_2)| &\le |f_k^\ell(r_1 e^{i\phi_0(t_1)}) - f_k^\ell(r_1 e^{i\phi_0(t_2)})| \\
&+ |f_k^\ell(r_1 e^{i\phi_0(t_2)}) - f_k^\ell(r_2 e^{i\phi_0(t_2)})| \\
(\left\|f_k^\ell(re^{i\phi_0(\cdot)})\right\|_\alpha \lesssim L\delta_k \text{ and } \left\|\partial_r f_k^\ell\right\|_\infty \lesssim LR_k^\alpha) \qquad &\lesssim L\delta_k |t_1 - t_2|^\alpha + LR_k^\alpha |r_1 - r_2| \\
(\dot\phi_0 \sim \lambda_k \text{ on } I_k^\ell) \qquad &\lesssim L\delta_k \cdot \lambda_k^{-\alpha} |\phi_0(t_1) - \phi_0(t_2)|^\alpha + LR_k^\alpha \cdot |r_1 - r_2| \\
(\delta_k = R_k^{1+\alpha}\lambda_k^\alpha \text{ and } |r_1 - r_2| \lesssim R_k) \qquad &\lesssim LR_k \cdot (R_k^\alpha |\phi_0(t_1) - \phi_0(t_2)|^\alpha + |r_1 - r_2|^\alpha) \\
&\sim LR_k |z_1 - z_2|^\alpha.
\end{aligned}$$

This finishes the proof that $\left\|f_k^\ell\right\|_\alpha \lesssim LR_k$, and the claim is proved. $\square$

Claim 4 provides us with local extensions $f_k^\ell : \Omega_m^\ell \to \mathbb{R}$. We can now use the partition of unity $\{\varphi_k^\ell\}$ from Claim 2 to define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f = \sum_{k,\ell} \varphi_k^\ell f_k^\ell. \tag{89}$$

Then f is well defined, because each φ_k^ℓ is supported on Ω_k^ℓ.

Since $\sum_{k,\ell} \varphi_k^\ell \equiv 1$ on Ω^0, and $f_k^\ell \circ \gamma_0 = f_0$ on each I_k^ℓ, we have that $f \circ \gamma_0 = f_0$ (recall that $\gamma_0 \subseteq \Omega^0$).

The function f is C^1 in r because each f_k^ℓ is, and

$$\partial_r f = \sum_{k,\ell} \partial_r \varphi_k^\ell f_k^\ell + \varphi_k^\ell \partial_r f_k^\ell. \tag{90}$$

By our estimates (84) and (88) on φ_k^ℓ and f_k^ℓ, and the local finiteness of the sums in (89) and (90),

$$|f(re^{i\theta})| \lesssim Lr^{1+\alpha} \quad \text{and} \quad |\partial_r f(re^{i\theta})| \lesssim Lr^\alpha \quad \text{for all } re^{i\theta} \in \mathbb{R}^2. \tag{91}$$

Thus, since $LR^\alpha = H^{1+\alpha/2}R^\alpha = H(HR^2)^{\alpha/2} \lesssim H$, condition (74) is satisfied by f. It follows from (87) that $f|_{B_{m/2}(0)} \equiv 0$.

To finish the proof it remains to show that f satisfies (75), i.e. that $\|\Gamma\|_\alpha \lesssim L$, where

$$\Gamma := f^2 + 2f \cot_{K_0} r + \partial_r f.$$

Using (3), we can also conclude from (85), (88), (89) and (90), that

$$\left\| f|_{\Omega_k^\ell} \right\|_\alpha \lesssim LR_k \qquad \text{and} \qquad \left\| \partial_r f|_{\Omega_k^\ell} \right\|_\alpha \lesssim L \qquad \text{for all } k, \ell. \tag{92}$$

Putting together (91),(92) and the estimates

$$\left\| \cot_{K_0} r|_{\Omega_k^\ell} \right\|_\infty \lesssim R_k^{-1} \quad \text{and} \quad \left\| \cot_{K_0} r|_{\Omega_k^\ell} \right\|_\alpha \lesssim R_k^{-1-\alpha},$$

one sees that for each k, ℓ, the following holds on Ω_k^ℓ:

$$\begin{aligned}
\|\Gamma\|_\alpha &\lesssim \|f\|_\alpha \|f\|_\infty + \|f\|_\alpha \|\cot_{K_0} r\|_\infty + \|f\|_\infty \|\cot_{K_0} r\|_\alpha + \|\partial_r f\|_\alpha \\
&\lesssim LR_k \cdot LR_k^{1+\alpha} + LR_k \cdot R_k^{-1} + LR_k^{1+\alpha} R_k^{-1-\alpha} + L \\
&\lesssim L(HR^2)^{1+\alpha/2} + L \\
&\lesssim L.
\end{aligned}$$

Our goal is to make this estimate global.

Given any $z, z' \in \mathbb{R}^2$, we would like to show that $|\Gamma(z) - \Gamma(z')| \lesssim L|z - z'|^\alpha$.

If $|z'|/|z| \leq 16$, then we can reduce to the local estimates by adding at most 4 intermediate points; so we assume henceforth $|z'|/|z| > 16$. Let k, ℓ, k', ℓ' be indices such that $z \in \Omega_k^\ell$ and $z' \in \Omega_{k'}^{\ell'}$. Then $k' \geq k+3$. For ease of notation let $\Omega = \Omega_k^\ell$ and $\Omega' = \Omega_{k'}^{\ell'}$.

Let w be any point in $\gamma_0 \cap \Omega$. If $\gamma_0 \cap \Omega = \varnothing$, then instead choose $w \in \Omega$ such that $f \equiv 0$ in a neighbourhood of w. Such w must exist in this case because of (87). Define w' similarly.

We have already shown that $\|\Gamma\|_\alpha \lesssim L$ on Ω and Ω', so it suffices to prove $|\Gamma(w') - \Gamma(w)| \lesssim L|w'-w|^\alpha$. But $|w'-w| \gtrsim R_{k'} - R_k \sim R_{k'}$ because $R_{k'} \geq 8R_k$. Hence we just need to show that

$$|\Gamma(w') - \Gamma(w)| \lesssim LR_{k'}^\alpha. \tag{93}$$

In case both $\gamma_0 \cap \Omega = \varnothing$ and $\gamma_0 \cap \Omega' = \varnothing$, the whole expression on the left of (93) vanishes. If only one of the sets is empty, say $\gamma_0 \cap \Omega'$, then $\Gamma(w') = 0$, and so by (91),

$$|\Gamma(w') - \Gamma(w)| = |\Gamma(w)| \lesssim L^2 R_k^{2+2\alpha} + LR_k^{1+\alpha} \cdot R_k^{-1} + LR_k^\alpha \lesssim LR_k^\alpha \leq LR_{k'}^\alpha$$

proving (93) in this case.

So we may assume both sets to be nonempty, so that there are $t \in I_k^\ell, t' \in I_{k'}^{\ell'}$ such that $w = \gamma_0(t) = \rho(t)e^{i\phi_0(t)}$ and $w' = \gamma_0(t') = \rho(t')e^{i\phi_0(t')}$, and therefore $f(w) = f_0(t)$ and $f(w') = f_0(t')$. We may also assume that $0 \leq t < t'$ (if $t < 0 < t'$ or $t' < 0 < t$ then add an intermediate point at $\gamma_0(0)$; the only other possibility is $t' < t \leq 0$ and can be dealt with similarly) and that

$$\ell = \ell' = +$$

(by construction, $I_k^- \cap [0,\infty) = \varnothing$ unless $k \in \{0,1\}$, and in these cases $I_k^- \cap [0,\infty) \subseteq I_k^+$).

Our intention is to use (62) of Proposition 3.11. Since $\rho(t') = |w'| \geq R_{k'}/2 \geq 4R_k \geq 2|w| = 2\rho(t)$, there are $t < s \leq s' < t'$ such that $\rho(s) = 2\rho(t)$ and $\rho(s') = \rho(t')/2$. It follows that

$$|\rho(t) - \rho(s)| \sim \rho(t) \sim R_k \quad \text{ and } \quad |\rho(t') - \rho(s')| \sim \rho(t') \sim R_{k'}. \tag{94}$$

Using (94) and the local estimates (92), we have

$$\begin{aligned}
\frac{f_0(t) - f_0(s)}{\rho(t) - \rho(s)} &= \frac{f(w) - f(\gamma_0(s))}{\rho(t) - \rho(s)} \\
&= \frac{f(\rho(t)e^{i\phi_0(t)}) - f(\rho(t)e^{i\phi_0(s)})}{\rho(t) - \rho(s)} + \frac{f(\rho(t)e^{i\phi_0(s)}) - f(\rho(s)e^{i\phi_0(s)})}{\rho(t) - \rho(s)} \\
&= O\left(\frac{LR_k \cdot R_k^\alpha}{R_k}\right) + \partial_r f(w) + O(LR_k^\alpha) \\
&= \partial_r f(w) + O(LR_k^\alpha)
\end{aligned}$$

and similarly

$$\frac{f_0(t') - f_0(s')}{\rho(t') - \rho(s')} = \partial_r f(w') + O(LR_{k'}^\alpha).$$

Since ρ does not vary by more than a factor of 4 on $[t,s]$ and $[s',t']$, we may now use (62) to obtain

$$\begin{aligned}
&|\partial_r f(w) + 2f(w)\cot_{K_0}(|w|) - \partial_r f(w') - 2f(w')\cot_{K_0}(|w'|)| \\
&= \left| \frac{f_0(t) - f_0(s)}{\rho(t) - \rho(s)} + 2f_0(t)\cot_{K_0}\rho(t) - \frac{f_0(t') - f_0(s')}{\rho(t') - \rho(s')} - 2f_0(t')\cot_{K_0}\rho(t') \right| + O(LR_{k'}^{\alpha}) \\
&\lesssim L \cdot |t' - t|^{\alpha} + LR_k^{1+\alpha}\lambda_k^{\alpha}|t-s|^{\alpha}/|\rho(t) - \rho(s)| + LR_{k'}^{1+\alpha}\lambda_{k'}^{\alpha}|t'-s'|^{\alpha}/|\rho(t') - \rho(s')| + LR_{k'}^{\alpha} \\
&\lesssim LR_{k'}^{\alpha} + LR_k^{1+\alpha}\lambda_k^{\alpha} \cdot R_k^{\alpha}/R_k + LR_{k'}^{1+\alpha}\lambda_{k'}^{\alpha} \cdot R_{k'}^{\alpha}/R_{k'} + LR_{k'}^{\alpha} \\
&\lesssim LR_{k'}^{\alpha}.
\end{aligned}$$

(in the third passage we have used (46) and (94), and the last inequality holds true because $\dot{\phi}_0 \lesssim 1/\sin_{K_0}\rho \sim 1/\rho$ and therefore $\lambda_k \lesssim R_k^{-1}$ and $\lambda_{k'} \lesssim R_{k'}^{-1}$).
Thus to prove (93) it suffices to show that $|f(w)^2 - f(w')^2| \lesssim LR_{k'}^{\alpha}$. By (91) and (92), we know that $\left\| f^2|_{\Omega} \right\|_{\alpha} \lesssim LR_k^{1+\alpha} \cdot LR_k \leq L(HR^2)^{1+\alpha/2} \lesssim L$, so

$$|f(w)^2 - f(w')^2| = |f(\tilde{w})^2 - f(w')^2| + O(LR_k^{\alpha}),$$

where $\tilde{w}$ is any point in Ω lying on the same radial ray as w' (such a point exists because $\ell = \ell'$). By (91), on $B_{R_{k'}}(0)$ we have $|\partial_r(f^2)| \lesssim LR_{k'}^{1+\alpha} \cdot LR_{k'}^{\alpha} = L^2R_{k'}^{1+2\alpha}$, and therefore

$$|f(\tilde{w})^2 - f(w')^2| \lesssim L^2R_{k'}^{1+2\alpha}(|w'| - |\tilde{w}|) \lesssim L^2R_{k'}^{2+2\alpha} \lesssim LR_{k'}^{\alpha}.$$

This finishes the proof of (93) and of the proposition. $\square$

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