# The slicing problem by Bourgain

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**Abstract**

In the context of his work on maximal functions in the 1980s, Jean Bourgain came across the following geometric question: Is there $c > 0$ such that for any dimension $n$ and any convex body $K \subseteq \mathbb{R}^n$ of volume one, there exists a hyperplane $H$ such that the $(n-1)$-dimensional volume of $K \cap H$ is at least $c$? This innocent and seemingly obvious question (which remains unanswered!) has established a new direction in high-dimensional geometry. It has emerged as an "engine" that inspired the discovery of many deep results and unexpected connections. Here we provide a survey of these developments, including many of Bourgain's results.

## Foreword by V. Milman: Some historical reminiscences

In August 1984 I visited Jean Bourgain for a couple of days in Brussels where he worked at the time. We intended to spend a year together at IHES, Paris (during the 1984-85 academic year). Jean was preparing his trip to Leningrad (now St. Petersburg) in September, and I wanted to see him before he left (I had many colleagues and friends there). When he brought me to the train station on my way back to Paris he proposed the following question: "Let $K$ be a centrally-symmetric convex body in $\mathbb{R}^n$; let $Vol(K) = 1$. Does there exist $u \in SL_n$ such that all hyperplane central sections of $u(K)$ will have around the same $(n-1)$-dimensional volume?"

To say more precisely for the non-experts: Is there a universal number $C$ (independent of anything, including the dimension $n$) such that for every $(n-1)$-dimensional subspace H the following holds:

$$\frac{1}{C} a \leq Vol_{n-1}(u(K) \cap H) \leq Ca? \tag{1}$$

Here $C$ means a universal constant, as usual. Jean added that the question had arisen in his work on maximal functions.

During the train trip back to Paris, I suddenly realised that some of our recent joint observations with Gromov (see Lemma 1 in [21]) on some consequences of the Brunn-Minkowski inequality may lead to the answer (see Lemma 2 in [21]).

I informed Jean about this upon arriving to Paris. This result of Jean on maximal functions was published in 1986 in Amer. J. Math [21]. I should note here that only a few years later did we learn about Hensley's paper [45] where isotropic position and Lemma 2 were already considered for problems of analytic number theory.

However going back to 1984/85, Jean asked me a few months later whether the number "a" in (1), which already depended on the body $K$, is actually uniformly independent of the dimension $n$, bounded from below? He knew how to prove that it is bounded from above (by its value for the euclidean ball of volume 1). "I don't need this information for my paper" - Jean said - "this number is cancelled in computations; but, I feel I should know it if I need to use it". Jean thought that I may see some geometric point from which it will easily follow, and indeed I thought at first it would be easy: If "a" for some K is extremely small this means that all central hyperplane sections have a very small volume for a body of volume 1. It looks very counter-intuitive, however, it is not yet proved, 35-years later.

The question appears in the "Remark" after Lemma 2 in [21]. Shortly after Alain Pajor and I produced some advanced study of the isotropic position and this problem [82] and demonstrated some equivalent problems. In the meantime Jean proved a lower bound of $\frac{1}{n^{0.25}\log n}$ (see [22]) and 20 years later Boaz Klartag improved upon it to obtain the better lower bound, of $\frac{1}{n^{0.25}}$ by a different approach.

It is surprising and striking how far-reaching and how consequential this problem has become. We will demonstrate this in this survey.

Jean had revisited the aforementioned problem many times, from his 1986 congress talk in Berkeley [23] to his works [24, 25, 26] in later years.

I was told once by Jean that he had spent more time on this problem and had dedicated more efforts to it than to any other problem he had ever worked on. A few months before his passing, Jean wrote to me again, inquiring about any recent progress. He wanted to know the answer before he would leave.

# 1 Introduction

The classical Busemann-Petty problem, which is closely related to the slicing problem, reads as follows: Let $K$ and $T$ be centrally-symmetric convex bodies in $\mathbb{R}^n$ and $Vol_{n-1}(K \cap \theta^\perp) \leq Vol_{n-1}(T \cap \theta^\perp)$ for all $\theta \in S^{n-1} = \{x \in \mathbb{R}^n\,;\,|x| = 1\}$. Does it follow that $Vol_n K \leq Vol_n T$? Here $\theta^\perp = \{x \in \mathbb{R}^n\,;\,\langle x, \theta\rangle = 0\}$ is the hyperplane orthogonal to $\theta$. This is Problem 1 in [30], where it is shown that the answer is affirmative when $K$ is an ellipsoid.

For general $K$ and $T$, the answer to the Busemann-Petty question turned out to be "yes" for dimensions $n \leq 4$. However, surprisingly, the intuition breaks and for dimensions $n \geq 5$ it does not hold (see the book by Gardner [39] and Koldobsky [63] for history and references). In fact, the intuition in high dimension fails so miserably and the computations are so difficult, that the counter-example in a sufficiently high dimension is simple to describe: Just take $K$ for the cube and $T$ for a Euclidean ball, as shown by K. Ball [5, 7]. Indeed, for $n \geq 10$, $K = [-1/2, 1/2]^n$ and for $T$ a Euclidean ball of volume $9/10$ centered at the origin in $\mathbb{R}^n$,

$$Vol_{n-1}(K \cap \theta^\perp) \leq \sqrt{2} < 0.9\sqrt{e} \approx \frac{\Gamma\left(\frac{n}{2}+1\right)^{(n-1)/n}}{\Gamma\left(\frac{n+1}{2}\right)} \cdot 0.9^{(n-1)/n} = Vol_{n-1}(T \cap \theta^\perp).$$

In order to overcome this obstruction, a question that looks more sensible to us today is the following:

**Question 1.1.** *Let $K, T \subseteq \mathbb{R}^n$ be centrally-symmetric convex bodies such that $Vol_{n-1}(K \cap \theta^\perp) \leq Vol_{n-1}(T \cap \theta^\perp)$ for all $\theta \in S^{n-1}$. Does it follow that $Vol_n K \leq C \cdot Vol_n T$ for some universal constant $C$?*

In particular, is it true that there exists a constant $c > 0$ (perhaps, very small) such that for every dimention $n$ and for any convex centrally-symmetric body $K \subset \mathbb{R}^n$, if $Vol_{n-1}(K \cap \theta^\perp) < c$ for every $\theta \in S^{n-1}$ then $Vol_n K \leq 1$? This is the essence of the slicing problem, sometimes referred to as the hyperplane conjecture. The assumption of central-symmetry is not very essential (see e.g. [50]), and Question 1.1 is in fact equivalent to the following:

**Question 1.2.** *Let $K \subset \mathbb{R}^n$ be a convex set of volume one. Does there exist a hyperplane $H \subset \mathbb{R}^n$, such that*

$$Vol_{n-1}(K \cap H) > 1/C$$

*for some universal constant $C > 0$, independent of the dimension $n$?*

This is known as *Bourgain's slicing problem.* It is not just a nice riddle; A positive answer would have several consequences in convex geometry. In fact, in some sense the hyperplane conjecture is the "opening gate" to a better understanding of uniform measures in high dimensions. It is simpler and it is implied by the *thin shell problem* of Anttila, Ball and Perisinnaki [2] and by the conjecture of Kannan, Lovász and Simonovits (KLS) on the *isoperimetric inequality* in convex sets [48], which we discuss below. In fact, the slicing problem appears virtually in any study of the uniform measure on convex sets in high dimension. Here is a sample of entirely equivalent formulations of Question 1.2 mostly taken from [82]. We write $A \simeq B$ if $cA \leq B \leq CA$ for some universal constants $c, C > 0$.

1. Let $K \subset \mathbb{R}^n$ be a convex body (i.e., a non-empty, bounded, open convex set). Does there exist an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$, with $Vol_n \mathcal{E} = Vol_n K$, such that $Vol_n(K \cap C\mathcal{E})/Vol_n(K) \geq 1/2$, where $C > 0$ is a universal constant?

2. Let $K \subset \mathbb{R}^n$ be a convex body. Select $n+2$ independent, random points according to the uniform measure on $K$. Let $p(K)$ be the probability that these $n+2$ points are the vertices of a convex polytope. Is it true that $(1 - p(K))^{1/n} \simeq 1/\sqrt{n}$? This question is known as the *Sylvester problem.*

3. Let $K \subset \mathbb{R}^n$ be a convex body of volume one. Is it true that there exists a volume-preserving, affine map $T : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$Vol_{n-1}(T(K) \cap H) \simeq 1$$

for any hyperplane through the origin $H \subset \mathbb{R}^n$?

4. Let $K \subset \mathbb{R}^n$ be a convex body. Denote by $\mathrm{Cov}(K)$ the covariance matrix of a random vector that is distributed uniformly in $K$. In Bourgain's notation, the *isotropic constant* of $K$ is defined as

$$L_K = \frac{\det(\mathrm{Cov}(K))^{\frac{1}{2n}}}{Vol_n(K)^{\frac{1}{n}}}. \tag{2}$$

The isotropic constant is invariant under invertible, affine transformations. It is known that $(2\pi e)^{-1/2} + o(1) \leq L_K$ for any convex body in $\mathbb{R}^n$ (the minimizer is the Euclidean ball or an ellipsoid). Is it true that $L_K < C$, for some universal constant $C > 0$, independent of the dimension?

However, let us now take a step back. The slicing problem is part of the study of measures in a high-dimensional space. One of the earliest results on probability distributions in high-dimensional spaces is of course the classical *central limit theorem*: The sum of independent random variables is approximately Gaussian when the number of variables approaches infinity, under quite general assumptions. In other words, for large $n$, suppose that $f_1, \dots, f_n$ are probability densities on the real line of mean zero and variance one, satisfying certain mild regularity conditions. Then the integral

$$\int_{H_t} \prod_{i=1}^{n} f_i(x_i)$$

with $H_t = \{x \in \mathbb{R}^n\, ; \, \sum_{i=1}^n x_i = t\sqrt{n}\}$, is approximately the Gaussian density $e^{-t^2/2}/\sqrt{2\pi}$. Therefore the value of this integral does not depend too much on the specific form of the densities we started with, and the behavior is asymptotically *universal*. This is a marvelous effect of universality in high dimensions, indicating that when viewed correctly, high-dimensional measures exhibit regularity and order rather than incomprehensible complications.

Another example for regularity in high dimensions is Dvoretzky's theorem, which asserts that any high-dimensional convex body has nearly Euclidean sections of a large dimension, see [81] and references therein for background. Thus the symmetries of the Euclidean ball appear, even though we made only minimal assumptions: only convexity and the high dimension. The central limit theorem and Dvoretzky's theorem are high-dimensional effects that lack clear analogs in low dimensions.

As it turns out, there are motifs in high-dimensional geometry which seem to compensate for the difficulties that arise from high dimensionality. One of these motifs is the *concentration of measure phenomenon*. Quite unexpectedly, a scalar Lipschitz function on a high-dimensional space behaves in many cases as if it were a constant function. For example, if we sample five random points from the $n$-dimensional unit sphere, for large $n$, and substitute them into a 1-Lipschitz function, then we will almost certainly obtain five numbers that are very close to one another. This phenomenon is reminiscent of the well-known geometric property that in the high-dimensional Euclidean sphere, "most of the mass is close to the equator, for any equator". This geometric property, which follows from the isoperimetric inequality, is unthinkable in, say, three dimensions. Since the second-named author's proof of Dvoretkzy's theorem in the 1970s, the concentration of measure has become a major tool in high-dimensional analysis.

It was a big surprise in the 1970s and 1980s that the asymptotic behavior (i.e., when the dimension increases) of high-dimensional normed spaces is "well organized" and not chaotic, as one could expect from the intuition which was based, perhaps, on exponential growth of entropy (=covering) for $n$-dimensional spaces. However, the concentration of measure balances the exponentially high entropy of $n$-dimensional spaces and leads to a "regularity" in high dimension, limiting the "geometric diversity" in high dimensions. The absolute constants involved in the analysis may balance the rate of exponential decay (coming from concentration) with the rate of exponential expansion (coming from covering/entropy). Surprisingly, both exponents have "roughly" the same order of decay via expansion by dimension and only a constant factor is needed in order to compensate and obtain a regularity result in high dimensions. The constant factors in the exponent are needed for compensation, and this explains the "isomorphic" nature of the results, the fact that absolute constants appear in their formulation.

So, a new intuition had to be created roughly four decades ago, and it was built upon results which showed very regular patterns. Today we may state that these results were observed roughly in two different forms:

(a) Geometric and structural results (e.g., Dvoretzky's theorem, Quotient of Subspace Theorem, Ramsey's theorem in Combinatorics).

(b) The uniform measure distribution (volume behavior) in high-dimensional convex bodies.

In both of these forms there is striking regularity and almost no pathology when the dimension increases. Bourgain's slicing problem had a major influence on (b), and the entire direction actually stemmed from his conjecture and his work. The Bourgain-Milman inequality [27] is one of the results from that period of time that is closest to a bridge between (a) and (b).

The spatial arrangement of volume due to the geometry of $\mathbb{R}^n$, for large $n$, imposes rigidity on convex sets and convexity-related measures. Convexity is one of the ways in which one may harness the concentration of measure phenomenon in order to formulate clean, non-trivial theorems. The Brunn-Minkowski inequality from the end of the $19^{th}$ century states that for any non-empty Borel sets $A, B \subseteq \mathbb{R}^n$,

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}, \tag{3}$$

where $A + B = \{x + y\,; x \in A, y \in B\}$ and $|A|$ is the volume of the set $A$. The Brunn-Minkowski inequality is a close relative of the isoperimetric inequality, and equality holds in (3) essentially only when $A$ and $B$ are congruent convex bodies. In addition to (1) above, let us mention another consequence of the Brunn-Minkowski theory: The reverse Hölder inequalities, proven by Berwald [13] and Borell (see [82] or Borell's papers [17, 18, 20]). For any convex body $K \subseteq \mathbb{R}^n$, a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ and $p, q > 0$,

$$\left( \int_K |f(x)|^p \frac{dx}{|K|} \right)^{1/p} \leq C \left( \int_K |f(x)|^q \frac{dx}{|K|} \right)^{1/q} \tag{4}$$

where $C = C_{p,q} > 0$ depends solely on $p, q$ and neither on $K$ nor on the dimension $n$. This amusing property of convex domains goes beyond linear functionals. Suppose now that $f$ :

$\mathbb{R}^n \to \mathbb{R}$ is an arbitrary polynomial of degree at most $d$. Bourgain proved in his paper [22] that (4) holds true in this case, with the constant $C$ depending only on $p, q$ and $d$, and not on the convex body or the dimension. These results serve as evidence for the general hypothesis, that in many respects the uniform measure on a high-dimensional convex body resembles a Gaussian measure.

In the same paper [22] Bourgain proved that the constant $C$ from Question 1.2 (or Question 1.1) may be replaced by $Cn^{0.25} \log n$. The logarithmic factor was later removed by the first-named author in [51]. We proceed with a more detailed account of the development of the study of the regularity of high-dimensional convexity-related measures, and the major influence that Jean Bourgain had on this development.

# 2 The isotropic position

The covariance matrix $\mathrm{Cov}(K) = (\mathrm{Cov}_{ij}(K))_{i,j=1,\dots,n}$ of a convex body $K \subset \mathbb{R}^n$ is given by

$$\mathrm{Cov}_{ij}(K) = \int_K x_i x_j \frac{dx}{Vol_n(K)} - \int_K x_i \frac{dx}{Vol_n(K)} \int_K x_j \frac{dx}{Vol_n(K)}. \tag{5}$$

When $Vol_n(K) = 1$, its isotropic constant satisfies $L_K^{2n} = \det \mathrm{Cov}(K)$, according to (2).

A convex body $K \subset \mathbb{R}^n$ of volume one is *isotropic* (or in *isotropic position*) if its barycenter lies at the origin and its covariance matrix is a scalar matrix. Any convex body $K \subset \mathbb{R}^n$ can be transformed into an isotropic convex body by applying an affine transformation of the form $Tx = \alpha \mathrm{Cov}(K)^{-1/2} x + v$ for appropriate $\alpha > 0$ and $v \in \mathbb{R}^n$. It follows from (2) that when $K$ is isotropic,

$$\mathrm{Cov}(K) = L_K^2 \cdot \mathrm{Id}. \tag{6}$$

In other words, when $K$ is isotropic, for any $\theta \in S^{n-1}$,

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2. \tag{7}$$

It follows from (6) that for any convex body $K \subseteq \mathbb{R}^n$ with $Vol_n(K) = 1$ and for any invertible linear map $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$L_K^2 \leq \frac{1}{n |\det T|^{2/n}} \int_K |Tx|^2 dx. \tag{8}$$

Indeed, it suffices to prove (8) in the case where $K$ is isotropic. In this case the right-hand side of (8) equals $L_K^2 \cdot \mathrm{Trace}[T^*T]/(n|\det T|^{2/n}) \geq L_K^2$, by the arithmetic-geometric means inequality.

When $K \subseteq \mathbb{R}^n$ is a convex body of volume one with barycenter at the origin, an alternative definition of $L_K^2$ is that it is the minimum of the right-hand side of (8) over all linear transformations of determinant one. Consequently, for such $K \subseteq \mathbb{R}^n$ there exists a linear map $T$ with $\det(T) = 1$ and

$$L_K^2 = \frac{1}{n} \int_K |Tx|^2 dx = \frac{1}{n} \int_0^\infty Vol_n(K \setminus \sqrt{s} T^{-1}(B^n)) ds \geq \frac{1}{n} \int_0^{\kappa_n^{-2/n}} \left[1 - \kappa_n s^{n/2}\right] ds = L_{B^n}^2,$$

where $B^n = \{x \in \mathbb{R}^n ; |x| = 1\}$ is the Euclidean unit ball and $\kappa_n = Vol_n(B^n)$. Since $L_{B^n}^2 = 1/(2\pi e) + o(1)$, we conclude that $L_K > c$ for some universal constant $c > 0$.

If $K \subset \mathbb{R}^n$ is an isotropic convex body, then for any two hyperplanes $H_1, H_2 \subseteq \mathbb{R}^n$ through the origin,

$$\frac{Vol_{n-1}(K \cap H_1)}{Vol_{n-1}(K \cap H_2)} \leq C \tag{9}$$

for a universal constant $C > 0$. This was proven by Hensley [45] in the case where $K$ is centrally-symmetric, and rediscovered by the second-named author in Lemma 2 in [21]. Fradelizi [38] eliminated the assumption that $K$ is centrally-symmetric, and obtained the sharp bound $C \leq \sqrt{6}$ in (9). In order to prove (9), one fixes a unit vector $\theta \in S^{n-1}$ and denotes

$$\rho(t) = Vol_{n-1}(K \cap (t\theta + \theta^{\perp})).$$

A crucial property that follows from the Brunn-Minkowski inequality is that $\rho$ is *log-concave*, that is, the function $\log \rho$ is a concave function (which is allowed to attain the value $-\infty$). Therefore the proof of (9) boils down to the proof of the following one-dimensional inequality: For any log-concave probability density $\rho : \mathbb{R} \to [0, \infty)$ with $\int t\rho(t)dt = 0$,

$$\frac{1}{\sqrt{12}} \leq \rho(0) \cdot \sqrt{\int_{-\infty}^{\infty} t^2 \rho(t) dt} \leq \frac{1}{\sqrt{2}}. \tag{10}$$

The space of one-dimensional, log-concave probability densities of mean zero and variance one is compact in the $L^1$-topology. A compactness argument shows that an inequality such as (10) holds true with *some* numerical constants. The sharp values of the constants in (10) are due to Fradelizi [38]. Consequently, whenever $K \subseteq \mathbb{R}^n$ is an isotropic convex body, for any hyperplane $H \subseteq \mathbb{R}^n$ through the origin,

$$\frac{1}{\sqrt{12} \cdot L_K} \leq Vol_{n-1}(K \cap H) \leq \frac{1}{\sqrt{2} \cdot L_K}. \tag{11}$$

The assumption that $H$ passes through the origin is not entirely necessary for the right-hand side inequality in (11), if one is willing to increase the constant. This follows from a version of inequality (10) where $\rho(0)$ is replaced by $\sup \rho$, see Fradelizi [38]. It follows that when $K \subseteq \mathbb{R}^n$ is convex and isotropic, for any hyperplane $H \subseteq \mathbb{R}^n$,

$$Vol_n(K \cap H) \leq \frac{1}{L_K}. \tag{12}$$

From (12) we obtain the relatively trivial bound $L_K \leq C\sqrt{n}$ for the isotropic constant. Indeed, since $K$ is convex and of volume one, it cannot have a width larger than $5\sqrt{n}$ in all directions, as otherwise $K - K$ would contain a Euclidean ball of volume larger than $4^n$, in contradiction to the Rogers-Shephard inequality [85]. We recall that this inequality states that $Vol_n(K - K) \leq 4^n Vol_n(K)$ for any convex body $K \subseteq \mathbb{R}^n$. Pick a direction in which the width of $K$ is at most $5\sqrt{n}$, and use Fubini's theorem to find a hyperplane $H$ orthogonal to this direction with $Vol_n(K \cap H) \geq 1/(5\sqrt{n})$. Now (12) shows that $L_K \leq 5\sqrt{n}$.

The idea demonstrated above, of reducing statements on convex bodies to one-dimensional inequalities pertaining to log-concave functions, is a common theme in convex geometry. For example, the reverse Hölder inequality (4) may be proven by reducing matters to a one-dimensional inequality with log-concave probability densities. A log-concave function in one dimension of a finite integral decays exponentially at infinity (see, e.g. [52, Lemma 2.1]). It follows (see [82] or [80]) that for any convex body $K \subseteq \mathbb{R}^n$ of volume one, the "$\psi_1$-norm" of a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\|f\|_{\psi_1(K)} \leq C\|f\|_{L^2(K)} \tag{13}$$

where $C > 0$ is a universal constant, where $\|f\|_{L^p(K)}^p = \int_K |f|^p$ and where for $\alpha \geq 1$,

$$\|f\|_{\psi_\alpha(K)} = \inf \left\{ \lambda > 0 \, ; \, \int_K \exp(|f/\lambda|^\alpha) \leq 2 \right\}. \tag{14}$$

The $\psi_1$-norm of a function $f$ is finite if its value distribution is sub-exponential, and the $\psi_2$-norm is finite if the distribution is sub-Gaussian. The contrast between $\psi_1$-norm and $\psi_2$-norm, or between sub-exponential tail and sub-Gaussian tail, lies at the heart of Bourgain's bound $L_K \leq Cn^{1/4} \log n$. Before proceeding with Bourgain's proof, let us provide a bit of background on $\psi_2$-processes and on certain results from the local theory of Banach spaces that are related to Bourgain's proof. Suppose that $\mu$ is a probability measure on $\mathbb{R}^n$, and denote

$$A := \sup_{\theta \in S^{n-1}} \|f_\theta\|_{\psi_2(\mu)} \tag{15}$$

where $f_\theta(x) = \langle x, \theta \rangle$ and where the $\psi_2$-norm of a function $f$ with respect to the measure $\mu$ is defined analogously to (14) above. A key result by Talagrand [95, 96] continuing the work of Fernique [35] states that for any norm $\| \cdot \|$ on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \|x\| d\mu(x) \leq CA \int_{\mathbb{R}^n} \|x\| d\gamma_n(x) \tag{16}$$

where $C > 0$ is a universal constant, and where $\gamma_n$ is the standard Gaussian measure on $\mathbb{R}^n$. The proof of inequality (16) involves concepts such as majorizing measures and generic chaining.

Bounds for the Gaussian integral of a norm, as on the right-hand side of (16), are of great importance in the local theory of Banach spaces. One of the most important and useful technical statements in this direction is the following theorem, which is a combination of three results, by Lewis [68], by Figiel and Tomczak-Jaegermann [36] and, the most non-trivial, by Pisier [90, 89] (see also the appendix of [27] for a complete proof):

**Theorem 2.1.** *For any norm $\| \cdot \|$ on $\mathbb{R}^n$ there exists an invertible linear transformation $T$ such that*

$$\int_{\mathbb{R}^n} \|Tx\| d\gamma_n(x) \cdot \int_{\mathbb{R}^n} \|(T^{-1})^* y\|_* d\gamma_n(y) \leq Cn \log d_{BM}$$

*where $\| \cdot \|_*$ is the dual norm and where $d_{BM}$ is the Banach-Mazur distance of the norm $\| \cdot \|$ from a Euclidean norm. The linear map $T$ determines the so-called $\ell$-position.*

When a norm $\|\cdot\|$ on $\mathbb{R}^n$ has $K$ as its unit ball, its Banach-Mazur distance from a Euclidean norm is

$$d_{BM} = d_{BM}(K) = \inf\{rs > 0\, ;\, \exists T : \mathbb{R}^n \to \mathbb{R}^n \text{ linear, with } r^{-1}B^n \subseteq T(K) \subseteq sB^n\}. \tag{17}$$

It is well-known that $d_{BM} \leq \sqrt{n}$, see e.g. [80]. We remark in passing that **up to logarithmic factors, the slicing problem is equivalent to the question of whether the isotropic position is an $\ell$-position**, as one may show, see [26] for related results. Theorem 2.1 is a central ingredient of the original proof of the Bourgain-Milman inequality [27], which states that for any convex body $K \subseteq \mathbb{R}^n$ with barycenter at the origin,

$$Vol_n(K)Vol_n(K^\circ) \geq c^n Vol_n(B^n)^2 \geq (c'/n)^n, \tag{18}$$

where $K^\circ = \{x \in \mathbb{R}^n\, ;\, \forall y \in K,\ |\langle x, y\rangle| \leq 1\}$ is the polar body, i.e., the unit ball of the dual norm. There are by now several proofs of (18) using methods and ideas from very different parts of mathematics. Kuperberg's proof relies on topology [65], Nazarov's proof on complex analysis [84], and the proof by Giannopoulos, Paouris and Vritsiou on transportation of measure via the logarithmic Laplace transform [42] as in Section 4 below. Inequality (18) is a converse to the Santaló inequality, which states that

$$Vol_n(K)Vol_n(K^\circ) \leq Vol_n(B^n)^2,$$

and may be proven via Steiner symmetrizations [75, 76]. A clever application of Hölder's inequality shows that $\int_{\mathbb{R}^n} \|x\| d\gamma_n(x) \geq cn \cdot v^{-1/n}$ where $v > 0$ is the volume of the unit ball of the norm $\|\cdot\|$ in $\mathbb{R}^n$. It thus follows from Theorem 2.1 and from the above that for any norm $\|\cdot\|$ on $\mathbb{R}^n$, there exists a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ of determinant one such that

$$\int_{\mathbb{R}^n} \|Tx\| d\gamma_n(x) \leq Cn \log n \cdot V^{1/n} \tag{19}$$

where now $V > 0$ is the volume of the unit ball of the dual norm $\|\cdot\|_*$ in $\mathbb{R}^n$.

Let us now return to the proof of Bourgain's bound for the isotropic constant. It follows from (7) and from the Markov-Chebyshev inequality that for any isotropic convex body $K \subseteq \mathbb{R}^n$,

$$Vol_n(K \cap (\sqrt{2n}L_K B^n)) = 1 - Vol_n(K \setminus (\sqrt{2n}L_K B^n)) \geq 1 - \frac{\int_K |x|^2 dx}{2nL_K^2} = \frac{1}{2}. \tag{20}$$

The first step of the proof is to use (20) in order to show the following: When replacing $K$ with $K \cap C\sqrt{n}L_K B^n$, the isotropic constant changes by a factor of at most $C$, and the new convex body is still roughly in isotropic position (up to a constant). Thus, it suffices to bound the isotropic constant of an isotropic convex body $K \subseteq \mathbb{R}^n$ which satisfies the additional assumption that

$$K \subseteq 10\sqrt{n}L_K B^n. \tag{21}$$

One corollary of (21) is that for any $\theta \in S^{n-1}$, the linear functional $f_\theta(x) = \langle x, \theta\rangle$ satisfies

$$\|f_\theta\|_{L^\infty(K)} \leq 10\sqrt{n}L_K.$$

The $\psi_1(K)$-norm of $f_\theta$ is at most $CL_K$, according to (7) and (13) above. There is a simple interpolation inequality between the $\psi_1$-norm and the $L^\infty$-norm that yields a bound for the $\psi_2$-norm. Namely, for any $\theta \in S^{n-1}$,

$$\|f_\theta\|_{\psi_2(K)} \leq \sqrt{\|f_\theta\|_{\psi_1(K)} \cdot \|f_\theta\|_{L^\infty(K)}} \leq \sqrt{CL_K \cdot 10\sqrt{n}L_K} = C'n^{1/4}L_K. \tag{22}$$

The proof of the interpolation inequality on the left-hand side of (22) is simple, note that when $\sup|f| \leq M$,

$$\int_K e^{|f/\sqrt{\lambda M}|^2} \leq \int_K e^{|f/\lambda|} \leq 2$$

if $\lambda \geq \|f\|_{\psi_1(K)}$. The next step in Bourgain's proof is to apply (8), and conclude that for any symmetric, positive-definite linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ with $\det T = 1$,

$$nL_K^2 \leq \int_K \langle Tx, x\rangle \leq \int_K \sup_{y \in TK} |\langle x, y\rangle| = \int_K \|Tx\|dx \tag{23}$$

where $\|x\| = \sup_{y\in K}|\langle x, y\rangle|$ is a norm on $\mathbb{R}^n$ whose unit ball is polar to $K \cap (-K)$. An interesting feature of the manouver (23) is the comparison between an integral with quadratic dependence on $x$, which is reflected in the square of $L_K$ on the left-hand side, and an integral whose dependence on $x$ is not quadratic but only linear. Next, thanks to (22) we may apply the Talagrand bound (16) and conclude that

$$nL_K^2 \leq \int_K \|Tx\|dx \leq Cn^{1/4}L_K \int_{\mathbb{R}^n} \|Tx\|d\gamma_n(x). \tag{24}$$

Inequality (24) is valid for any linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ of determinant one (the assumption that $T$ is symmetric and positive-definite is immaterial due to the symmmetries of the Gaussian measure). We may now choose $T$ to be a map leading to $\ell$-position, and apply Pisier's bound in the form of inequality (19) above. This shows that

$$nL_K^2 \leq Cn^{1/4}L_K \cdot n\log n \cdot Vol_n(K \cap (-K))^{1/n} \leq C'n^{1/4}L_K \cdot n\log n.$$

This completes the proof of Bourgain's bound $L_K \leq Cn^{1/4}\log n$.

In his paper [21], Bourgain claimed a positive answer to the slicing problem in the case where $K \subseteq \mathbb{R}^n$ is *unconditional*, i.e., when for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$(x_1, \ldots, x_n) \in K \qquad \Longleftrightarrow \qquad (|x_1|, \ldots, |x_n|) \in K.$$

In this case, one may use the Loomis-Whitney inequality [70], which is valid for any compact set in $\mathbb{R}^n$:

$$Vol_n(K) \leq \prod_{i=1}^n Vol_{n-1}(Proj_{e_i^\perp}K)^{1/(n-1)}, \tag{25}$$

where $Proj_{e_i^\perp}$ is the orthogonal projection onto the hyperplane $e_i^\perp$, and $e_i$ is the $i^{th}$-standard unit vector in $\mathbb{R}^n$. When $K$ is convex and unconditional, $Proj_{e_i^\perp}K = K \cap e_i^\perp$. Hence (11) and (25) imply that $L_K \leq 1/\sqrt{2}$ when $K$ is convex and unconditional (this numerical constant may be improved). Moreover, if $K \subseteq T$ and $T$ is an unconditional convex body such that $Vol_n(T)/Vol_n(K) \leq A^n$, it is known that $L_T \leq CA$, see [82] or the recent book by Brazitikos, Giannopoulos, Valettas and Vritsiou [12], a large part of which is concerned with the slicing problem.

In addition to unconditional bodies, there are other classes of convex bodies for which an affirmative answer to the slicing problem is known. These include zonoids [86], their duals, more generally subspaces and quotients of $L_p$ spaces [8, 46, 47, 77], unit balls of Schatten class norms [64], random convex bodies [58], 2-convex bodies and other examples described in [12].

In [24] Bourgain proved the boundness of the isotropic constant for "$\psi_2$-bodies" which are convex bodies for which the $\psi_1$-estimate (13) can be upgraded to a $\psi_2$-estimate. That is, for a convex body $K \subseteq \mathbb{R}^n$ of volume one with barycenter at the origin, and for $1 \leq \alpha \leq 2$, we write $b_\alpha(K)$ to be the minimum $b > 0$ such that for any linear functional $f : \mathbb{R}^n \to \mathbb{R}$,

$$\|f\|_{\psi_\alpha(K)} \leq b\|f\|_{L^2(K)}.$$

Thus $b_1(K) \leq C$ according to (13), while for ellipsoids $\mathcal{E} \subseteq \mathbb{R}^n$ we have $b_2(\mathcal{E}) \leq C$. Bourgain proved that $L_K \leq Cb_2(K) \log b_2(K)$, and the logarithmic factor was later removed by the first-named author and E. Milman in [62] using methods related to those described in Section 4 below. The current state of the art is the bound $L_K \leq C\sqrt{b_\alpha(K)^\alpha n^{1-\alpha/2}}$ for any $1 \leq \alpha \leq 2$, from [62].

A class of convex bodies in high dimensions with favorable properties is the class of convex bodies of a *finite volume ratio*, a notion introduced by Szarek and Tomczak-Jaegermann [93, 94]. These are centrally-symmetric convex bodies $K \subseteq \mathbb{R}^n$ that contain an ellipsoid $\mathcal{E}$ such that $Vol_n(K)/Vol_n(\mathcal{E}) \leq C^n$ for a universal constant $C$. Dvoretzky's theorem assert that an arbitrary convex body in $\mathbb{R}^n$ has a $k$-dimensional section that is approximately Euclidean for $k$ of the order of magnitude of $\log n$. This estimate is dramatically improved in the class of finite-volume ratio bodies, and it was proven by Kashin [49] and then using this terminology by Szarek and Tomczak-Jaegermann [93, 94] that such bodies contain an approximately Euclidean section of dimension $k \geq cn$. In our joint work with Bourgain [25, 26], we proved that the validity of the hyperplane conjecture in the class of finite-volume ratio bodies, would imply its validity in the class of all convex bodies. This is proven via a method based on Steiner symmetrization.

We move on to describe yet another equivalent formulation of the slicing problem, which is also related to Steiner symmetrizations. Let $K \subseteq \mathbb{R}^n$ be a convex body and let $H = h^\perp \subseteq \mathbb{R}^n$ be a hyperplane, where $h \in S^{n-1}$ is a unit vector. Define the *Steiner symmetral of $K$ with respect to $H$* as the set:

$$S_H(K) = \left\{ x + th\,; x \in H, t \in \mathbb{R},\ K \cap (x + \mathbb{R}h) \neq \emptyset\ ,\ |t| \leq \frac{1}{2} Meas\{K \cap (x + \mathbb{R}h)\} \right\}$$

where $Meas$ is the one-dimensional Lebesgue measure in the line $x + \mathbb{R}h$. Steiner symmetrization preserves the volume of the set $K$ and it transforms convex sets to convex sets. Applying consecutive Steiner symmetrizations with respect to a seqeunce of hyperplanes makes $K$ "more symmetric", or "closer to a Euclidean ball". It was proven in [59] that for any convex body $K \subseteq \mathbb{R}^n$ with $Vol_n(K) = Vol_n(B^n)$ there exist $3n$ Steiner symmetrizations that transform $K$ into a convex body $\tilde{K}$ with

$$\frac{1}{C} B^n \subseteq \tilde{K} \subseteq CB^n. \tag{26}$$

When $\tilde{K}$ satisfies (26) we say that it is an "isomorphic Euclidean ball". If one applies only $n - \ell$ symmetrizations for some positive $\ell$, then there exists an $\ell$-dimensional projection of $K$ that remains unchanged in the symmetrization process. Hence at least $n - O(1)$ symmetrizations are required to arrive at an isomorphic Euclidean ball as in (26), or even at an isomorphic ellipsoid. However, there exist certain special convex bodies, such as the unit cube, that can be transformed into an isomorphic ellipsoid using fewer symmetriztions. Given a convex body $K \subset \mathbb{R}^n$ and a function $c(\varepsilon)$ $(0 < \varepsilon < 1)$, we say that "$K$ is $c(\varepsilon)$-symmetrizable" if for any $\varepsilon > 0$ there exist $\lfloor \varepsilon n \rfloor$ Steiner symmetrizations that transform $K$ into a convex body $\tilde{K}$ with

$$d_{BM}(K) < c(\varepsilon)$$

where $d_{BM}(K)$ is the Banach-Mazur distance between $K$ and a Euclidean ball, defined analogously to (17) above. For example, the cube $[-1, 1]^n$ is $c(\varepsilon)$-symmetrizable for $c(\varepsilon) = C\sqrt{|\log \varepsilon|/\varepsilon}$. Suppose that we are allowed to remove $10\%$ of the mass of a convex body $K$. Can we now apply only $\varepsilon n$ Steiner symmetrizations, and obtain a body that resembles an ellipsoid, up to a universal constant?

**Question 2.1.** *Does there exist $C, d > 0$, such that for any dimension $n$ and for any convex body $K \subseteq \mathbb{R}^n$, there exists a convex body $T \subseteq K$ with $Vol_n(T) > 0.9 \cdot Vol_n(K)$ such that $T$ is $(C/\varepsilon)^d$-symmetrizable?*

In [60] it is proven that **Question 2.1 has an affirmative answer if and only if Bourgain's hyperplane conjecture holds true**.

# 3 Distribution of volume in convex bodies

The assumption that $K$ is *convex* was used in Bourgain's proof through the $\psi_1$-bound (13), the fact that the distribution of values of a linear functional on a convex set has a uniformly subexponential tail. In fact, instead of dealing with the uniform measure on a given convex body $K \subseteq \mathbb{R}^n$ of volume one, we may consider a more general situation: Suppose that $\mu$ is a probability measure supported on $K$ whose continuous density is denoted by $f$. Assume that $\mu$ satisfies a $\psi_1$-condition: For any linear functional $f : \mathbb{R}^n \to \mathbb{R}$,

$$\|f\|_{\psi_1(\mu)} \leq A\|f\|_{L^2(\mu)},$$

for some parameter $A > 0$, where the definition of the $\|\cdot\|_{\psi_1(\mu)}$ norm is analogous to (14). A straightforward adaptation of Bourgain's argument (see the Appendix of [57]) shows that under these assumptions there exists a hyperplane $H \subseteq \mathbb{R}^n$ with

$$\int_H f \geq \frac{c(A)}{n^{1/4} \log n}, \tag{27}$$

where $c(A) > 0$ depends solely on $A$. Up to the logarithmic factor, the estimate "$n^{1/4} \log n$" in (27) is sharp, as shown by the first named author and Koldobsky [57]. Thus Bourgain's bound for the slicing problem is sharp up to logarithmic factors, if all that one takes from convexity is the uniform $\psi_1$-estimate for linear functionals.

Nevertheless, there is more to say about the distribution of volume in convex bodies beyond the sub-exponential tail of linear functionals. We begin by discussing the point of view emphasized by K. Ball [6], that connects between volume distribution of convex bodies and that of log-concave measures. Recall that a function $\rho : \mathbb{R}^n \to [0, \infty)$ is log-concave if $-\log \rho$ is a convex function, which is allowed to attain the value $+\infty$. For example, the characteristic function of a convex body, that equals one on the body and vanishes elsewhere, is a log-concave function.

A Borel measure on $\mathbb{R}^n$ is log-concave if it is supported in an affine subspace with a log-concave density in this subspace. It was proven by Borell [19] that a finite, Borel measure $\mu$ on $\mathbb{R}^n$ is a log-concave measure if and only if the following Brunn-Minkowski type inequality holds true: For any compacts $A, B \subseteq \mathbb{R}^n$ and for any $0 < \lambda < 1$,

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}. \tag{28}$$

It follows from (28) that the push-forward of a log-concave measure under a linear map, is again log-concave. In particular, by projecting the uniform measure on a convex body to a lower-dimensional subspace, we obtain a log-concave measure on this subspace. Given an integrable log-concave function $\rho : \mathbb{R}^n \to \mathbb{R}$ with $\rho(0) > 0$, define

$$K(\rho) = \left\{ x \in \mathbb{R}^n \, ; \, \int_0^\infty \rho(tx) t^n dt \geq \frac{\rho(0)}{n+1} \right\}. \tag{29}$$

As shown by K. Ball [6], the set in (29) is always convex. The convexity of $K(\rho)$ is closely related to the Busemann inequality [29], see [82].

The convex body $K(\rho)$ seems to represent rather well the volume distribution of the measure $\mu$ whose density is $\rho$. For example, if the barycenter of $\mu$ lies at the origin, so does the barycenter of $K(\rho)$. As in [51], we define the isotropic constant of a log-concave function $\rho : \mathbb{R} \to [0, \infty)$ with $0 < \int \rho < \infty$ as

$$L_\rho = \left( \frac{\sup \rho}{\int \rho} \right)^{1/n} \cdot \det \mathrm{Cov}(\rho)^{1/(2n)}$$

where the covariance matrix $\mathrm{Cov}(\rho)$ is defined analogously to (5) above. It was proven by Ball [6] (see also [51]) that we always have

$$L_\rho \simeq L_{K(\rho)}. \tag{30}$$

Thus **the slicing problem is equivalent to the problem of bounding the isotropic constant of an arbitrary log-concave measure** $\mu$ on $\mathbb{R}^n$.

A new era in the study of volume distribution in high-dimensional convex sets began in 2005 when Paouris [88] found applications of the following property: For any absolutely-continuous, log-concave probability measure $\mu$ on $\mathbb{R}^n$, there exists $\theta \in S^{n-1}$ with

$$\left( \int_{\mathbb{R}^n} |\langle x, \theta \rangle|^n d\mu(x) \right)^{1/n} \simeq \sqrt{\int_{\mathbb{R}^n} |x|^2 d\mu(x)}. \tag{31}$$

This property is proven by associating a certain convex body with the measure $\mu$ similarly to (29) above, see [61] or (better) the short argument in [1]. Note that this property of log-concave measures does not follow from the $\psi_1$-bound for linear functionals, used in Bourgain's proof (e.g., look at the random variable $X = R\Gamma$ where $R$ is a standard one-dimensional Gaussian, and $\Gamma$ is an $n$-dimensional standard Gaussian independent of $R$). Recalling that the orthogonal projection of the uniform measure of a convex body is always log-concave, we conclude the following from (31): For any isotropic convex body $K \subseteq \mathbb{R}^n$ and for any $\ell$-dimensional subspace $E \subseteq \mathbb{R}^n$, there exists a unit vector $\theta \in E$ with

$$\left( \int_K |\langle x, \theta \rangle|^\ell dx \right)^{1/\ell} \simeq \sqrt{\ell} L_K. \tag{32}$$

Note that an isotropic convex body $K$ is a $\psi_2$-body if and only if (32) holds true for any $\theta \in \mathbb{R}^n$ and for any $\ell$. Thus property (32) may be viewed as a weak form of a $\psi_2$-estimate, which is valid for any convex body. Paouris used (32) in order to prove the following large deviation estimate:

**Theorem 3.1** (Paouris [88])**.** *Let $K \subseteq \mathbb{R}^n$ be an isotropic convex body of volume one. Then for any $t > 1$,*

$$Vol_n(\{x \in K \, ; \, |x| \geq C t L_K \sqrt{n}\}) \leq e^{-t\sqrt{n}} \tag{33}$$

*where $C > 0$ is a universal constant.*

In order to appreciate Theorem 3.1, recall from (20) that

$$Vol_n(\{x \in K \, ; \, |x| \leq 2 L_K \sqrt{n}\}) \geq 1/2,$$

i.e., at least half of the mass of $K$ is located in a ball of radius $2L_K\sqrt{n}$ centered at the origin. Theorem 3.1 tells us that only a tiny fraction, just an $e^{-\sqrt{n}}$-fraction of the mass of $K$, is located outside a ball of radius $CL_K\sqrt{n}$. This effect is a precursor to the *thin shell estimate* for isotropic convex bodies that we will discuss shortly. The Paouris proof of Theorem 3.1 applies the second-named author's estimates for the Dvoretzky theorem (see [81]) in the context of the norm

$$\|y\|_{L^p(K)} = \|\langle \cdot, y \rangle\|_{L^p(K)} = \left( \int_K |\langle x, y \rangle|^p dx \right)^{1/p} \qquad (y \in \mathbb{R}^n).$$

The unit ball of the dual norm is denoted by $Z_p(K) \subseteq \mathbb{R}^n$ and it is referred to as the *$L_p$-centroid body of $K$*, see also Lutwak and Zhang [71]. If $K$ is isotropic, then the set $Z_2(K)$ is a Euclidean ball of radius $L_K$. In the case where $K$ is centrally-symmetric, we have $Z_\infty(K) = K$. The $\psi_1$-estimate for linear functionals on convex bodies (13) is equivalent to the assertion that

$$Z_p(K) \subseteq C p Z_2(K) \qquad \text{for all } p \geq 1. \tag{34}$$

The idea of Paouris was to apply the quantitative theory of Dvoretzky's theorem, due to the second-named author [81], to the $L_p$-centroid bodies. Together with the estimate (34), this quantitative theory yields the following: Suppose that $K \subseteq \mathbb{R}^n$ is an isotropic convex body and $1 \leq \ell \leq c\sqrt{n}$. Then for a random $\ell$-dimensional subspace $E \subseteq \mathbb{R}^n$, with high probability, the orthogonal projection of $Z_\ell(K)$ onto $E$ denoted by

$$\mathrm{Proj}_E(Z_\ell(K)) \tag{35}$$

is an isomorphic Euclidean ball. In other words, the convex body in (35) contains a Euclidean ball of radius $r$ centered at the origin, and it is contained in a Euclidean ball of radius $Cr$. We may now invoke (32) and conclude that $r$ has the order of magnitude of $\sqrt{\ell}L_K$. Thus by the quantitative estimates revolving around Dvoretzky's theorem, due to Litvak, Milman and Schechtman [69],

$$\sqrt{\ell}L_K \simeq r \simeq \left( \int_{S^{n-1}} \|y\|_{L^\ell(K)}^\ell d\sigma(y) \right)^{1/\ell} \simeq \sqrt{\frac{\ell}{n}} \left( \int_K |x|^\ell dx \right)^{1/\ell}, \tag{36}$$

where $\sigma$ is the rotationally-invariant probability measure on $S^{n-1}$. Thus (36) yields estimates for $L_p$-moments of the Euclidean norm for all $p \leq c\sqrt{n}$, which imply that only a fraction of at most $e^{-\sqrt{n}}$ of the volume of $K$, is located outside a ball of radius $CL_K\sqrt{n}$.

The tension between $\psi_1$-estimates and $\psi_2$-estimates for convex bodies, going back to Bourgain's work in the 1980s, is a central issue in the analysis of the slicing problem. Recall that the inclusion (34) follows from the $\psi_1$-bound, while a $\psi_2$-estimate with constant $A$ would yield that $Z_p(\mu) \subseteq CA\sqrt{p}Z_2(\mu)$. In this respect, it is worthwhile to mention yet another equivalent formulation of the hyperplane conjecture, which may be extracted from [62, Remark 3.3]: **Question 1.2 has an affirmative answer if and only if for any isotropic convex body $K \subseteq \mathbb{R}^n$ and any $1 \leq p \leq n$,**

$$\boldsymbol{Vol^{1/n}(Z_p(\mu)) \simeq \sqrt{p} \cdot Vol^{1/n}(Z_2(\mu))}.$$

A question by the second-named author (see [10, 86, 87]) asks whether for any convex body $K \subseteq \mathbb{R}^n$ there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ for which

$$\|\varphi\|_{\psi_2(K)} \leq C\|\varphi\|_{L^2(K)},$$

with a universal constant $C$. In other words, does any convex body have at least one direction with a uniformly sub-Gaussian tail? In some sense, a direction where the tail is approximately exponential resembles a "cone-like behavior" of the convex body (see [52]), and the question is whether there always exists a direction in which better, sub-gaussian behavior is observed. It was proven by the first-named author in [52] that the answer is affirmative up to logarithmic factors. The logarithmic factor that the proof in [52] yielded is $\log^5(t+1)$ (in formula (37) below), and it was subsequently improved to $\log^2(t+1)$ in Giannopoulos, Pajor and Paouris [40] and then to $\log(t+1)$ in Giannopoulos, Paouris and Valettas [41]:

**Theorem 3.2.** *Let $n \geq 1$ be an integer, and let $K \subset \mathbb{R}^n$ be a convex body of volume one. Then there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that for any $t \geq 1$,*

$$Vol_n\left(\{x \in K; |\varphi(x)| \geq t\|\varphi\|_{L_1(K)}\}\right) \leq e^{-c\frac{t^2}{\log(t+1)}} \tag{37}$$

*where $c > 0$ is a universal constant.*

In the case of unconditional convex bodies, the logarithmic factor in (37) is not needed at all, as proven in Bobkov and Nazarov [15]. In general it is not known whether the logarithmic factor is necessary, or equivalently, whether any convex body admits at least one uniformly sub-Gaussian direction. We move on to discuss the question of the existence of *approximately Gaussian directions*. A conjecture that appears in the works of Anttila, Ball and Perissinaki [2] and Brehm and Voigt [28] suggests that any high-dimensional convex body admits at least one approximately Gaussian direction. That is, it was conjectured that whenever $X$ is a random vector in $\mathbb{R}^n$, uniformly distributed in some convex body, then there exists $0 \neq \theta \in \mathbb{R}^n$ such that the random variable

$$\langle X, \theta \rangle$$

is approximately a Gaussian random variable. The degree of the approximation is expected to improve when the dimension $n$ increases. The conjecture clearly holds true in the case where $X$ is uniform in an $n$-dimensional cube, by the classical central limit theorem, and in the case where $X$ is uniform in a Euclidean ball or an ellipsoid, by the so-called Maxwell observation. The conjecture has turned out to be true in general, as proven by the first-named author [53]:

**Theorem 3.3** ("Central limit theorem for convex bodies"). *There exists a sequence $\varepsilon_n \searrow 0$ for which the following holds: let $K \subset \mathbb{R}^n$ be a convex body, and let $X$ be a random vector that is distributed uniformly in $K$. Then there exist a unit vector $\theta \in S^{n-1}$, $t_0 \in \mathbb{R}$ and $V > 0$ such that*

$$\sup_{A \subseteq \mathbb{R}} \left| \mathbb{P}\{ \langle X, \theta \rangle \in A \} - \frac{1}{\sqrt{2\pi V}} \int_A e^{-\frac{(t-t_0)^2}{2V}} dt \right| \leq \varepsilon_n,$$

*where the supremum runs over all measurable sets $A \subseteq \mathbb{R}$.*

*Moreover, if the convex body $K \subseteq \mathbb{R}^n$ is an isotropic body of volume one, then there exists a subset $\Theta \subseteq S^{n-1}$ with $\sigma(\Theta) \geq 1 - \varepsilon_n$ such that for all $\theta \in \Theta$,*

$$\sup_{A \subseteq \mathbb{R}} | \mathbb{P}\{ \langle X, \theta \rangle \in A \} - \mathbb{P}\{ Z \in A \} | \leq \varepsilon_n,$$

*where $Z$ is a Gaussian random variable of mean zero and variance $L_K^2$.*

The bound obtained in [54] for $\varepsilon_n$ is $\varepsilon_n \leq C/n^\alpha$ where $C, \alpha > 0$ are universal constants and $\alpha \geq 1/15$. The optimal exponent $\alpha$ remains unknown.

Theorem 3.3 exposes a universal property of high-dimensional convex bodies: they all have approximately Gaussian one-dimensional marginals. Moreover, most of these marginals of a high-dimensional convex body, with the isotropic normalization, are approximately Gaussian. In fact, this phenomena is not restricted to one-dimensional marginals. As was proven by Eldan and the first-named author [32], when one projects the uniform measure of an istoropic convex body $K \subset \mathbb{R}^n$ to a random $k$-dimensional subspace $E$ with $k \leq cn^\alpha$, the probability measure obtained in $E$ has a density that is approximately Gaussian, both in total variation sense, and in the sense that the ratio between this density and a Gaussian density in $E$ is very close to 1 in large parts of the subspace $E$. Here, $c, \alpha > 0$ are universal constants.

Interestingly, the Gaussian approximation property of convex bodies may be reformulated in terms of a *thin shell condition*, according to a beautiful general principle that goes back to Sudakov [92] and to Diaconis and Freedman [31] (see also Anttila, Ball and Perissinaki [2], Bobkov [14] and von Weizsäcker [97]). This principle reads as follows: suppose that $X$ is any random vector in $\mathbb{R}^n$ with finite second moments, normalized to have mean zero and identity covariance. Then most of the one-dimensional marginals of $X$ are approximately Gaussian *if and only if* the random variable $|X|/\sqrt{n}$ is concentrated around the value one, i.e.,

$$\mathbb{E}\left(\frac{|X|}{\sqrt{n}} - 1\right)^2 \leq \varepsilon,$$

for a small number $\varepsilon > 0$. These assumptions imply that the Kolmogorov distance between a typical marginal of $X$ and a Gaussian distribution is bounded by $C(\varepsilon + n^{-\alpha})$ for universal constants $C, \alpha > 0$, see the formulation in [55]. In other words, typical marginals are approximately Gaussian if and only if most of the mass of $X$ is concentrated in a "thin spherical shell" whose radius is $\sqrt{n}$ and whose width is much smaller than $\sqrt{n}$. Theorem 3.3 is therefore parallel to the estimate

$$\frac{\sigma_K^2}{n} := \mathbb{E}\left(\frac{|X|}{\sqrt{n}L_K} - 1\right)^2 \leq \frac{C}{n^\alpha}, \tag{38}$$

valid for any random vector $X$ that is distributed uniformly in an isotropic convex body $K \subseteq \mathbb{R}^n$, where $C, \alpha > 0$ are universal constants. Thus most of the volume of a convex body in high dimensions, with the isotropic normalization, is contained in a thin spherical shell, whose width is much smaller than its radius. This complements the Paouris large deviation bound, Theorem 3.1 above. See Figure 1 for an illustration.

The parameter $\alpha$ from (38) is related to the width of the thin spherical shell that contains most of the mass of an isotropic convex body $K \subseteq \mathbb{R}^n$. The argument in [54] leads to the estimate $\alpha \geq 1/6$, which was improved to $\alpha \geq 1/4$ by Fleury [37], to $\alpha \geq 1/3$ by Guédon and Milman [44] and then to $\alpha \geq 1/2$ by Lee and Vempala [66] who built upon a stochastic localization technique of Eldan [34]. In terms of the *thin shell parameter* $\sigma_K$ defined in (38), the current best bound due to Lee and Vempala [66] is that for any isotropic convex body $K \subseteq \mathbb{R}^n$,

$$\sigma_K \leq C n^{1/4} \tag{39}$$

where $C > 0$ is a universal constant. The "$1/4$" in (39) perhaps reminds us of the best known result for the slicing problem $L_K \leq Cn^{1/4}$ from [51], which is up to logarithmic factors due to Bourgain. This is not fully a coincidence. It was proven by Eldan and the first-named author in [33] that

$$\sup_{K \subseteq \mathbb{R}^n} L_K \leq C \sup_{K \subseteq \mathbb{R}^n} \sigma_K. \tag{40}$$

Thus any progress on the thin-shell parameter $\sigma_K$ beyond the bound (39) would automatically lead to progress in the slicing problem. It was conjectured in Anttila, Ball and Perissinaki [2] and in Bobkov and Koldobsky [16] that $\sigma_K$ is bounded by a universal constant, perhaps up to a logarithmic factor. In view of (40), **the thin shell conjecture would imply the hyperplane conjecture**.

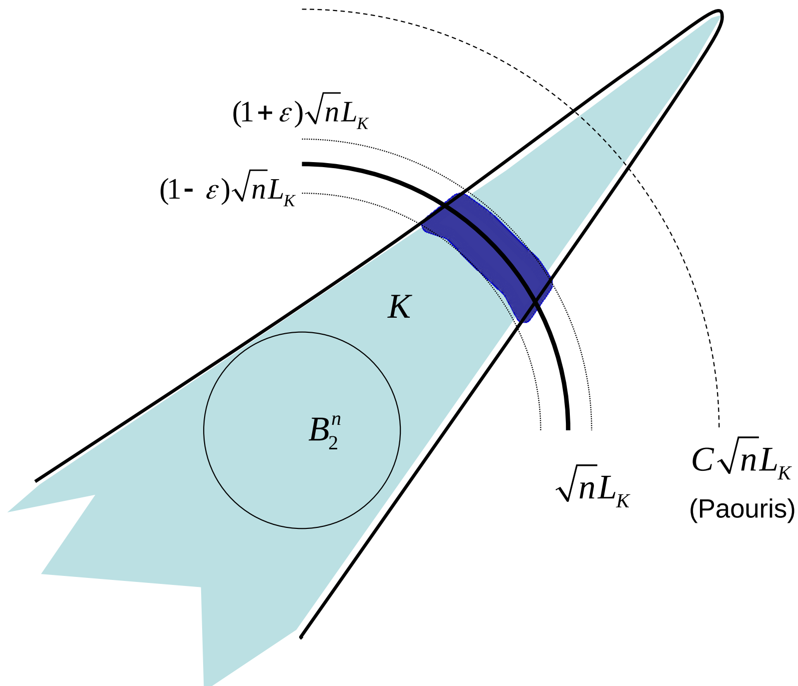


Figure 1: The thin-shell and large deviation regimes (illustration by S. Artstein-Avidan).

We move on to a brief discussion of further developments related to the isoperimetric problem in convex bodies, see e.g. the recent survey by Lee and Vempala [66] for a thorough treatment. In addition to the isotropic constant $L_K$ and the thin shell parameter $\sigma_K$, an important quantity related to an isotropic convex body $K \subseteq \mathbb{R}^n$ is its *isoperimetric constant* or *Cheeger constant*, defined as follows:

$$\frac{1}{\psi_K} := L_K \cdot \inf_{A \subseteq \overline{K}} \frac{Vol_{n-1}(K \cap \partial A)}{\min\{Vol_n(A), Vol_n(K \setminus A)\}} \tag{41}$$

where the infimum runs over all subsets $A \subseteq \overline{K}$ with smooth boundary, and where we recall that in this paper a convex body $K$ is an open set and $\overline{K}$ is its closure. In the infimum in (41), we partition $K$ into two parts so as to minimize the surface area of the interface between them; we do not include in the surface area the part of $\partial A$ that lies on the boundary of $K$, only the interface between the two parts inside the convex, open set $K$. The reason for the normalization in (41) is the following chain of inequalities:

$$\sup_{K \subseteq \mathbb{R}^n} L_K \leq C \sup_{K \subseteq \mathbb{R}^n} \sigma_K \leq \tilde{C} \sup_{K \subseteq \mathbb{R}^n} \psi_K \leq \bar{C} n^{1/4}, \tag{42}$$

where the suprema run over all isotropic convex bodies in $\mathbb{R}^n$, and where the last inequality was proven by Lee and Vempala [66]. It was conjectured in Kannan, Lovász and Simonovits

(KLS) [48] that $\psi_K \leq C$ for any isotropic convex body $K \subseteq \mathbb{R}^n$, where $C > 0$ is a universal constant. This is a stronger conjecture than slicing, in view of (42). Ball and Nguyen [9] proved the bound $L_K \leq \exp(C\psi_K^2)$ for any isotropic convex body in $\mathbb{R}^n$. Eldan reduced the study of the isoperimetric KLS conjecture to the thin-shell conjecture, up to logarithmic factors. Denoting $\psi_n = \sup_{K \subseteq \mathbb{R}^n} \psi_K$ and $\sigma_n = \sup_{K \subseteq \mathbb{R}^n} \psi_K$ it was proven in Eldan's breakthrough paper [34] that

$$\psi_n \leq C \sqrt{\log n \cdot \sum_{\ell=1}^{n} \frac{\sigma_\ell^2}{\ell}},$$

where $C > 0$ is a universal constant. It follows that **up to factors logarithmic in the dimension, the thin shell conjecture is equivalent to the isoperimetric KLS conjecture**. We summarize this section by noting that current progress in the thin shell and KLS conjectures stops at $n^{1/4}$, which is the best known bound for the seemingly-innocent slicing problem.

# 4 Bound for the isotropic constant

In this section we provide some details regarding the logarithmic improvement of Bourgain's bound for the isotropic constant. This improvement is related to the following theorem due to the first-named author [51], the so-called *isomorphic version of the slicing problem*:

**Theorem 4.1.** *Let $K \subset \mathbb{R}^n$ be a convex body and $0 < \varepsilon < 1$. Then there exists a convex body $T \subset \mathbb{R}^n$ such that*

*(i)* $(1-\varepsilon)T \subseteq K \subseteq (1+\varepsilon)T$.

*(ii)* $L_T < C/\sqrt{\varepsilon}$*, where $C > 0$ is a universal constant.*

In [56] it is proven that $T$ from Theorem 4.1 can be additionally assumed to be a projective image of $K$. Recall that the projective image of a polytope is itself a polytope with the same number of vertices and faces.

The Paouris large deviation estimate, which is Theorem 3.1 above, implies the following: For any convex bodies $K, T \subseteq \mathbb{R}^n$, if

$$\left(1 - \frac{1}{\sqrt{n}}\right) T \subseteq K \subseteq \left(1 + \frac{1}{\sqrt{n}}\right) T \tag{43}$$

then

$$L_K \simeq L_T. \tag{44}$$

Indeed, since isotropic constants are invariant under affine transformations, we may assume that $K$ is an isotropic convex body. Theorem 3.1 shows that at most an $e^{-10\sqrt{n}}$-fraction of the volume of $K$ is located outside the ball $C\sqrt{n}L_K B^n$. It thus follows from (43) that at most an $e^{-\sqrt{n}}$-fraction of the volume of $T$ is located outside this ball. The variational characterization (8) of the isotropic constant of $T$ now implies that $L_T \leq CL_K$, and (44) follows by symmetry. Consequently, by substituting $\varepsilon = 1/\sqrt{n}$ in Theorem 4.1, we conclude that for any convex body $K \subseteq \mathbb{R}^n$,

$$L_K \leq Cn^{1/4},$$

as advertised. Let us now elaborate on the ideas behind the proof of Theorem 4.1. We are given a convex body $K \subseteq \mathbb{R}^n$. After translation, we may assume that the barycenter of $K$ lies at the origin. Consider the *logarithmic Laplace transform*

$$F(x) = F_K(x) = \log \int_{\mathbb{R}^n} e^{\langle x,y \rangle} dy.$$

The function $F$ is smooth and convex in $\mathbb{R}^n$, by the Cauchy-Schwartz inequality. In fact, for any $x \in \mathbb{R}^n$, denoting by $\mu_x$ the probability measure on $K$ with density $y \mapsto e^{\langle x,y \rangle - F(x)} 1_K(y)$, we have

$$\nabla F(x) = \mathrm{bar}(\mu_x) \qquad \text{and} \qquad \nabla^2 F(x) = \mathrm{Cov}(\mu_x), \tag{45}$$

where $\mathrm{bar}(\mu_x) = \int y d\mu_x(y)$ is the barycenter of $\mu_x$ and where $\nabla^2 F(x)$ is the Hessian matrix of $F$. Note that $\nabla F(x) \in K$ for any $x \in \mathbb{R}^n$, since $\mu_x$ is a measure supported on $K$ and hence its barycenter is in $K$ by convexity. The Hessian of $F$ is positive-definite everywhere, according to (45). This convexity property of $F$ implies that the map $x \mapsto \nabla F(x)$ is a diffeomorphism from $\mathbb{R}^n$ onto an open subset of $K$ (which is in fact $K$ itself).

Fix $0 < \varepsilon < 1$ as in the formulation of Theorem 4.1. We may use the point of view of "transportation of measure", and change variables as follows:

$$\int_{\varepsilon n K^\circ} \det \mathrm{Cov}(\mu_x) dx = \int_{\varepsilon n K^\circ} \det \nabla^2 F(x) dx \overset{\text{“}y=\nabla F(x)\text{''}}{=} \int_{\nabla F(\varepsilon n K^\circ)} 1 dy \leq Vol_n(K),$$

as $\nabla F(\varepsilon n K^\circ) \subseteq \nabla F(\mathbb{R}^n) \subseteq K$. In particular, there exists $x \in \varepsilon n K^\circ$ such that

$$\det \mathrm{Cov}(\mu_x) \leq \frac{Vol_n(K)}{Vol_n(\varepsilon n K^\circ)} = \varepsilon^{-n} \frac{Vol_n(K)^2}{n^n Vol_n(K) Vol_n(K^\circ)} \leq \left(\frac{C}{\varepsilon}\right)^n Vol_n(K)^2, \tag{46}$$

where the last passage follows from the Bourgain-Milman inequality (18). Let us take a closer look at the probability measure $\mu_x$. It is a log-concave measure with density

$$\rho(y) = e^{\langle x,y \rangle - F(x)} 1_K(y).$$

Since $x \in \varepsilon n K^\circ$, we know that $|\langle x,y \rangle| \leq \varepsilon n$ for all $y \in K$. Hence,

$$\frac{\sup_K \rho}{\inf_K \rho} = \frac{\sup_{y \in K} e^{\langle x,y \rangle}}{\inf_{y \in K} e^{\langle x,y \rangle}} \leq \frac{e^{\varepsilon n}}{e^{-\varepsilon n}} = e^{2\varepsilon n}. \tag{47}$$

Recall the convex body $K(\rho)$ associated with the log-concave density $\rho$ via formula (29). It follows from (29) and (47) that

$$(1 - C\varepsilon)K \subseteq K(\rho) \subseteq (1 + C\varepsilon)K$$

for some universal constant $C > 0$. We still need to show that $L_{K(\rho)} \leq \tilde{C}/\sqrt{\varepsilon}$, so that Theorem 4.1 would follow with $T = K(\rho)$. In view of (30), it suffices to show that

$$L_\rho \leq C/\sqrt{\varepsilon}. \tag{48}$$

However, since the barycenter of $K$ lies at the origin, we know that $\nabla F(0) = 0$ by (45). Since $F$ is a convex function, its critical points are global minimum points, and hence $F(x) \geq F(0) = \log Vol_n(K)$ for any $x \in \mathbb{R}^n$. Consequently, by (46),

$$\begin{aligned} L_\rho = (\sup \rho)^{1/n} \cdot \det \mathrm{Cov}(\rho)^{1/(2n)} &= \exp\left(\frac{\sup_{y \in K} \langle x, y \rangle - F(x)}{n}\right) \cdot \det \mathrm{Cov}(\rho)^{1/(2n)} \\ &\leq \exp\left(\frac{\varepsilon n - \log Vol_n(K)}{n}\right) \cdot \left(\left(\frac{C}{\varepsilon}\right)^n Vol_n(K)^2\right)^{1/(2n)} \leq \frac{\tilde{C}}{\sqrt{\varepsilon}}. \end{aligned}$$

This completes the proof of (48), as well as our sketch of the proof of Theorem 4.1.

It is possible to view the hyperplane conjecture as a strong conjectural version of the Bourgain-Milman inequality and of the *M-ellipsoid theory* due to the second-named author. We present two interpretations of this point of view. The first interpretation is related to the strong slicing conjecture, which suggests that for any convex body $K \subseteq \mathbb{R}^n$,

$$L_K \leq L_{\Delta^n} = \frac{(n!)^{\frac{1}{n}}}{(n+1)^{\frac{n+1}{2n}} \cdot \sqrt{n+2}}, \tag{49}$$

where $\Delta^n \subseteq \mathbb{R}^n$ is any simplex whose vertices span $\mathbb{R}^n$ and add up to zero. This conjecture holds true in two dimensions. See also Rademacher [91] for supporting evidence. On the other hand, the Mahler conjecture suggests that for any convex body $K \subseteq \mathbb{R}^n$ containing the origin in its interior,

$$Vol_n(K) Vol_n(K^\circ) \geq Vol_n(\Delta^n) \cdot Vol_n((\Delta^n)^\circ) = \frac{(n+1)^{n+1}}{(n!)^2}. \tag{50}$$

In two dimensions the conjecture was proven by Mahler [73], see also Meyer [74], and see Barthe and Fradelizi [11] for the case of convex bodies with symmetries. The Bourgain-Milman inequality established (50) up to a factor of $c^n$, for a universal constant $c > 0$. In [56] it is shown that **the strong version (49) of Bourgain's slicing conjecture implies Mahler's conjecture (50)**. Let us also mention in passing that in the centrally-symmetric case, the strong version of Bourgain's slicing conjecture is that the isotropic constant is maximized for the cube. If this is true, then an old conjecture by Minkowski would follow, see Magazinov [72] and also Autissier [3]. The Minkowski conjecture suggests that for any lattice $L \subseteq \mathbb{R}^n$ of unit covolume and for any $x \in \mathbb{R}^n$ there exists $y \in L$ with $\prod_{i=1}^n |x_i - y_i| \leq 2^{-n}$.

There is also a second interpretation of the relationship between the slicing problem and the notion of the $M$-ellipsoid and the Bourgain-Milman inequality. For a convex body $K \subseteq \mathbb{R}^n$, an ellipsoid $\mathcal{E} \subseteq \mathbb{R}^n$ is called an *$M$-ellipsoid for $K$ with constant $A$* if

$$Vol_n(\mathcal{E}) = Vol_n(K) \qquad \text{and} \qquad N(K, \mathcal{E}) \cdot N(\mathcal{E}, K) \leq e^{An}.$$

Here, $N(K,T) = \inf\{N\,;\, \exists x_1,\ldots,x_N \in \mathbb{R}^n, K \subseteq \bigcup_{i=1}^N (x_i + T)\}$ is the covering number of $K$ by $T$, the minimal number of translates of $T$ that may cover $K$. The second-named author proved [78, 79] that there exists a universal constant $C > 0$, such that any convex body $K \subseteq \mathbb{R}^n$ has an $M$-ellipsoid with constant $C$. This fact plays an important role in Asymptotic Geometric Analysis. For example, assume the normalization $Vol_n(K) = Vol_n(B^n)$, and let $u \in SL_n$ be such that $u(\mathcal{E}) = B^n$. Set $K_1 = u(K)$. Then for any $0 < \lambda < 1$, with high probability of choosing a random $\lfloor \lambda n \rfloor$-dimensional subspace $E \subseteq \mathbb{R}^n$, the convex body

$$\mathrm{Proj}_E(K_1)$$

is a convex body of finite volume-ratio, with a volume-ratio constant depending solely on $\lambda$. It was observed by K. Ball [4] that for an isotropic convex body $K \subseteq \mathbb{R}^n$, the Euclidean ball of volume one is an $M$-ellipsoid for $K$ with a constant depending solely on $L_K$. Hence, a positive solution to the slicing problem, i.e., a universal bound on $L_K$, would imply the theorem on the existence of the $M$-ellipsoid.

Note that the $M$-ellipsoid is an isomorphic notion: if $\mathcal{E}$ is an $M$-ellipsoid for $K$ with constant $A$ and $K/2 \subseteq T \subseteq 2K$, then a homothetic copy of $\mathcal{E}$ of volume $Vol_n(T)$ is also an $M$-ellipsoid of $T$ with constant $4\alpha$. Therefore Theorem 4.1 implies the existence of an $M$-ellipsoid for any $K$, with some universal constant $C > 0$.

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