A Central Limit Theorem for Convex Sets

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Abstract. We show that there exists a sequence $\varepsilon_n \searrow 0$ for which the following holds: Let $K \subset \mathbb{R}^n$ be a compact, convex set with a non-empty interior. Let X be a random vector that is distributed uniformly in K. Then there exist a unit vector θ in $\mathbb{R}^n$, $t_0 \in \mathbb{R}$ and $\sigma > 0$ such that

$$\sup_{A \subset \mathbb{R}} \left| Prob\{ \langle X, \theta \rangle \in A \} - \frac{1}{\sqrt{2\pi\sigma}} \int_A e^{-\frac{(t-t_0)^2}{2\sigma^2}} dt \right| \leq \varepsilon_n, \qquad (*)$$

where the supremum runs over all measurable sets $A \subset \mathbb{R}$, and where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^n$. Furthermore, under the additional assumptions that the expectation of X is zero and that the covariance matrix of X is the identity matrix, we may assert that most unit vectors θ satisfy $(*)$, with $t_0 = 0$ and $\sigma = 1$. Corresponding principles also hold for multi-dimensional marginal distributions of convex sets.

1. Introduction

We begin with an example. Let $n \geq 1$ be an integer, and consider the cube $Q = [-\sqrt{3}, \sqrt{3}]^n \subset \mathbb{R}^n$. Suppose that $X = (X_1, ..., X_n)$ is a random vector that is distributed uniformly in the cube Q. Then $X_1, ..., X_n$ are independent, identically-distributed random variables of mean zero and variance one. Consequently, the classical central limit theorem states that the distribution of the random variable

$$\frac{X_1 + ... + X_n}{\sqrt{n}}$$

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is close to the standard normal distribution, when n is large. Moreover, suppose we are given $\theta_1, ..., \theta_n \in \mathbb{R}$ with $\sum_{i=1}^n \theta_i^2 = 1$. Then under mild conditions on the θ_i's (such as Lindeberg's condition, see, e.g., [13, Section VIII.4]), the distribution of the random variable

$$\langle \theta, X \rangle = \sum_{i=1}^n \theta_i X_i$$

is approximately gaussian, provided that the dimension n is large. For background on the classical central limit theorem we refer the reader to, e.g., [13] and [50].

Let us consider a second example, no less fundamental than the first. We denote by $|\cdot|$ the standard Euclidean norm in $\mathbb{R}^n$, and let $\sqrt{n+2}\, D^n = \{x \in \mathbb{R}^n; |x| \leq \sqrt{n+2}\}$ be the Euclidean ball of radius $\sqrt{n+2}$ around the origin in $\mathbb{R}^n$. We also write $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$ for the unit sphere in $\mathbb{R}^n$. Suppose that $Y = (Y_1, ..., Y_n)$ is a random vector that is distributed uniformly in the ball $\sqrt{n+2}\, D^n$. Then $Y_1, ..., Y_n$ are identically-distributed random variables of mean zero and variance one, yet they are not independent. Nevertheless, it was already observed by Maxwell that for any $\theta = (\theta_1, ..., \theta_n) \in S^{n-1}$, the distribution of the random variable

$$\langle \theta, Y \rangle = \sum_{i=1}^n \theta_i Y_i$$

is close to the standard normal distribution, when n is large. See, e.g., [12] for the history of the latter fact and for more information.

There is a wealth of central limit theorems in probability theory that ensure normal approximation for a sum of many independent, or weakly dependent, random variables. Our first example, that of the cube, fits perfectly into this framework. The approach we follow in this paper relates more to the second example, that of the Euclidean ball, where the "true source" of the gaussian approximation may be attributed to geometry. The geometric condition we impose on the distribution of the random variables is that of convexity. We shall see that convexity may substitute for independence in certain aspects of the phenomenon represented by the classical central limit theorem.

A function $f : \mathbb{R}^n \to [0, \infty)$ is log-concave if

$$f(\lambda x + (1-\lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}$$

for all $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$. That is, f is log-concave when $\log f$ is concave on the support of f. Examples of interest for log-concave functions include characteristic functions of convex sets, the gaussian density, and several densities from statistical mechanics. In this manuscript, we consider random vectors in $\mathbb{R}^n$ that are distributed according to a log-concave density. Thus, our treatment includes as a special case the uniform distribution on an arbitrary compact, convex set with a non-empty interior.

We say that a function $f : \mathbb{R}^n \to [0, \infty)$ is isotropic if it is the density of a random vector with zero mean and identity covariance matrix. That is, f is isotropic when

$$\int_{\mathbb{R}^n} f(x)dx = 1, \quad \int_{\mathbb{R}^n} xf(x)dx = 0 \text{ and } \int_{\mathbb{R}^n} \langle x, \theta \rangle^2 f(x)dx = |\theta|^2$$

for all $\theta \in \mathbb{R}^n$. Any log-concave function with $0 < \int f < \infty$ may be brought to an isotropic position via an affine map, that is, $f \circ T$ is isotropic for some affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ (see, e.g., [34]). Suppose that X and Y are two random variables attaining values in some measure space Ω (here Ω will always be $\mathbb{R}$ or $\mathbb{R}^n$ or a subspace $E \subset \mathbb{R}^n$). We define their total-variation distance as

$$d_{TV}(X, Y) = 2 \sup_{A \subset \Omega} | \, Prob\{X \in A\} \, - \, Prob\{Y \in A\} \, | \, ,$$

where the supremum runs over all measurable sets $A \subset \Omega$. Note that $d_{TV}(X, Y)$ equals the L^1-distance between the densities of X and Y, when these densities exist. Let σ_{n-1} stand for the unique rotationally-invariant probability measure on S^{n-1}, also referred to as the uniform probability measure on the sphere S^{n-1}.

Theorem 1.1 *There exist sequences $\varepsilon_n \searrow 0, \delta_n \searrow 0$ for which the following holds: Let $n \geq 1$, and let X be a random vector in $\mathbb{R}^n$ with an isotropic, log-concave density. Then there exists a subset $\Theta \subset S^{n-1}$ with $\sigma_{n-1}(\Theta) \geq 1 - \delta_n$, such that for all $\theta \in \Theta$,*

$$d_{TV} \left(\langle X, \theta \rangle \, , \, Z \right) \leq \varepsilon_n,$$

where $Z \sim N(0, 1)$ is a standard normal random variable.

We have the bounds $\varepsilon_n \leq C \left(\frac{\log \log(n+2)}{\log(n+1)} \right)^{1/2}$ and $\delta_n \leq \exp\left(-cn^{0.99}\right)$ for ε_n and δ_n from Theorem 1.1, where $c, C > 0$ are universal constants. The quantitative estimate we provide for ε_n is rather poor. While Theorem 1.1 seems to be a reasonable analog of the classical central limit theorem for the category of log-concave densities, we are still lacking the precise Berry-Esseen type bound. A plausible guess might be that the logarithmic dependence should be replaced by a power-type decay, in the bound for ε_n.

Theorem 1.1 implies the result stated in the abstract of this paper, which does not require isotropicity; indeed, recall that any log-concave density can be made isotropic by applying an appropriate affine map. Thus, any log-concave density in high dimension has at least one almost-gaussian marginal. When the log-concave density is also isotropic, we can assert that, in fact, the vast majority of its marginals are approximately gaussian.

An inherent feature of Theorem 1.1 is that it does not provide a specific unit vector $\theta \in S^{n-1}$ for which $\langle X, \theta \rangle$ is approximately normal. This is inevitable: We clearly cannot take $\theta = (1, 0, ..., 0)$ in the example of the cube above, and hence there is no fixed unit vector that suits all isotropic, log-concave densities. Nevertheless, under additional symmetry assumptions, we can identify a unit vector that always works.

Borrowing terminology from Banach space theory, we say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is unconditional if

$$f(x_1, ..., x_n) = f(|x_1|, ..., |x_n|) \quad \text{for all} \quad x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

That is, f is unconditional when it is invariant under coordinate reflections.

Theorem 1.2 *There exists a sequence $\varepsilon_n \searrow 0$ for which the following holds: Let $n \geq 1$, and let $f : \mathbb{R}^n \to [0, \infty)$ be an unconditional, isotropic, log-concave function. Let $X = (X_1, ..., X_n)$ be a random vector in $\mathbb{R}^n$ that is distributed according to the density f. Then,*

$$d_{TV}\left(\frac{X_1 + ... + X_n}{\sqrt{n}} \, , \, Z \right) \leq \varepsilon_n$$

where $Z \sim N(0,1)$ is a standard normal random variable.

We provide the estimate $\varepsilon_n \leq \frac{C}{(\log(n+1))^{1/5}}$ for ε_n from Theorem 1.2. Multi-dimensional versions of Theorem 1.1 are our next topic. For integers k, n with $1 \leq k \leq n$, let $G_{n,k}$ stand for the grassmannian of all k-dimensional subspaces in $\mathbb{R}^n$. Let $\sigma_{n,k}$ be the unique rotationally-invariant probability measure on $G_{n,k}$. Whenever we refer to the uniform measure on $G_{n,k}$, and whenever we select a random k-dimensional subspace in $\mathbb{R}^n$, we always relate to the probability measure $\sigma_{n,k}$ defined above. For a subspace $E \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, let $Proj_E(x)$ stand for the orthogonal projection of x onto E. A standard gaussian random vector in a k-dimensional subspace $E \subset \mathbb{R}^n$ is a random vector X that satisfies $Prob\{X \in A\} = (2\pi)^{-k/2} \int_A \exp(-|x|^2/2) dx$ for any measurable set $A \subset E$.

Theorem 1.3 *There exists a universal constant $c > 0$ for which the following holds: Let $n \geq 3$ be an integer, and let X be a random vector in $\mathbb{R}^n$ with an isotropic, log-concave density. Let $\varepsilon > 0$ and suppose that $1 \leq k \leq c\varepsilon^2 \frac{\log n}{\log \log n}$ is an integer. Then there exists a subset $\mathcal{E} \subset G_{n,k}$ with $\sigma_{n,k}(\mathcal{E}) \geq 1 - e^{-cn^{0.99}}$ such that for any $E \in \mathcal{E}$,*

$$d_{TV}\left(Proj_E(X) \, , \, Z_E \right) \leq \varepsilon,$$

where Z_E is a standard gaussian random vector in the subspace E.

That is, most k-dimensional marginals of an isotropic, log-concave function, are approximately gaussian with respect to the total-variation metric, provided that $k << \frac{\log n}{\log \log n}$. Note the clear analogy between Theorem 1.3 and Milman's precise quantitative theory of Dvoretzky's theorem, an analogy that dates back to Gromov [18, Section 1.2]. Readers that are not familiar with Dvoretzky's theorem are referred to, e.g., [15, Section 4.2], to [33] or to [28]. Dvoretzky's theorem shows that k-dimensional geometric projections of an n-dimensional convex body are ε-close to a Euclidean ball, provided that $k < c\varepsilon^2 \log n$. Theorem 1.3 states that k-dimensional marginals, or measure-projections, of an n-dimensional convex body are ε-close to gaussian when $k < c\varepsilon^2 \log n / (\log \log n)$. Thus, according to Dvoretzky's theorem, the geometric shape of the support of the marginal distribution may be approximated by a very regular body – a Euclidean ball, or an ellipsoid – whereas Theorem 1.3 demonstrates that the marginal distribution itself is very regular; it is approximately normal.

More parallels between Theorem 1.3 and Dvoretzky's theorem are apparent from the proof of Theorem 1.3 below. We currently do not know whether there exists a single subspace that satisfies both the conclusion of Theorem 1.3 and the conclusion of Dvoretzky's theorem simultaneously; both theorems show that a "random subspace" works with large probability, but with respect to different Euclidean structures. The logarithmic dependence on the dimension is known to be tight in Milman's form of Dvoretzky's theorem. However, we have no reason to believe that the quantitative estimates in Theorem 1.3 are the best possible.

There are several mathematical articles where Theorem 1.1 is explicitly conjectured. Brehm and Voigt suggest Theorem 1.1 as a conjecture in [7], where they write that this conjecture appears to be "known among specialists". Anttila, Ball and Perissinaki formulated the same conjecture in [1], independently and almost simultaneously with Brehm and Voigt. Anttila, Ball and Perissinaki also proved the conjecture for the case of uniform distributions on convex sets whose modulus of convexity and diameter satisfy certain quantitative assumptions. Gromov wrote a remark in [18, Section 1.2] that seems related to Theorem 1.1 and Theorem 1.3, especially in view of the techniques we use here. Following [1] and [7], significant contributions regarding the central limit problem for convex sets were made by Bastero and Bernués [3], Bobkov [4], Bobkov and Koldobsky [5], Brehm and Voigt [7], Brehm, Hinow, Vogt and Voigt [8], Koldobsky and Lifshits [24], E. and M. Meckes [30], E. Milman [31], Naor and Romik [36], Paouris [37], Romik [44], S. Sodin [48], Wojtaszczyk [53] and others.

Let us explain a few ideas from our proof. We begin with a general principle that goes back to Sudakov [51] and to Diaconis and Freedman [11] (see also the expositions of Bobkov [4] and von Weizsäcker [52]. A sharpening for the case of convex bodies was obtained by Anttila, Ball and Perissinaki [1]). This principle reads as follows: Suppose X is any random vector in $\mathbb{R}^n$ with zero mean and identity covariance matrix. Then most of the marginals of X are approximately gaussian, if and only if the random variable $|X|/\sqrt{n}$ is concentrated around the value one. In other words, typical marginals are approximately gaussian if and only if most of the mass is concentrated on a "thin spherical shell" of radius $\sqrt{n}$ and width much smaller than $\sqrt{n}$. Therefore, to a certain extent, our task is essentially reduced to proving the following:

Theorem 1.4 *Let $n \geq 1$ be an integer and let X be a random vector with an isotropic, log-concave density in $\mathbb{R}^n$. Then for all $0 \leq \varepsilon \leq 1$,*

$$Prob\left\{ \left| \frac{|X|}{\sqrt{n}} - 1 \right| \geq \varepsilon \right\} \leq Cn^{-c\varepsilon^2},$$

where $c, C > 0$ are universal constants.

A significantly superior estimate to that of Theorem 1.4, for the case where ε is a certain universal constant greater than one, is given by Paouris [39], [40]. It would be interesting to try and improve the bound in Theorem 1.4 also for smaller values of ε.

Returning to the sketch of the proof, suppose that we are given a random vector X in $\mathbb{R}^n$ with an isotropic, log-concave density. We need to show that most of its marginals are almost-gaussian. Select a random k-dimensional subspace $E \subset \mathbb{R}^n$, for a certain integer k. We use a concentration of measure inequality – in a way similar to Milman's proof of Dvoretzky's theorem – to show that with large probability of choosing the subspace E, the distribution of the random vector $Proj_E(X)$ is approximately spherically-symmetric. This step is carried out in Section 3, and it is also outlined by Gromov [18, Section 1.2].

Fix a subspace E such that $Proj_E(X)$ is approximately spherically-symmetric. In Section 4 we use the Fourier transform to conclude that the approximation by a spherically-symmetric distribution actually holds in the stronger L^∞-sense, after convolving with a gaussian. In Section 5 we show that the gaussian convolution has only a minor effect, and we obtain a spherically-symmetric approximation to $Proj_E(X)$ in the total-variation, L^1-sense. Thus, we obtain a density in the subspace E that has two properties: It is log-concave, by Prékopa-Leindler, and it is also approximately radial. A key observation is that such densities are necessarily very close to the uniform distribution on the sphere; this observation boils down to estimating the asymptotics of some one-dimensional integral. At this point, we further project our density, that is already known to be close to the uniform distribution on a sphere, to any lower-dimensional subspace. By Maxwell's principle we obtain an approximately gaussian distribution in this lower-dimensional subspace. This completes the rough sketch of our proof.

Throughout this paper, unless stated otherwise, the letters $c, C, c', \tilde{C}$ etc. denote positive universal constants, that are not necessarily the same in different appearances. The symbols $C, C', \bar{C}, \tilde{C}$ etc. denote universal constants that are assumed to be sufficiently large, while $c, c', \bar{c}, \tilde{c}$ etc. denote sufficiently small universal constants. We abbreviate $\log$ for the natural logarithm, $\mathbb{E}$ for expectation, $Prob$ for probability and Vol for volume.

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2. Some background on log-concave functions

Here we gather some useful facts pertaining mostly to log-concave densities. For more information about log-concave functions, the reader is referred to, e.g., [2], [22] and [29]. The raison d'être of log-concave densities on $\mathbb{R}^n$ stems from the classical Brunn-Minkowski inequality and its generalizations. Let $E \subset \mathbb{R}^n$ be a subspace, and let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable function. We denote the marginal of f with respect to the subspace E by

$$\pi_E(f)(x) = \int_{x+E^\perp} f(y)dy \quad (x \in E)$$

where $x+E^\perp$ is the affine subspace in $\mathbb{R}^n$ that is orthogonal to E and passes through x. The Prékopa-Leindler inequality (see [42], [26], [43] or the first pages of [41]), which is a functional version of Brunn-Minkowski, implies that $\pi_E(f)$ is log-concave whenever f is log-concave and integrable. Therefore, when f is isotropic and log-concave, $\pi_E(f)$ is also isotropic and log-concave. A further consequence of the Prékopa-Leindler inequality, is that when f and g are integrable log-concave functions on $\mathbb{R}^n$, so is their convolution $f * g$. (The latter result actually goes back to [10], [27] and [47].)

Lemma 2.1 *Let $n \geq 1$ be an integer, and let X be a random vector in $\mathbb{R}^n$ with a log-concave density. Assume that $F : \mathbb{R}^n \to [0, \infty)$ is an even, convex function, such that $F(tx) = tF(x)$ for all $t > 0, x \in \mathbb{R}^n$. Denote $E = \sqrt{\mathbb{E}|F(X)|^2}$. Then,*

(i) $Prob\{F(X) \geq tE\} \leq 2e^{-t/10}$ *for all* $t \geq 0$.

Additionally, let $0 < \varepsilon \leq \frac{1}{2}$, and let $M > 0$ satisfy $Prob\{F(X) \geq M\} \leq \varepsilon$. Then,

(ii) $Prob\{F(X) \geq tM\} \leq (1-\varepsilon)\left(\frac{\varepsilon}{1-\varepsilon}\right)^{(t+1)/2}$ *for all* $t \geq 1$.

Lemma 2.1 is the well-known Borell's lemma (see its elegant proof in [6] or [35, Theorem III.3]). Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable function.

For $\theta \in S^{n-1}$ and $t \in \mathbb{R}$ we define $H_{\theta,t} = \{x \in \mathbb{R}^n; \langle x, \theta \rangle \leq t\}$ and

$$M_f(\theta, t) = \int_{H_{\theta,t}} f(x)dx. \tag{1}$$

The function M_f is continuous in θ and t, non-decreasing in t, and its derivative $\frac{\partial M_f}{\partial t}$ is the Radon transform of f. Thus, in principle, one may recover the function f from a complete knowledge of M_f. Clearly, for any subspace $E \subset \mathbb{R}^n$,

$$M_{\pi_E(f)}(\theta, t) = M_f(\theta, t) \quad \text{for all} \quad \theta \in S^{n-1} \cap E, t \in \mathbb{R}. \tag{2}$$

Moreover, let $\theta \in S^{n-1}$, let $E = \mathbb{R}\theta$ be the one-dimensional subspace spanned by θ, and denote $g = \pi_E(f)$. Then

$$g(t\theta) = \frac{\partial}{\partial t} M_{\pi_E(f)}(\theta, t) = \frac{\partial}{\partial t} M_f(\theta, t) \tag{3}$$

for all points $t \in \mathbb{R}$ where, say, $g(t\theta)$ is continuous.

Lemma 2.2 *Let $n \geq 1$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, log-concave function. Fix $\theta \in S^{n-1}$. Then,*

(i) For $t \geq 0$ we have $1 - 2e^{-|t|/10} \leq M_f(\theta, t) \leq 1$.
(ii) For $t \leq 0$ we have $0 \leq M_f(\theta, t) \leq 2e^{-|t|/10}$.

Proof: Let X be a random vector with density f. Then $\mathbb{E}|\langle X, \theta \rangle|^2 = 1$. We use Lemma 2.1(i), with the function $F(x) = |\langle x, \theta \rangle|$, to deduce the desired inequalities. $\square$

The space of all isotropic, log-concave functions in a fixed dimension is a compact space, with respect to, e.g., the L^1-metric. In particular, one-dimensional log-concave functions are quite rigid. For instance, suppose that $g : \mathbb{R} \to [0, \infty)$ is an isotropic, log-concave function. Then (see Hensley [20] and also, e.g., [29, Lemma 5.5] or [14]),

$$\frac{1}{10} \leq g(0) \leq \sup_{x \in \mathbb{R}} g(x) \leq 1. \tag{4}$$

We conclude that for any log-concave, isotropic function $f : \mathbb{R}^n \to [0, \infty)$,

$$|M_f(\theta, t) - M_f(\theta, s)| \leq |t - s| \quad \text{for all } s, t \in \mathbb{R},\ \theta \in S^{n-1}. \tag{5}$$

To prove (5), we set $E = \mathbb{R}\theta$ and $g = \pi_E(f)$. Then g is isotropic and log-concave, hence $\sup g \leq 1$ by (4). Note that g is continuous in the interior of its support, since it is a log-concave function. According to (3), the function $t \mapsto g(t\theta)$ is the derivative of the function $t \mapsto M_f(\theta, t)$, and (5) follows.

Our next proposition is essentially taken from Anttila, Ball and Perissinaki [1], yet we use the extension to the non-even case which is a particular case of a result of Bobkov [4, Proposition 3.1]. A function $g : S^{n-1} \to \mathbb{R}$ is L-Lipshitz, for $L > 0$, if $|g(x) - g(y)| \leq L|x - y|$ for all $x, y \in S^{n-1}$.

Proposition 2.3 *Let $n \geq 1$ be an integer. Let $t \in \mathbb{R}$ and let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, log-concave function. Then, the function*

$$\theta \mapsto M_f(\theta, t) \quad (\theta \in S^{n-1})$$

is C-Lipshitz on S^{n-1}. Here, $C > 0$ is a universal constant.

The proof of Proposition 2.3 in [4] involves analysis of two-dimensional log-concave functions. A beautiful argument yielding Proposition 2.3, for the case where f is an even function, appears in [1]. The approach in [1] is based on an application of Busemann's theorem in dimension $n+1$, which leads to the conclusion that $\theta \mapsto |\theta| M_f(t, \theta/|\theta|)^{-1}$ is a norm on $\mathbb{R}^n$ for any fixed $t \geq 0$.

3. Techniques from Milman's proof of Dvoretzky's theorem

It is well-known that for large n, the uniform probability measure σ_{n-1} on the unit sphere S^{n-1} satisfies strong concentration inequalities. This concentration of measure phenomenon is one of the main driving forces in high-dimensional convex geometry, as was first demonstrated by Milman in his proof of Dvoretzky's theorem (see [32] or [15, Section 4.2]). Our next proposition is essentially taken from Milman's work, though the precise formulation we use is due to Gordon [16], [17] (see also [45], [46] or [36, Theorem 6]).

Proposition 3.1 *Let $n \geq 1$ be an integer, let $L > 0$, $0 < \varepsilon \leq 1/2$, and let $g : S^{n-1} \to \mathbb{R}$ be an L-Lipshitz function. Denote $M = \int_{S^{n-1}} g(x) d\sigma_{n-1}(x)$. Assume that $1 \leq k \leq \hat{c}\varepsilon^2 n$ is an integer. Suppose that $E \in G_{n,k}$ is a random subspace, i.e., E is distributed according to the probability measure $\sigma_{n,k}$ on $G_{n,k}$. Then, with probability greater than $1 - \exp\left(-c\varepsilon^2 n\right)$,*

$$|g(\theta) - M| \leq \varepsilon L \quad \text{for all } \theta \in S^{n-1} \cap E. \tag{1}$$

Here, $0 < c, \hat{c} < 1$ are universal constants.

Our use of "Dvoretzky's theorem type" arguments in the next lemma is inspired by the powerful methods of Paouris in [38], [39], [40].

Lemma 3.2 *Let $n \geq 1$ be an integer, let $A \geq 1, 0 < \delta \leq \frac{1}{2}$ and let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, log-concave function. Assume that $1 \leq \ell \leq c\delta A^{-1} \log n$ is an integer, and let E be a random ℓ-dimensional subspace in $\mathbb{R}^n$. Then with probability greater than $1 - e^{-cn^{1-\delta}}$,*

$$\sup_{\theta \in S^{n-1} \cap E} M_f(\theta, t) \leq e^{-A\ell} + \inf_{\theta \in S^{n-1} \cap E} M_f(\theta, t) \quad \text{for all } t \in \mathbb{R}. \tag{2}$$

Here, $0 < c < 1$ is a universal constant.

Proof: We may assume that n exceeds a given universal constant, since otherwise, for a suitable choice of a small universal constant c, there is no ℓ with $1 \leq \ell \leq c\delta A^{-1} \log n$. Fix a real number t. According to Proposition 2.3, the function $\theta \mapsto M_f(\theta, t)$ is C-Lipshitz on S^{n-1}. Let $E \in G_{n,\ell}$ be a random subspace, uniformly distributed in $G_{n,\ell}$. We would like to apply Proposition 3.1 with $k = \ell, L = C$ and $\varepsilon = \frac{1}{2} n^{-\delta/2}$. Note that for this choice of parameters,

$$k = \ell \leq c\delta A^{-1} \log n \leq \hat{c}\varepsilon^2 (\log 1/\varepsilon)^2 n \quad \text{and} \quad 2\varepsilon L \leq e^{-2c\delta \log n} \leq e^{-2A\ell},$$

provided that c is a sufficiently small, positive universal constant, and that n is greater than some universal constant. Hence the appeal to Proposition 3.1 is legitimate. From the conclusion of that proposition, with probability larger than $1 - e^{-c' n^{1-\delta}}$ of selecting E,

$$\sup_{\theta \in S^{n-1} \cap E} M_f(\theta, t) \leq e^{-2A\ell} + \inf_{\theta \in S^{n-1} \cap E} M_f(\theta, t). \tag{3}$$

For any fixed $t \in \mathbb{R}$, the estimate (3) holds with probability greater than $1 - e^{-c' n^{1-\delta}}$. Denote $I = \{ i \cdot e^{-2A\ell} ; i = -\lceil e^{30A\ell} \rceil, ..., \lceil e^{30A\ell} \rceil \}$. Then, with probability greater than $1 - e^{-\bar{c} n^{1-\delta}}$, we obtain

$$\forall t \in I, \quad \sup_{\theta \in S^{n-1} \cap E} M_f(\theta, t) \leq e^{-2A\ell} + \inf_{\theta \in S^{n-1} \cap E} M_f(\theta, t). \tag{4}$$

Indeed, the estimate for the probability follows from the inequality $(2e^{30A\ell} + 3) e^{-c' n^{1-\delta}} \leq e^{-\bar{c} n^{1-\delta}}$.

Fix an ℓ-dimensional subspace $E \subset \mathbb{R}^n$ that satisfies (4). Select $\theta_1, \theta_2 \in S^{n-1} \cap E$. We will demonstrate that for any $t \in \mathbb{R}$,

$$M_f(\theta_1, t) \leq e^{-A\ell} + M_f(\theta_2, t). \tag{5}$$

To that end, note that when $|t| \geq 20A\ell$, by Lemma 2.2,

$$|M_f(\theta_1, t) - M_f(\theta_2, t)| \leq 2e^{-|t|/10} \leq 2e^{-2A\ell} \leq e^{-A\ell}. \tag{6}$$

Hence (5) holds for $|t| \geq 20A\ell$. We still need to consider the case where $|t| < 20A\ell$. In this case, $|t| \leq e^{20A\ell}$ and hence there exists $t_0 \in I$ with $|t - t_0| \leq \frac{1}{2} \cdot e^{-2A\ell}$. According to (5) from Section 2, the function $t \mapsto M_f(\theta_i, t)$ is 1-Lipshitz for $i = 1, 2$. Therefore, by using (4), we conclude (5) also for the case where $|t| < 20A\ell$. Thus (5) holds for all $t \in \mathbb{R}$, under the assumption that E satisfies (4).

Recall that $\theta_1, \theta_2 \in S^{n-1} \cap E$ are arbitrary, hence we may take the supremum over θ_1 and the infimum over θ_2 in (5). We discover that whenever the subspace E satisfies (4), it necessarily also satisfies (2). The probability for a random ℓ-dimensional subspace $E \subset \mathbb{R}^n$ to satisfy (4) was shown to be greater than $1 - e^{-\bar{c} n^{1-\delta}}$. The lemma thus follows. $\square$

Remark. For the case where f is even, Lemma 3.2 follows from a direct application of Dvoretzky's theorem in Milman's form. Indeed, in this case, $\theta \mapsto |\theta| M_f(\theta,t)^{-1}$ is a norm, and Lemma 3.2 asserts that this norm is almost Hilbertian when restricted to certain random subspaces.

4. Almost spherical log-concave functions

A large portion of this section is devoted to proving the following proposition.

Proposition 4.1 *There exist universal constants $C_0, C > 1$ and $0 < c < 1$ for which the following holds: Let $n \geq 1$ be an integer and let $f : \mathbb{R}^n \to [0,\infty)$ be an isotropic, log-concave function. Assume that*

$$\sup_{\theta \in S^{n-1}} M_f(\theta,t) \leq e^{-C_0 n} + \inf_{\theta \in S^{n-1}} M_f(\theta,t) \quad \textit{for all} \;\; t \in \mathbb{R}. \tag{1}$$

Suppose that Y is a random vector in $\mathbb{R}^n$ with density f. Then for all $0 < \varepsilon < 1$,

$$Prob\left\{ \left| \frac{|Y|}{\sqrt{n}} - 1 \right| \geq \varepsilon \right\} \leq Ce^{-c\varepsilon^2 n}. \tag{2}$$

For $n \geq 1$ and $v > 0$ we define $\gamma_{n,v} : \mathbb{R}^n \to [0,\infty)$ to be the function

$$\gamma_{n,v}(x) = \frac{1}{(2\pi v)^{n/2}} \exp\left(-\frac{|x|^2}{2v} \right). \tag{3}$$

Then $\gamma_{n,v}$ is the density of a gaussian random vector in $\mathbb{R}^n$ with expectation zero and covariance matrix that equals vId, where Id is the identity matrix. We write $O(n)$ for the group of orthogonal transformations of $\mathbb{R}^n$.

Lemma 4.2 *Let $n \geq 1$ be an integer, let $\alpha \geq 5$, and let $f : \mathbb{R}^n \to [0,\infty)$ be an isotropic, log-concave function. Assume that*

$$\sup_{\theta \in S^{n-1}} M_f(\theta,t) \leq e^{-5\alpha n} + \inf_{\theta \in S^{n-1}} M_f(\theta,t) \quad \textit{for all} \;\; t \in \mathbb{R}. \tag{4}$$

Denote $g = f * \gamma_{n,1}$, where $*$ stands for convolution. Then,

$$\sup_{\theta \in S^{n-1}} g(t\theta) \leq e^{-\alpha n} + \inf_{\theta \in S^{n-1}} g(t\theta) \quad \textit{for all} \;\; t \geq 0. \tag{5}$$

Proof: We will show that the Fourier transform of f is almost spherically-symmetric. As usual, we define

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi, x \rangle} f(x) dx \quad (\xi \in \mathbb{R}^n),$$

where $i^2 = -1$. Let $r > 0$, and fix $\xi_1, \xi_2 \in \mathbb{R}^n$ with $|\xi_1| = |\xi_2| = r$. Denote by $E_1 = \mathbb{R}\xi_1, E_2 = \mathbb{R}\xi_2$ the one-dimensional subspaces spanned by ξ_1, ξ_2, respectively. From (3) of Section 2 we see that $\pi_{E_j}(f)(t\xi_j/|\xi_j|) = \frac{\partial}{\partial t} M_f(\xi_j/|\xi_j|, t)$ for $j = 1, 2$ and for all t in the interior of the support of the log-concave function $t \mapsto \pi_{E_j}(f)(t\xi_j/|\xi_j|)$. By integrating by parts we obtain

$$\begin{aligned} \hat{f}(\xi_1) - \hat{f}(\xi_2) &= \int_{-\infty}^{\infty} \left[\pi_{E_1}(f) \left(t \frac{\xi_1}{|\xi_1|} \right) - \pi_{E_2}(f) \left(t \frac{\xi_2}{|\xi_2|} \right) \right] e^{-2\pi i r t} dt \\ &= 2\pi i r \int_{-\infty}^{\infty} \left[M_f \left(\frac{\xi_1}{|\xi_1|}, t \right) - M_f \left(\frac{\xi_2}{|\xi_2|}, t \right) \right] e^{-2\pi i r t} dt, \qquad (6) \end{aligned}$$

as the boundary terms clearly vanish. From Lemma 2.2 we have

$$\left| M_f \left(\frac{\xi_1}{|\xi_1|}, t \right) - M_f \left(\frac{\xi_2}{|\xi_2|}, t \right) \right| \leq 2e^{-|t|/10} \quad \text{for all } \ t \in \mathbb{R}. \qquad (7)$$

According to (6), (7) and to our assumption (4), we conclude that for any $r > 0$ and $\xi_1, \xi_2 \in \mathbb{R}^n$ with $|\xi_1| = |\xi_2| = r$,

$$\begin{aligned} & |\hat{f}(\xi_1) - \hat{f}(\xi_2)| \qquad (8) \\ & \leq 2\pi r \left[80\alpha n \cdot e^{-5\alpha n} + \int_{|t| > 40\alpha n} 2e^{-|t|/10} dt \right] \leq r e^{-2\alpha n}, \end{aligned}$$

where we made use of the fact that $\alpha n \geq 5$. A standard computation (e.g. [49, page 6]) shows that $\widehat{\gamma_{n,1}}(\xi) = e^{-2\pi^2 |\xi|^2}$. Recall that we define $g = f * \gamma_{n,1}$, and hence $\hat{g}(\xi) = e^{-2\pi^2|\xi|^2} \cdot \hat{f}(\xi)$. We thus deduce from (8) that for any $\xi_1, \xi_2 \in \mathbb{R}^n$,

$$|\hat{g}(\xi_1) - \hat{g}(\xi_2)| \leq e^{-2\pi^2 r^2} r e^{-2\alpha n} \quad \text{whenever } \ |\xi_1| = |\xi_2| = r > 0. \qquad (9)$$

Let $x \in \mathbb{R}^n$, and let $U \in O(n)$ be an orthogonal transformation. By using the inverse Fourier transform (see, e.g. [49, Chapter I]) and applying (9), we get

$$\begin{aligned} |g(x) - g(Ux)| &= \left| \int_{\mathbb{R}^n} [\hat{g}(\xi) - \hat{g}(U\xi)] e^{2\pi i \langle x, \xi \rangle} d\xi \right| \\ &\leq \int_{\mathbb{R}^n} e^{-2\pi^2|\xi|^2} |\xi| e^{-2\alpha n} d\xi \leq e^{-2\alpha n} \int_{\mathbb{R}^n} e^{-\pi|\xi|^2} d\xi = e^{-2\alpha n}. \qquad (10) \end{aligned}$$

Since $x \in \mathbb{R}^n$ and $U \in O(n)$ are arbitrary, from (10) we conclude (5). $\square$

Let $f : [0, \infty) \to [0, \infty)$ be a log-concave function with $0 < \int_0^\infty f < \infty$, that is continuous on $[0, \infty)$ and C^2-smooth on $(0, \infty)$. For $p > 1$, denote by $t_p(f)$ the unique $t > 0$ for which $f(t) > 0$ and also

$$(\log f)'(t) = \frac{f'(t)}{f(t)} = -\frac{p-1}{t}. \tag{11}$$

Lemma 4.3 *$t_p(f)$ is well-defined, under the above assumptions on f and p.*

Proof: We need to explain why a solution t to equation (11) exists and is unique, for all $p > 1$. To that end, note that f is a log-concave function with finite, positive mass, hence it decays exponentially fast at infinity (this is a very simple fact; see, e.g., [23, Lemma 2.1]). Therefore, the function $\varphi(t) = t^{p-1} f(t)$ satisfies

$$\lim_{t \to 0^+} \varphi(t) = \lim_{t \to \infty} \varphi(t) = 0.$$

The function φ is continuous, non-negative, not identically zero, and tends to zero at 0 and at ∞. Consequently, φ attains its positive maximum at some finite point $t_0 > 0$. Then $\varphi(t_0) > 0$ and $\varphi'(t_0) = 0$, since φ is C^2-smooth. On the other hand, f is log-concave, and $t \mapsto t^{p-1}$ is strictly log-concave, hence φ is strictly log-concave on its support. Therefore, there is at most one point where φ is non-zero and φ' vanishes. We conclude that there exists exactly one point $t_0 > 0$ such that $f(t_0) > 0$ and

$$\varphi'(t_0) = t_0^{p-2} \left[(p-1) f(t_0) + t_0 f'(t_0) \right] = 0.$$

Thus a finite, positive t that solves (11) exists and is unique. $\square$

Let us mention a few immediate properties of the quantity $t_p(f)$. First, $f(t_p(f)) > 0$ for all $p > 1$. Second, suppose that f is a continuous, log-concave function on $[0, \infty)$, C^2-smooth on $(0, \infty)$, with $0 < \int f < \infty$. Then,

$$f(t) \geq e^{-(n-1)} f(0) \quad \text{for any} \quad 0 \leq t \leq t_n(f). \tag{12}$$

Indeed, if $f(0) = 0$ then (12) is trivial. Otherwise, $f(0) > 0$ and $f(t_n(f)) > 0$, hence f is necessarily positive on $[0, t_n(f)]$ by log-concavity. Therefore $\log f$ is finite and continuous on $[0, t_n(f)]$, and C^2-smooth in $(0, t_n(f))$. Additionally, $\log f$ is concave, hence $(\log f)'$ is non-increasing in $(0, t_n(f))$. From the definition (11) we deduce that $(\log f)'(t) \geq -(n-1)/t_n(f)$ for all $0 < t < t_n(f)$, and (12) follows.

Furthermore, since $(\log f)'$ is non-increasing on the interval in which it is defined, then $(\log f)'(t) \leq -(n-1)/t_n(f)$ for $t > t_n(f)$ for which $f(t) > 0$. We conclude that for any $\alpha \geq 1$,

$$f(t) \leq e^{-(\alpha-1)(n-1)} f(t_n(f)) \quad \text{when} \quad t \geq \alpha t_n(f). \tag{13}$$

Note that $t_p(f)$ behaves well under scaling of f. Indeed, let f be a continuous, log-concave function on $[0,\infty)$, C^2-smooth on $(0,\infty)$, with $0 < \int f < \infty$. For $\delta > 0$, denote $\tau_\delta(x) = \delta x$. From the definition (11) we see that for any $p > 1$,

$$t_p(f \circ \tau_\delta) = \delta^{-1} \cdot t_p(f). \tag{14}$$

Lemma 4.4 *Let $n \geq 2$, and let $f, g : [0,\infty) \to [0,\infty)$ be continuous, log-concave functions, C^2-smooth on $(0,\infty)$, such that $f(0) > 0, g(0) > 0$ and $\int f < \infty, \int g < \infty$. Assume that for any $t \geq 0$,*

$$|f(t) - g(t)| \leq e^{-5n} \min\{f(0), g(0)\}. \tag{15}$$

Then,

$$\left(1 - e^{-n}\right) t_n(g) \leq t_n(f) \leq \left(1 + e^{-n}\right) t_n(g).$$

Proof: Set $\delta = t_n(f)$. According to (14), both the conclusions and the requirements of the lemma are invariant when we replace f and g with $f \circ \tau_\delta$ and $g \circ \tau_\delta$, respectively. We apply this replacement, and assume from now on that $t_n(f) = 1$.

Inequality (12) and our assumption that $f(0) > 0$ show that $f(t) \geq e^{-n} f(0) > 0$ for $0 \leq t \leq 1$. We combine this inequality with (15) to obtain the bound $|g(t)/f(t) - 1| \leq e^{-4n}$ for all $0 \leq t \leq 1$. In particular, g is positive on $[0,1]$. Denote $f_0 = \log f, g_0 = \log g$. Then for all $0 \leq t \leq 1$,

$$-2e^{-4n} < \log(1 - e^{-4n}) \leq g_0(t) - f_0(t) \leq \log(1 + e^{-4n}) < e^{-4n}. \tag{16}$$

Next, we claim that

$$g_0'(t) \geq f_0'\left(t + e^{-2n}\right) - 4e^{-2n} \quad \text{for all } \ 0 < t \leq 1 - e^{-2n}. \tag{17}$$

Indeed, assume by contradiction that (17) does not hold. Then there exists $0 < t_0 \leq 1 - e^{-2n}$ for which $g_0'(t_0) < f_0'(t_0 + e^{-2n}) - 4e^{-2n}$. From our assumptions, f and g are log-concave, hence f_0 and g_0 are concave, and hence f_0' and g_0' are non-increasing on $(0,1)$. Therefore, for $t \in (t_0, t_0 + e^{-2n})$,

$$g_0'(t) \leq g_0'(t_0) < f_0'(t_0 + e^{-2n}) - 4e^{-2n} \leq f_0'(t) - 4e^{-2n}. \tag{18}$$

Denote $t_1 = t_0 + e^{-2n}$. Then $[t_0, t_1] \subset [0,1]$ and by (18),

$$[f_0(t_1) - g_0(t_1)] \, - \, [f_0(t_0) - g_0(t_0)] \, > \, 4e^{-2n} \cdot (t_1 - t_0) \, = \, 4e^{-4n},$$

in contradiction to (16). Thus, our momentary assumption – that (17) does not hold – was false, and hence (17) is proved.

From the definition (11) we see that $f_0'(1) = (\log f)'(1) = -(n-1)$. Recall once again that g_0' is non-increasing. By applying the case $t = 1 - e^{-2n}$ in (17), we conclude that for $0 < s < 1 - 4e^{-2n}$,

$$\begin{aligned} &g_0'(s) \geq g_0'(1 - e^{-2n}) \geq f_0'(1) - 4e^{-2n} = -(n-1) - 4e^{-2n} \\ &\geq -(n-1)\left(1 + 4e^{-2n}\right) \geq -\frac{n-1}{1-4e^{-2n}} > -\frac{n-1}{s}. \end{aligned} \tag{19}$$

From (19) we conclude that $g'(s)/g(s) = g_0'(s) \neq -\frac{n-1}{s}$ for all $0 < s < 1 - 4e^{-2n}$. The definition (11) shows that

$$t_n(g) \geq 1 - 4e^{-2n}.$$

Recalling the scaling argument above, we see that we have actually proved that

$$t_n(g) \geq (1 - 4e^{-2n}) t_n(f),$$

whenever the assumptions of the lemma hold. However, these assumptions are symmetric in f and g. Hence,

$$t_n(g) \geq (1 - 4e^{-2n}) t_n(f) \quad \text{and also} \quad t_n(f) \geq (1 - 4e^{-2n}) t_n(g)$$

for any functions f, g that satisfy the assumptions of the lemma. Since $1 + e^{-n} \geq 1/(1 - 4e^{-2n})$ for $n \geq 2$, the lemma is proved. $\square$

Our next lemma is a standard application of the Laplace asymptotic method, and is similar to, e.g., [22, Lemma 2.1] and [23, Lemma 2.5]. We will make use of the following well-known bound: For $\alpha, \delta > 0$,

$$\int_\delta^\infty e^{-\alpha \frac{t^2}{2}} dt = \frac{1}{\sqrt{\alpha}} \int_{\delta\sqrt{\alpha}}^\infty e^{-\frac{t^2}{2}} dt \leq \frac{\sqrt{2\pi}}{\sqrt{\alpha}} e^{-\alpha \frac{\delta^2}{2}}. \tag{20}$$

The inequality in (20) may be proved, for example, by computing the Laplace transform of the gaussian density and applying Markov's inequality (e.g., [50, Section 1.3]).

Lemma 4.5 *Let $n \geq 2$ be an integer, and let $f : [0,\infty) \to [0,\infty)$ be a continuous, log-concave function, C^2-smooth on $(0,\infty)$, with $0 < \int_0^\infty f < \infty$. Then for $0 \leq \varepsilon \leq 1$,*

$$\int_{t_n(f)(1-\varepsilon)}^{t_n(f)(1+\varepsilon)} t^{n-1} f(t) dt \geq \left(1 - Ce^{-c\varepsilon^2 n}\right) \int_0^\infty t^{n-1} f(t) dt, \tag{21}$$

where $C > 1$ and $0 < c < 1$ are universal constants.

Proof: We begin with a scaling argument. A glance at (14) and (21) assures us that both the validity of the assumptions and the validity of the conclusions of the present lemma, are not altered when we replace f with $f \circ \tau_\delta$, for any $\delta > 0$. Hence, we may switch from f to $f \circ \tau_{t_n(f)}$, and reduce matters to the case $t_n(f) = 1$. Thus $f(1) > 0$. Multiplying f by an appropriate positive constant, we may assume that $f(1) = 1$.

We denote $\psi(t) = (n-1)\log t + \log f(t) \ \ (t > 0)$, where we set $\psi(t) = -\infty$ whenever $f(t) = 0$. Since $f(1) = 1$, then $\psi(1) = 0$. Additionally, $\psi'(1) = 0$ because $t_n(f) = 1$. The function ψ is concave, and therefore it attains its maximum at 1. Let $s_0, s_1 > 0$ be the minimal positive numbers for which $\psi(1 - s_0) = -1$ and $\psi(1 + s_1) = -1$. Such s_0 and s_1 exist since ψ is continuous, $\psi(1) = 0$ and $\psi(t) \to -\infty$ when $t \to 0$ (because of $\log t$) and when $t \to \infty$ (because of $\log f$, since f is log-concave with $0 < \int f < \infty$).

We may suppose that $n \geq 100$; for an appropriate choice of a large universal constant C, the right hand side of (21) is negative for $n < 100$, and hence the lemma is obvious for $n < 100$. Denote $m = \inf\{t > 0; f(t) \neq 0\}$ and $M = \sup\{t > 0; f(t) \neq 0\}$. Since $t_n(f) = 1$, necessarily $m < 1$ and $M > 1$. Then, for $m < t < M$,

$$\psi''(t) = -\frac{n-1}{t^2} + (\log f)''(t) \leq -\frac{n-1}{t^2}, \tag{22}$$

since $\log f$ is concave and hence $(\log f)'' \leq 0$. From (22) we obtain, in particular, the inequality $\psi''(t) \leq -\frac{n-1}{4}$ for $m < t < \min\{2, M\}$. Recalling that $\psi(1) = \psi'(1) = 0$, we see that $\psi(t) \leq -\frac{n-1}{8}(t-1)^2$ for all $0 < t < 2$. Therefore $\psi(1 - 4/\sqrt{n}) \leq -1$ and $\psi(1 + 4/\sqrt{n}) \leq -1$, and consequently

$$s_0 \leq \frac{4}{\sqrt{n}} \quad \text{and} \quad s_1 \leq \frac{4}{\sqrt{n}}. \tag{23}$$

Since $n \geq 100$, then (23) implies that $s_0, s_1 \leq \frac{1}{2}$. Recall that the function ψ is concave, hence ψ' is non-increasing. The relations $\psi(1 - s_0) = \psi(1 + s_1) = -1, \psi(1) = 0$ thus imply that

$$\psi'(1 - s_0) \geq \frac{1}{s_0} \quad \text{and} \quad \psi'(1 + s_1) \leq -\frac{1}{s_1}. \tag{24}$$

Examination of (22) shows us that $\psi''(t) \leq -(n-1)$ for $m < t \leq 1 - s_0$. By definition, $\psi(1 - s_0) = -1$. We thus conclude from (24) that $\psi(1 - s_0 - t) \leq -1 - \frac{t}{s_0} - \frac{n-1}{2}t^2$ for $0 < t < 1 - s_0$. Fix $0 \leq \varepsilon \leq 1$. Then,

$$\int_0^{1-s_0-\varepsilon} e^{\psi(t)} dt \leq e^{-1} \int_\varepsilon^\infty e^{-\frac{t}{s_0} - (n-1)\frac{t^2}{2}} dt \tag{25}$$

$$\leq \min\left\{ s_0 e^{-\frac{\varepsilon}{s_0}}, \int_\varepsilon^\infty e^{-(n-1)\frac{t^2}{2}} dt \right\} \leq \min\left\{ s_0 e^{-\frac{\varepsilon}{s_0}}, \frac{e^{-(n-1)\frac{\varepsilon^2}{2}}}{\sqrt{(n-1)/(2\pi)}} \right\}$$

where we used (20) to estimate the last integral. Next, observe again that $\psi''(t) \leq -\frac{n-1}{4}$ for all $m < t < \min\{2, M\}$, by (22). We use (24), as well as the fact that $\psi(1+s_1) = -1$, to obtain

$$\psi(1+s_1+t) \leq -1 - \frac{t}{s_1} - \frac{n-1}{8}t^2 \quad \text{for } 0 \leq t \leq 1 - s_1. \tag{26}$$

Consequently,

$$\int_{1+s_1+\varepsilon}^{2} e^{\psi(t)} dt \leq e^{-1} \int_{\varepsilon}^{\infty} e^{-\frac{t}{s_1} - (n-1)\frac{t^2}{8}} dt \tag{27}$$

$$\leq \min\left\{ s_1 e^{-\frac{\varepsilon}{s_1}}, \int_{\varepsilon}^{\infty} e^{-(n-1)\frac{t^2}{8}} dt \right\} \leq \min\left\{ s_1 e^{-\frac{\varepsilon}{s_1}}, \frac{e^{-(n-1)\frac{\varepsilon^2}{8}}}{\sqrt{(n-1)/(8\pi)}} \right\}$$

by (20). Since $s_1 \leq \frac{1}{2}$, we deduce from (26) that $\psi(2) \leq -\frac{1}{2s_1} - \frac{n-1}{32}$. Recall that ψ' is non-increasing, that $\psi'(1) = 0$ and that $\psi''(t) \leq -\frac{n-1}{4}$ for $1 < t < \min\{2, M\}$. Therefore, $\psi'(t) \leq -\frac{n-1}{4}$ whenever $2 \leq t < M$. Thus we realize that $\psi(2+t) \leq \left(-\frac{1}{2s_1} - \frac{n-1}{32}\right) - \frac{n-1}{4}t$ for $t \geq 0$. Hence,

$$\int_{2}^{\infty} e^{\psi(t)} dt \leq e^{-\frac{1}{2s_1} - \frac{n-1}{32}} \int_{0}^{\infty} e^{-\frac{n-1}{4}t} dt \leq \frac{8s_1}{n-1} e^{-\frac{n-1}{32}}. \tag{28}$$

Let $s = s_0 + s_1$. Then, by the definition of s_0 and s_1,

$$\int_{0}^{\infty} e^{\psi(t)} dt \geq \int_{1-s_0}^{1+s_1} e^{\psi(t)} dt \geq \int_{1-s_0}^{1+s_1} e^{-1} dt = e^{-1}s. \tag{29}$$

The inequalities we gathered above will allow us to prove (21). Note that (21) is trivial when $\varepsilon \leq \frac{4}{\sqrt{n}}$; for an appropriate choice of a large constant C, the right-hand side of (21) is negative in this case. We may thus restrict our attention to the case where $\frac{4}{\sqrt{n}} < \varepsilon < 1$. Hence, $s_0 + \varepsilon \leq 2\varepsilon$ and $s_1 + \varepsilon \leq 2\varepsilon$, by (23). We add (25), (27) and (28) to get

$$\int_{|t-1| \geq 2\varepsilon} e^{\psi(t)} dt \leq \min\left\{ s e^{-\varepsilon/s}, \frac{20}{\sqrt{n}} e^{-\frac{\varepsilon^2 n}{20}} \right\} + \frac{20s}{n} \cdot e^{-n/100}. \tag{30}$$

Division of (30) by (29) yields,

$$\frac{\int_{|t-1| \geq 2\varepsilon} \exp(\psi(t)) dt}{\int_0^\infty \exp(\psi(t)) dt} \leq 60 \min\left\{ e^{-\varepsilon/s}, \frac{e^{-\frac{\varepsilon^2 n}{20}}}{s\sqrt{n}} \right\} + 40 e^{-n/100}. \tag{31}$$

In order to establish (21) and complete the proof, it is sufficient to show that

$$\int_{|t-1| \geq 2\varepsilon} \exp(\psi(t)) dt \leq 100 e^{-\varepsilon^2 n/100} \int_0^\infty \exp(\psi(t)) dt. \tag{32}$$

According to (23), we know that $s = s_0 + s_1 \le \frac{10}{\sqrt{n}}$. In the case where

$$\varepsilon > 10 \frac{\sqrt{\log \frac{10}{s\sqrt{n}}}}{\sqrt{n}},$$

we have $\frac{1}{s\sqrt{n}} < \exp\left(\frac{\varepsilon^2 n}{100}\right)$ and hence the estimate (32) follows from (31) by choosing the "$\frac{e^{-\frac{\varepsilon^2 n}{20}}}{s\sqrt{n}}$" term in the minimum in (31). In the complementary case, we have

$$\varepsilon \le 10 \frac{\sqrt{\log \frac{10}{s\sqrt{n}}}}{\sqrt{n}} \le \frac{100}{sn},$$

since $\sqrt{\log t} \le t$ for $t \ge 1$. In this case, $\varepsilon/s \ge \frac{1}{100}\varepsilon^2 n$, and (32) follows by selecting the "$e^{-\varepsilon/s}$" term in (31). Hence (32) is proved for all cases. The proof is complete. $\square$

The following lemma is standard, and is almost identical, for example, to [35, Appendix V.4]. For a random vector X in $\mathbb{R}^n$, we denote its covariance matrix by $Cov(X)$.

Lemma 4.6 *Let $n \ge 1$ be an integer, let $A, r, \alpha, \beta > 0$ and let X be a random vector in $\mathbb{R}^n$ with $\mathbb{E}X = 0$ and $Cov(X) = \beta Id$. Assume that the density of X is log-concave, and that*

$$Prob\left\{ \left| \frac{|X|}{r} - 1 \right| \ge \varepsilon \right\} \le Ae^{-\alpha\varepsilon^2 n} \quad \textit{for} \ \ 0 \le \varepsilon \le 1. \tag{33}$$

Then,

(i) For all $0 \le \varepsilon \le 1, \quad Prob\left\{ \left| \frac{|X|}{\sqrt{\beta n}} - 1 \right| \ge \varepsilon \right\} \le C' e^{-c'\varepsilon^2 n}.$

(ii) $\left| \frac{r}{\sqrt{\beta n}} - 1 \right| \le \frac{C}{\sqrt{n}}$ *provided that $n \ge C$.*

Here, $C, C', c' > 0$ are constants that depend solely on A and α.

Proof: By a simple scaling argument, we may assume that $\beta = 1$; otherwise, replace the function $f(x)$ with the function $\beta^{n/2} f(\beta^{1/2} x)$. In this proof, c, C, C' etc. stand for constants depending only on A and α. We begin by proving (ii). Since $\sqrt{\mathbb{E}|X|^2} = \sqrt{n}$, Lemma 2.1(i) implies that

$$Prob\left\{ |X| \ge t\sqrt{n} \right\} \le 2e^{-t/10} \quad \text{for all } t > 0.$$

Therefore,

$$\left|n-r^2\right| \leq \mathbb{E}\left||X|^2-r^2\right| = \int_0^\infty Prob\left\{\left||X|^2-r^2\right| \geq t\right\} dt \tag{34}$$

$$\leq \int_0^{r^2} A\exp\left(-\frac{\alpha t^2 n}{8r^4}\right) dt + \int_{r^2}^\infty \min\left\{Ae^{-\alpha n}, 2\exp\left(-\frac{\sqrt{t}}{10\sqrt{n}}\right)\right\} dt$$

$$\leq C\frac{r^2}{\sqrt{n}} + C'n^3Ae^{-\alpha n} + C''e^{-cn} < C\frac{r^2}{\sqrt{n}} + \tilde{C}e^{-\tilde{c}n},$$

provided that $n > C$. From (34) we deduce (ii). To prove (i), it is enough to consider the case where $\varepsilon \geq \frac{C}{\sqrt{n}}$. In this case, by (ii),

$$Prob\left\{\left||X|-\sqrt{n}\right| \geq \varepsilon\sqrt{n}\right\} \leq Prob\left\{\left||X|-r\right| \geq C'\varepsilon r\right\}$$

and (i) follows from (33) for the range $0 < \varepsilon < 1/C'$. By adjusting the constants, we establish (i) for the entire range $0 \leq \varepsilon \leq 1$. $\square$

Lemma 4.7 *Let $n \geq 1$ be an integer, let $\beta > 0$, and let $f : \mathbb{R}^n \to [0,\infty)$ be a log-concave function that is the density of a random vector with zero mean and with covariance matrix that equals βId. Then*

$$f(0) \geq e^{-n}\sup_{x\in\mathbb{R}^n} f(x) \geq \left(\frac{c}{\sqrt{\beta}}\right)^n$$

where $0 < c < 1$ is a universal constant.

Proof: The inequality $f(0) \geq e^{-n}\sup f$ is proved in [14, Theorem 4]. By our assumptions, $\int_{\mathbb{R}^n}|x|^2 f(x)dx = \beta n$. Markov's inequality entails

$$\int_{\sqrt{2\beta n}D^n} f(x)dx \geq \frac{1}{2}.$$

Therefore,

$$\sup_{x\in\mathbb{R}^n} f \geq \frac{1}{Vol(\sqrt{2\beta n}D^n)}\int_{\sqrt{2\beta n}D^n} f(x)dx \geq (C\beta)^{-n/2}\cdot\frac{1}{2},$$

since $Vol(\sqrt{n}D^n) \leq \tilde{C}^n$ (see, e.g., [41, page 11]). $\square$

Proof of Proposition 4.1: Recall our assumption (1) and our desired conclusion (2) from the formulation of the proposition. We assume that n is greater than some large universal constant, since otherwise (2) is obvious for an appropriate choice of constants $C, c > 0$. Denote $g = f * \gamma_{n,1}$, the convolution of f and $\gamma_{n,1}$. Then g is log-concave, and is the density of a random vector with mean zero and covariance matrix $2Id$. By Lemma 4.7,

$$g(0) \geq \bar{c}^n. \tag{35}$$

We set $C_0 = 25\,(1 + \log 1/\bar{c})$ where $0 < \bar{c} < 1$ is the constant from (35). Our assumption (1) is precisely the basic requirement of Lemma 4.2, for $\alpha = C_0/5 \geq 5$. By the conclusion of that lemma,

$$\sup_{\theta \in S^{n-1}} g(t\theta) \leq e^{-5n} g(0) + \inf_{\theta \in S^{n-1}} g(t\theta) \quad \text{for all } \ t \geq 0, \tag{36}$$

since $e^{-C_0 n/5} \leq e^{-5n} g(0)$, according to the definition of C_0 and (35). The function g is C^∞-smooth, since $g = f * \gamma_{n,1}$ with $\gamma_{n,1}$ being C^∞-smooth. Additionally, since $0 < \int g < \infty$ then for some $A, B > 0$,

$$g(x) \leq A e^{-B|x|} \quad \text{for all } \ x \in \mathbb{R}^n \tag{37}$$

(see, e.g., [23, Lemma 2.1]). For $\theta \in S^{n-1}$ and $t \geq 0$, we write $g_\theta(t) = g(t\theta)$. Then g_θ is log-concave, continuous on $[0,\infty)$, C^∞-smooth on $(0,\infty)$ and integrable on $[0,\infty)$ by (37). In addition, $g_\theta(0) = g(0) > 0$ by (35). Fix $\theta_0 \in S^{n-1}$, and denote $r_0 = t_n(g_{\theta_0})$. According to (36), for any $\theta \in S^{n-1}$ and $t \geq 0$,

$$|g_\theta(t) - g_{\theta_0}(t)| \leq e^{-5n} g(0) = e^{-5n} \min\{g_\theta(0), g_{\theta_0}(0)\}.$$

Thus the functions g_θ and g_{θ_0} satisfy the assumptions of Lemma 4.4, for any $\theta \in S^{n-1}$. By the conclusion of that lemma, for any $\theta \in S^{n-1}$,

$$(1 - e^{-n}) r_0 \leq t_n(g_\theta) \leq (1 + e^{-n}) r_0,$$

because $r_0 = t_n(g_{\theta_0})$. We deduce that for any $10 e^{-n} \leq \varepsilon \leq 1$ and $\theta \in S^{n-1}$,

$$(1+\varepsilon) r_0 \geq \left(1 + \frac{\varepsilon}{2}\right) t_n(g_\theta) \quad \text{and} \quad (1-\varepsilon) r_0 \leq \left(1 - \frac{\varepsilon}{2}\right) t_n(g_\theta). \tag{38}$$

For $0 \leq \varepsilon \leq 1$ let $A_\varepsilon = \{x \in \mathbb{R}^n; \big||x| - r_0\big| \leq \varepsilon r_0\}$. We will prove that for all $0 \leq \varepsilon \leq 1$,

$$\int_{A_\varepsilon} g(x) dx \geq 1 - C e^{-c\varepsilon^2 n}. \tag{39}$$

Note that (39) is obvious for $\varepsilon < 10 e^{-n} \leq \frac{10}{\sqrt{n}}$, since in this case $1 - C e^{-c\varepsilon^2 n} \leq 0$ for an appropriate choice of universal constants $c, C > 0$. We still need to deal with the case $10 e^{-n} \leq \varepsilon \leq 1$. To that end, note that g_θ satisfies the requirements of Lemma 4.5 for any $\theta \in S^{n-1}$ by the discussion above. We will integrate in polar coordinates and use (38) as well as Lemma 4.5. This yields

$$\begin{aligned}\int_{A_\varepsilon} g(x) dx &\geq \int_{S^{n-1}} \int_{(1-\varepsilon/2) t_n(g_\theta)}^{(1+\varepsilon/2) t_n(g_\theta)} t^{n-1} g_\theta(t) dt d\theta \\ &\geq \left(1 - C e^{-c\varepsilon^2 n}\right) \int_{S^{n-1}} \int_0^\infty t^{n-1} g_\theta(t) dt d\theta = 1 - C e^{-c\varepsilon^2 n},\end{aligned}$$

since $\int_{\mathbb{R}^n} g = 1$. This completes the proof of (39).

Let $X_1, X_2, ...$ be a sequence of independent, real-valued, standard gaussian random variables. By the classical central limit theorem,

$$Prob \left\{ \sum_{i=1}^{m} X_i^2 \leq m \right\} \stackrel{m\to\infty}{\longrightarrow} \frac{1}{2}.$$

Consequently, $1/C' \leq Prob\{\sum_{i=1}^n X_i^2 \leq n\} \leq 1 - 1/C'$ for some universal constant $C' > 0$. Denote $X = (X_1, ..., X_n)$. Then X is distributed according to the density $\gamma_{n,1}$ in $\mathbb{R}^n$. We record the bound just mentioned:

$$\frac{1}{C'} \leq Prob\{|X|^2 \leq n\} \leq 1 - \frac{1}{C'}. \tag{40}$$

Let Y be another random vector in $\mathbb{R}^n$, independent of X, that is distributed according to the density f. Since the density of X is an even function, then for any measurable sets $I, J \subset [0, \infty)$ with $Prob\{|X| \in I\} > 0$ and $Prob\{|Y| \in J\} > 0$,

$$Prob \left\{ \langle X, Y \rangle \geq 0 \textit{ given that } |X| \in I, |Y| \in J \right\} = \frac{1}{2}. \tag{41}$$

Additionally, the random vector $X + Y$ has g as its density, because $g = f * \gamma_{n,1}$. Therefore (39) translates to

$$Prob \left\{ \big| |X + Y| - r_0 \big| > \varepsilon r_0 \right\} \leq C e^{-c\varepsilon^2 n} \quad \text{for all } \ 0 \leq \varepsilon \leq 1. \tag{42}$$

Since X and Y are independent, we conclude from (40), (41) and (42) that for all $0 < \varepsilon < 1$,

$$\begin{aligned} & Prob \left\{ |Y|^2 \geq r_0^2 (1 + \varepsilon)^2 - n \right\} \\ & \leq 2C' Prob \left\{ |Y|^2 \geq r_0^2 (1 + \varepsilon)^2 - n, |X| \geq \sqrt{n}, \langle X, Y \rangle \geq 0 \right\} \\ & \leq 2C' Prob \left\{ |X + Y|^2 \geq r_0^2 (1 + \varepsilon)^2 \right\} \leq C \exp\left(-c\varepsilon^2 n\right), \end{aligned} \tag{43}$$

and similarly,

$$\begin{aligned} & Prob \left\{ |Y|^2 \leq r_0^2 (1 - \varepsilon)^2 - n \right\} \\ & \leq 2C' Prob \left\{ |Y|^2 \leq r_0^2 (1 - \varepsilon)^2 - n, |X| \leq \sqrt{n}, \langle X, Y \rangle \leq 0 \right\} \\ & \leq 2C' Prob \left\{ |X + Y| \leq r_0 (1 - \varepsilon) \right\} \leq C \exp\left(-c\varepsilon^2 n\right). \end{aligned} \tag{44}$$

Next, we estimate r_0. Recall that the density of $X + Y$ is log-concave, $\mathbb{E}(X + Y) = 0$ and $Cov(X + Y) = 2Id$. We invoke Lemma 4.6(ii), based on (42), and conclude that $3n/2 \leq r_0^2 \leq 3n$, under the legitimate assumption that $n > C$. Denote $r = \sqrt{r_0^2 - n}$. Then $\sqrt{n/2} \leq r \leq \sqrt{2n}$ and

$$r^2(1 + 10\varepsilon)^2 \geq r_0^2(1 + \varepsilon)^2 - n, \qquad r_0^2(1 - \varepsilon)^2 - n \geq r^2(1 - 10\varepsilon)^2,$$

for $0 \le \varepsilon \le 1/10$. Therefore, (43) and (44) imply that for any $0 < \varepsilon < \frac{1}{10}$,

$$Prob\left\{ r^2(1-10\varepsilon)^2 \le |Y|^2 \le r^2(1+10\varepsilon)^2 \right\} \ge 1 - 2Ce^{-c\varepsilon^2 n}.$$

After adjusting the constants, we see that

$$\forall 0 \le \varepsilon \le 1, \quad Prob\left\{ \left| \frac{|Y|}{r} - 1 \right| \ge \varepsilon \right\} \le C' e^{-c'\varepsilon^2 n}. \tag{45}$$

Recall that Y is distributed according to the density f, which is an isotropic, log-concave function. We may thus apply Lemma 4.6(i), based on (45), and conclude (2). The proposition is proved. □

We proceed to discuss applications of Proposition 4.1. The following lemma is usually referred to as the Johnson-Lindenstrauss dimension reduction lemma [21]. We refer, e.g., to [9, Lemma 2.2] for an elementary proof. Recall that we denote by $Proj_E(x)$ the orthogonal projection of x onto E, whenever x is a point in $\mathbb{R}^n$ and $E \subset \mathbb{R}^n$ is a subspace.

Lemma 4.8 *Let $1 \le k \le n$ be integers, and let $E \in G_{n,k}$ be a random k-dimensional subspace. Let $x \in \mathbb{R}^n$ be a fixed vector. Then for all $0 \le \varepsilon \le 1$,*

$$Prob\left\{ \left| |Proj_E(x)| - \sqrt{\frac{k}{n}}|x| \right| \ge \varepsilon\sqrt{\frac{k}{n}}|x| \right\} \le Ce^{-c\varepsilon^2 k} \tag{46}$$

where $c, C > 0$ are universal constants.

Proof of Theorem 1.4: We use the constant $C_0 \ge 1$ from Proposition 4.1, and the constant c from Lemma 3.2. Let $\ell = \lfloor \frac{c}{100C_0}\log n \rfloor$ and fix $0 \le \varepsilon \le 1/3$. We may assume that $\ell \ge 1$; otherwise, n is smaller than some universal constant and the conclusion of the theorem is obvious. We assume that X is a random vector in $\mathbb{R}^n$ whose density is an isotropic, log-concave function to be denoted by f. Let $E \in G_{n,\ell}$ be a fixed subspace that satisfies

$$\sup_{\theta \in S^{n-1}\cap E} M_f(\theta, t) \le e^{-C_0\ell} + \inf_{\theta \in S^{n-1}\cap E} M_f(\theta, t) \quad \text{for all } \ t \in \mathbb{R}. \tag{47}$$

Denote $g = \pi_E(f)$. Then (47) translates, with the help of (2) from Section 2, to

$$\sup_{\theta \in S^{n-1}\cap E} M_g(\theta, t) \le e^{-C_0\ell} + \inf_{\theta \in S^{n-1}\cap E} M_g(\theta, t) \quad \text{for all } \ t \in \mathbb{R}. \tag{48}$$

The function g is an isotropic, log-concave function, and it is the density of $Proj_E(X)$. We invoke Proposition 4.1, for ℓ and g, based on (48). By the conclusion of that proposition,

$$Prob\left\{ \left| \frac{|Proj_E(X)|}{\sqrt{\ell}} - 1 \right| \ge \varepsilon \right\} \le C' e^{-c'\varepsilon^2 \ell}, \tag{49}$$

under the assumption that the subspace E satisfies (47). Suppose that $F \in G_{n,\ell}$ is a random ℓ-dimensional subspace in $\mathbb{R}^n$, independent of X. Recall our choice of the integer ℓ. According to Lemma 3.2, with probability greater than $1 - e^{-cn^{0.99}}$, the subspace $E = F$ satisfies (47). We conclude from (49) that

$$Prob\left\{ \left| \frac{|Proj_F(X)|}{\sqrt{\ell}} - 1 \right| \geq \varepsilon \right\} \leq C' e^{-c'\varepsilon^2 \ell} + e^{-cn^{0.99}} \leq \tilde{C} e^{-\tilde{c}\varepsilon^2 \ell},$$

where the last inequality holds as $\ell \leq \log n$ and $0 \leq \varepsilon \leq 1/3$. Since X and F are independent, then by Lemma 4.8,

$$Prob\left\{ \left| |Proj_F(X)| - \sqrt{\frac{\ell}{n}}|X| \right| \geq \varepsilon \sqrt{\frac{\ell}{n}}|X| \right\} \leq \hat{C} e^{-\hat{c}\varepsilon^2 \ell}.$$

To summarize, with probability greater than $1 - \bar{C} e^{-\bar{c}\varepsilon^2 \ell}$ we have

(i) $(1-\varepsilon)\sqrt{\ell} \leq |Proj_F(X)| \leq (1+\varepsilon)\sqrt{\ell}$, and also

(ii) $(1+\varepsilon)^{-1}\sqrt{\frac{n}{\ell}}|Proj_F(X)| \leq |X| \leq (1-\varepsilon)^{-1}\sqrt{\frac{n}{\ell}}|Proj_F(X)|$.

Hence,

$$Prob\left\{ \frac{1-\varepsilon}{1+\varepsilon} \leq \frac{|X|}{\sqrt{n}} \leq \frac{1+\varepsilon}{1-\varepsilon} \right\} \geq 1 - \bar{C} e^{-\bar{c}\varepsilon^2 \ell}. \tag{50}$$

Note that $\frac{1+\varepsilon}{1-\varepsilon} \leq 1 + 3\varepsilon$ and $1 - 3\varepsilon \leq \frac{1-\varepsilon}{1+\varepsilon}$, and recall that $0 \leq \varepsilon \leq \frac{1}{3}$ was arbitrary, and that $\ell = \lfloor \frac{c}{100C_0} \log n \rfloor$. By adjusting the constants, we deduce from (50) that the inequality in the conclusion of the theorem is valid for all $0 \leq \varepsilon \leq 1$. The theorem is thus proved. $\square$

The following lemma may be proved via a straightforward computation. Nevertheless, we will present a shorter, indirect proof that is based on properties of the heat kernel, an idea we borrow from [7, Theorem 3.1].

Lemma 4.9 *Let $n \geq 1$ be an integer and let $\alpha, \beta > 0$. Then,*

$$\int_{\mathbb{R}^n} |\gamma_{n,\alpha}(x) - \gamma_{n,\beta}(x)|\, dx \leq C\sqrt{n} \left| \frac{\beta}{\alpha} - 1 \right|, \tag{51}$$

where $C > 0$ is a universal constant.

Proof: The integral on the left-hand side of (51) is never larger than 2. Consequently, the lemma is obvious when $\frac{\beta}{\alpha} > 2$ or when $\frac{\beta}{\alpha} < \frac{1}{2}$, and hence we may assume that $\frac{1}{2}\alpha \leq \beta \leq 2\alpha$. Moreover, in this case both the left-hand side and the right-hand side of (51) are actually symmetric in α and β up to a factor of at most 2. Therefore, we may assume that

$\alpha < \beta \leq 2\alpha$ (the case $\beta = \alpha$ is obvious). For $t > 0$ and for a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, we define

$$(P_t f)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y) dy \quad (x \in \mathbb{R}^n)$$

whenever the integral converges. Then $(P_t)_{t>0}$ is the heat semigroup on $\mathbb{R}^n$. We will make use of the following estimate: For any smooth, integrable function $f : \mathbb{R}^n \to \mathbb{R}$ and any $t > 0$,

$$\int_{\mathbb{R}^n} |(P_t f)(x) - f(x)|\, dx \leq 2\sqrt{t} \int_{\mathbb{R}^n} |\nabla f(x)| dx. \tag{52}$$

An elegant proof of the inequality (52), in a much more general setting, is given by Ledoux [25, Section 5]. It is straightforward to verify that

$$\int_{\mathbb{R}^n} |\nabla \gamma_{n,\alpha}(x)| dx = \frac{1}{(2\pi\alpha)^{n/2}} \int_{\mathbb{R}^n} \frac{|x|}{\alpha} e^{-\frac{|x|^2}{2\alpha}} dx$$
$$\leq \frac{1}{\alpha} \left(\frac{1}{(2\pi\alpha)^{n/2}} \int_{\mathbb{R}^n} |x|^2 e^{-\frac{|x|^2}{2\alpha}} dx \right)^{1/2} = \sqrt{\frac{n}{\alpha}}.$$

Consequently, (52) implies that

$$\int_{\mathbb{R}^n} \left| P_{\frac{\beta-\alpha}{2}} \left(\gamma_{n,\alpha}\right)(x) - \gamma_{n,\alpha}(x) \right| dx \leq 2\sqrt{\frac{\beta-\alpha}{2}} \sqrt{\frac{n}{\alpha}}. \tag{53}$$

It is well-known and easy to prove that $\gamma_{n,\beta} = P_{\frac{\beta-\alpha}{2}}(\gamma_{n,\alpha})$. Since $\alpha < \beta \leq 2\alpha$, then (53) implies (51). The lemma is proved. $\square$

We are now able to prove Theorem 1.2 by combining the classical Berry-Esseen bound with Theorem 1.4.

Proof of Theorem 1.2: We may assume that n exceeds a given universal constant. Let f and X be as in the assumptions of Theorem 1.2. According to Theorem 1.4,

$$Prob \left\{ \left| \frac{|X|}{\sqrt{n}} - 1 \right| \geq \varepsilon \right\} \leq C n^{-c\varepsilon^2} \quad \text{for all } \ 0 \leq \varepsilon \leq 1. \tag{54}$$

The case $\varepsilon = \sqrt{2} - 1$ in (54) shows that $\delta_0 := Prob\left\{|X| \geq \sqrt{2n}\right\} \leq C n^{-c/4} \leq n^{-c/10}$, under the legitimate assumption that n exceeds a certain universal constant. By (54) and by Lemma 2.1(ii),

$$\mathbb{E}\left| \frac{|X|^2}{n} - 1 \right| = \int_0^\infty Prob\left\{ \left| \frac{|X|^2}{n} - 1 \right| \geq t \right\} dt \tag{55}$$
$$\leq \int_0^1 C' n^{-c't^2} dt + \int_1^\infty (1-\delta_0) \left(\frac{\delta_0}{1-\delta_0} \right)^{(\sqrt{\frac{1+t}{2}}+1)/2} dt \leq \frac{C''}{\sqrt{\log n}}.$$

Let $\delta_1, ..., \delta_n$ be independent Bernoulli random variables, that are also independent of X, such that $Prob\{\delta_i = 1\} = Prob\{\delta_i = -1\} = 1/2$ for $i = 1, ..., n$. For $t \in \mathbb{R}$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$ denote

$$P(x; t) = Prob\left\{ \frac{\sum_{i=1}^n \delta_i x_i}{\sqrt{n}} \leq t \right\}.$$

We write

$$\Phi_{\sigma^2}(t) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{t} \exp\left(-\frac{t^2}{2\sigma^2} \right) dt$$

for $\sigma > 0$ and $t \in \mathbb{R}$. By the Berry-Esseen bound (see, e.g., [13, Section XVI.5] or [50, Section 2.1.30]), for any $x \in \mathbb{R}^n$,

$$\sup_{t \in \mathbb{R}} \left| P(x; t) - \Phi_{|x|^2/n}(t) \right| \leq C \frac{\sum_{i=1}^n |x_i|^3}{|x|^3}, \tag{56}$$

where $C > 0$ is a universal constant. Since f is unconditional, the random variable $\left(\sum_{i=1}^n X_i\right)/\sqrt{n}$ has the same law of distribution as the random variable $\left(\sum_{i=1}^n \delta_i X_i\right)/\sqrt{n}$. For $t \in \mathbb{R}$ we set

$$P(t) = Prob\left\{ \frac{\sum_{i=1}^n X_i}{\sqrt{n}} \leq t \right\} = Prob\left\{ \frac{\sum_{i=1}^n \delta_i X_i}{\sqrt{n}} \leq t \right\}.$$

We denote the expectation over the random variable X by $\mathbb{E}_X$. Then $P(t) = \mathbb{E}_X P(X; t)$ by the complete probability formula. For $i = 1, ..., n$, the random variable X_i has mean zero, variance one, and its density is a log-concave function. Consequently, $\mathbb{E}|X_i|^2 = 1$, and by Lemma 2.1(i), for any $1 \leq i \leq n$,

$$Prob\left\{ |X_i| \geq 20 \log n \right\} \leq 2e^{-2\log n} = \frac{2}{n^2}.$$

Therefore, with probability greater than $1 - \frac{2}{n}$ of selecting X,

$$|X_i| \leq 20 \log n \quad \text{for all} \ \ 1 \leq i \leq n. \tag{57}$$

Fix $t \in \mathbb{R}$. We substitute into (56) the information from (57), and from the case $\varepsilon = 1/2$ in (54). We see that with probability greater than $1 - Cn^{-c/4} - \frac{2}{n}$ of selecting X,

$$\left| P(X; t) - \Phi_{\frac{|X|^2}{n}}(t) \right| \leq C \frac{\sum_{i=1}^n |X_i|^3}{|X|^3} \leq C' \frac{(\log n)^3}{\sqrt{n}}.$$

Since always $0 \leq P(X; t) \leq 1$ and $0 \leq \Phi_1(t) \leq 1$, we conclude that

$$\mathbb{E}_X \left| P(X; t) - \Phi_{\frac{|X|^2}{n}}(t) \right| \leq C' \frac{(\log n)^3}{\sqrt{n}} + 2Cn^{-c/4} + \frac{2}{n} < \frac{C'}{n^{c'}}. \tag{58}$$

According to Lemma 4.9, for any $x \in \mathbb{R}^n$,

$$\left| \Phi_{\frac{|x|^2}{n}}(t) \,-\, \Phi_1(t) \right| \leq \int_{-\infty}^{\infty} \left| \gamma_{1,\frac{|x|^2}{n}}(s) - \gamma_{1,1}(s) \right| ds \leq \hat{C} \left| \frac{|x|^2}{n} - 1 \right|,$$

and therefore by (55)

$$\mathbb{E}_X \left| \Phi_{\frac{|X|^2}{n}}(t) \,-\, \Phi_1(t) \right| \leq \hat{C} \mathbb{E}_X \left| \frac{|X|^2}{n} - 1 \right| \leq \frac{C}{\sqrt{\log n}}. \qquad (59)$$

Recall that $P(t) = \mathbb{E}_X P(X;t)$ and that t is an arbitrary real number. We apply Jensen's inequality, and then combine (58) and (59) to obtain

$$\forall t \in \mathbb{R}, \quad |\, P(t) \,-\, \Phi_1(t) \,| \leq \mathbb{E}_X \,|\, P(X;t) \,-\, \Phi_1(t) \,| \leq \frac{C}{\sqrt{\log n}}. \qquad (60)$$

The random variable $(X_1 + ... + X_n)/\sqrt{n}$ has mean zero, variance one and a log-concave density. Its cumulative distribution function $P(t) = Prob\,\{(X_1 + ... + X_n)/\sqrt{n} \leq t\}$ satisfies (60). Therefore, we may invoke [8, Theorem 3.3], and conclude from (60) that

$$d_{TV}\left(\frac{X_1 + ... + X_n}{\sqrt{n}} \,,\, Z \right) \leq \check{C} \left(\frac{C \log \frac{C}{\sqrt{\log n}}}{\sqrt{\log n}} \right)^{1/2} = \check{C} \frac{\sqrt{\log \log n}}{(\log n)^{1/4}},$$

where $Z \sim N(0,1)$ is a standard gaussian random variable. The theorem follows, with $\varepsilon_n \leq C(\log\log(n+2))^{1/2}/(\log(n+1))^{1/4}$. $\square$

Remarks.

1. Suppose that f is a log-concave density in high dimension that is isotropic and unconditional. In Theorem 1.2, we were able to describe an explicit one-dimensional marginal of f that is approximately normal. It seems possible to identify some multi-dimensional subspaces $E \subset \mathbb{R}^n$, spanned by specific sign-vectors, such that $\pi_E(f)$ is guaranteed to be almost-gaussian. We did not pursue this direction.
2. Under the assumptions of Theorem 1.2, we proved that $\langle X, \theta \rangle$ is approximately gaussian when $\theta = (1, ..., 1)/\sqrt{n}$. A straightforward adaptation of the proof of Theorem 1.2 shows that $\langle X, \theta \rangle$ is approximately gaussian under the weaker assumption that $|\theta_1|, ..., |\theta_n|$ are rather small (as in Lindeberg's condition).
3. Theorem 1.1, with a worse bound for ε_n, follows by combining Theorem 1.4 with the methods in [1], and then applying [8, Theorem 3.3]. We will deduce Theorem 1.1 from the stronger Theorem 1.3 in the next section.

5. Multi-dimensional marginals

The next few pages are devoted to the proof of the following lemma.

Lemma 5.1 *Let $n \geq 2$ be an integer, let $\alpha \geq 10$, and let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, log-concave function. Denote $g = f * \gamma_{n,n^{-30\alpha}}$. Then,*

$$\int_{\mathbb{R}^n} |g(x) - f(x)| dx \leq \frac{C}{n^{\alpha/10}},$$

where $C > 0$ is a universal constant.

We begin with an addendum to Lemma 4.5. Rather than appealing to the Laplace asymptotic method once again, we will base our proof on an elegant observation by Bobkov regarding one-dimensional log-concave functions.

Lemma 5.2 *Let $n \geq 2$ be an integer, let $\alpha \geq 5$ and let $f : [0, \infty) \to [0, \infty)$ be a log-concave function with $\int f < \infty$. Denote $t_0 = \sup\{t > 0; f(t) \geq e^{-\alpha n} f(0)\}$. Then,*

$$\int_0^{t_0} t^{n-1} f(t) dt \geq \left(1 - e^{-\alpha n/8}\right) \int_0^\infty t^{n-1} f(t) dt. \tag{1}$$

Proof: If $\int f = 0$ then $f \equiv 0$ almost everywhere and (1) is trivial. Thus, we may suppose that $\int f > 0$. Moreover, we may assume that f is continuous on $[0, \infty)$ and C^2-smooth on $(0, \infty)$, by approximation (for example, convolve f with $\gamma_{1,\varepsilon}$ on $\mathbb{R}$, restrict the result to $[0, \infty)$, and let ε tend to zero). Since $0 < \int f < \infty$ then f decays exponentially fast at infinity, and $0 < \int_0^\infty t^{n-1} f(t) dt < \infty$. Multiplying f by a positive constant, we may assume that $\int_0^\infty t^{n-1} f(t) dt = 1$.

For $t > 0$, denote,

$$\phi(t) = t^{n-1} f(t) \quad \text{and} \quad \Phi(t) = \int_0^t \phi(s) ds.$$

Then ϕ is a log-concave function with $\int \phi = 1$. Recall the definition of $t_n(f)$, that is, (11) from Section 4. According to that definition, $\phi'(t_n(f)) = 0$. Denote $M = f(t_n(f)) > 0$. Then $M \geq e^{-(n-1)} f(0)$ by (12) from Section 4, and hence

$$t_0 \geq t_1 := \sup\left\{t > 0; f(t) \geq e^{-(\alpha-1)(n-1)} M\right\},$$

where t_0 is defined in the formulation of the lemma. Since $M > 0$ and since f is continuous and vanishes at infinity, the number t_1 is finite, greater than

$t_n(f)$, and satisfies $f(t_1) = e^{-(\alpha-1)(n-1)}M$. From (13) of Section 4 we see that $t_1 \leq \alpha t_n(f)$. Therefore,

$$\phi(t_1) = \phi(t_n(f)) \cdot \left(\frac{t_1}{t_n(f)}\right)^{n-1} \cdot \frac{f(t_1)}{M}$$
$$\leq \phi(t_n(f)) \cdot \alpha^{n-1} \cdot e^{-(\alpha-1)(n-1)} \leq \phi(t_n(f)) e^{-\alpha n/8} = e^{-\alpha n/8} \cdot \max \phi,$$

where $\phi(t_n(f)) = \max \phi$ because ϕ is log-concave, $\phi(t_n(f)) > 0$ and $\phi'(t_n(f)) = 0$. Let $\Phi^{-1} : (0,1) \to (0,\infty)$ stand for the inverse function to Φ. A useful fact we learned from Bobkov's work [4, Lemma 3.2] is that the function $\psi(t) = \phi(\Phi^{-1}(t))$ is concave on $(0,1)$. (To see this, differentiate ψ twice, and use the inequality $(\log \phi)'' \leq 0$.)

Since ϕ attains its maximum at $t_n(f)$, then ψ attains its maximum at $\Phi(t_n(f))$. The function ψ is non-negative and concave on $(0,1)$, hence for $t \geq \Phi(t_n(f))$ and $0 < \varepsilon < 1$,

$$\psi(t) \leq \varepsilon \cdot \max \psi \quad \Rightarrow \quad t \geq 1 - \varepsilon.$$

Equivalently, for $s \geq t_n(f)$ and $0 < \varepsilon < 1$, the inequality $\phi(s) \leq \varepsilon \cdot \max \psi = \varepsilon \cdot \max \phi$ implies the bound $\Phi(s) \geq 1 - \varepsilon$. We have shown that $t_1 \geq t_n(f)$ satisfies $\phi(t_1) \leq e^{-\alpha n/8} \max \phi$, and hence we conclude that $\Phi(t_1) \geq 1 - e^{-\alpha n/8}$. Recalling that $t_0 \geq t_1$, the lemma follows. □

Corollary 5.3 *Let $n \geq 2$ be an integer, let $\alpha \geq 5$, and let $f : \mathbb{R}^n \to [0,\infty)$ be a log-concave function with $\int f = 1$. Denote $K = \{x \in \mathbb{R}^n; f(x) \geq e^{-\alpha n} f(0)\}$. Then,*

$$\int_K f(x)dx \geq 1 - e^{-\alpha n/8}.$$

Proof: For $\theta \in S^{n-1}$ set

$$I(\theta) = \{t \geq 0; f(t\theta) \geq e^{-\alpha n} f(0)\} = \{t \geq 0; t\theta \in K\}.$$

By log-concavity, $I(\theta)$ is a (possibly infinite) interval in $[0,\infty)$ containing zero. For $t \geq 0$ and $\theta \in S^{n-1}$ we denote $f_\theta(t) = f(t\theta)$. Then f_θ is log-concave. Since $\int f = 1$, then, e.g., by [23, Lemma 2.1] we know that f decays exponentially fast at infinity and $\int f_\theta < \infty$. Next, we integrate in polar coordinates and use Lemma 5.2. This yields

$$\int_K f(x)dx = \int_{S^{n-1}} \int_0^{\sup I(\theta)} t^{n-1} f_\theta(t) dt d\theta$$
$$\geq \left(1 - e^{-\alpha n/8}\right) \int_{S^{n-1}} \int_0^\infty t^{n-1} f_\theta(t) dt d\theta = 1 - e^{-\alpha n/8}.$$

□

Lemma 5.4 *Let $n \geq 1$ be an integer and let X be a random vector in $\mathbb{R}^n$ with an isotropic, log-concave density. Suppose that $K \subset \mathbb{R}^n$ is convex with $Prob\{X \in K\} \geq \frac{9}{10}$. Then,*

$$\frac{1}{10} D^n \subset K.$$

Proof: Assume the contrary. Since K is convex, then there exists $\theta \in S^{n-1}$ such that $K \subset \{x \in \mathbb{R}^n; \langle x, \theta \rangle < 1/10\}$. Hence,

$$Prob\left\{\langle X, \theta \rangle \leq \frac{1}{10}\right\} \geq Prob\left\{X \in K\right\} \geq \frac{9}{10}. \tag{2}$$

Denote $E = \mathbb{R}\theta$, the one-dimensional line spanned by θ, and let $g = \pi_E(f)$. Then g is log-concave and isotropic, hence $\sup g \leq 1$ by (4) of Section 2. Since g is the density of the random variable $\langle X, \theta \rangle$ and $\sup g \leq 1$, then

$$Prob\left\{0 \leq \langle X, \theta \rangle \leq \frac{1}{10}\right\} = \int_0^{1/10} g(t)dt \leq \frac{1}{10}. \tag{3}$$

An appeal to [4, Lemma 3.3] – a result that essentially goes back to Grünbaum and Hammer [19] – shows that

$$Prob\{\langle X, \theta \rangle < 0\} \leq 1 - \frac{1}{e} < \frac{4}{5}. \tag{4}$$

After adding (4) to (3), we arrive at a contradiction to (2). This completes the proof. □

For two sets $A, B \subset \mathbb{R}^n$ we write $A + B = \{x + y; x \in A, y \in B\}$ and $A - B = \{x - y; x \in A, y \in B\}$ to denote their Minkowski sum and difference.

Lemma 5.5 *Let $n \geq 2$ be an integer, let $\alpha \geq 10$, and let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, log-concave function. Consider the sets $K_0 = \{x \in \mathbb{R}^n; f(x) \geq e^{-\alpha n} f(0)\}$ and $K = \{x \in \mathbb{R}^n; \exists y \notin K_0, |x - y| \leq n^{-3\alpha}\}$. Then,*

$$\int_K f(x)dx \leq \frac{C}{n^\alpha}$$

where $C > 0$ is a universal constant.

Proof: Let μ be the probability measure on $\mathbb{R}^n$ whose density is f. By Corollary 5.3,

$$\mu(K_0) = \int_{K_0} f(x)dx \geq 1 - e^{-\alpha n/8} \geq \frac{9}{10}. \tag{5}$$

The set K_0 is convex, since f is log-concave. According to (5) and Lemma 5.4,

$$\frac{1}{10}D^n \subset K_0. \tag{6}$$

By the definition, $K = (\mathbb{R}^n \setminus K_0) + n^{-3\alpha}D^n$. Since $D^n \subset -10K_0$, then

$$K \subset (\mathbb{R}^n \setminus K_0) - 10n^{-3\alpha}K_0 \subset \mathbb{R}^n \setminus \left(1 - n^{-2\alpha}\right) K_0, \tag{7}$$

because K_0 is convex and $10n^{-3\alpha} \le n^{-2\alpha}$. We use (6) and Lemma 4.7 for $\beta = 1$. This implies the estimate

$$\mu\left(\frac{D^n}{20}\right) = \int_{\frac{D^n}{20}} f(x)dx \ge e^{-\alpha n} f(0) \cdot Vol\left(\frac{D^n}{20}\right) \ge \left(\frac{c' e^{-\alpha}}{\sqrt{n}}\right)^n, \tag{8}$$

where we also used the standard estimate $Vol(D^n) \ge (c/\sqrt{n})^n$. The inclusion (6) and the convexity of K_0 entail that

$$\left(2n^{-2\alpha}\right)\frac{D^n}{20} + \left(1 - 2n^{-2\alpha}\right) K_0 \subset \left(1 - n^{-2\alpha}\right) K_0.$$

Therefore, according to the Prékopa-Leindler inequality,

$$\mu\left(\left(1 - n^{-2\alpha}\right) K_0\right) \ge \mu\left(\frac{D^n}{20}\right)^{2n^{-2\alpha}} \cdot \mu\left(K_0\right)^{1-2n^{-2\alpha}}. \tag{9}$$

We combine (7), (9), (8) and (5) to obtain

$$\begin{aligned} \mu(K) &\le \mu\left(\mathbb{R}^n \setminus \left(1 - n^{-2\alpha}\right) K_0\right) = 1 - \mu\left(\left(1 - n^{-2\alpha}\right) K_0\right) \\ &\le 1 - \left(\left(\frac{c' e^{-\alpha}}{\sqrt{n}}\right)^n\right)^{2n^{-2\alpha}} \cdot \left(1 - e^{-\alpha n/8}\right)^{1-2n^{-2\alpha}} \le \frac{C'}{n^{\alpha}}, \end{aligned}$$

for some universal constant $C' > 0$ (the verification of the last inequality is elementary and routine). The lemma is thus proved. $\square$

Proof of Lemma 5.1: By approximation, we may assume that f is continuously differentiable. Denote $\psi = \log f$ (with $\psi = -\infty$ when $f = 0$). Then ψ is a concave function. Consider the sets $K_0 = \{x \in \mathbb{R}^n; f(x) \ge e^{-\alpha n} f(0)\}$ and $K = \{x \in \mathbb{R}^n; \exists y \notin K_0, |x - y| < n^{-4\alpha}\}$. The first step of the proof is to show that

$$\{x \in K_0; |\nabla\psi(x)| > n^{5\alpha}\} \subset K. \tag{10}$$

Note that $f(0) > 0$ by [14, Theorem 4], and hence $f(x) > 0$ for all $x \in K_0$. Consequently, ψ is finite on K_0, and $\nabla\psi$ is well-defined on K_0. In order to prove (10), let us pick $x \in K_0$ such that $|\nabla\psi(x)| > n^{5\alpha}$. Set $\theta = \nabla\varphi(x)/|\nabla\varphi(x)|$. To prove (10), it suffices to show that

$$x - n^{-4\alpha}\theta \notin K_0,$$

by the definition of K. According to the definition of K_0, it is enough to prove that

$$f\left(x - n^{-4\alpha}\theta\right) < e^{-\alpha n} f(0). \tag{11}$$

We thus focus on proving (11). We may assume that $f(x - n^{-4\alpha}\theta) > 0$ since otherwise (11) holds trivially. By concavity, $\varphi(t) := \psi(x + t\theta) = \log f(x + t\theta)$ is finite for $-n^{-4\alpha} \le t \le 0$, and

$$\varphi'(0) = \langle \nabla\psi(x), \theta \rangle = |\nabla\psi(x)| > n^{5\alpha}.$$

Since φ is concave, then φ' is non-increasing. Consequently, $\varphi'(t) > n^{5\alpha}$ for $-n^{-4\alpha} \le t \le 0$. Hence,

$$\varphi(0) - \varphi(-n^{-4\alpha}) > n^{5\alpha} \cdot n^{-4\alpha} = n^{\alpha} \ge \alpha n + 1, \tag{12}$$

as $\alpha \ge 10$ and $n \ge 2$. Recall that $f(0) \ge e^{-n} f(x)$ by [14, Theorem 4] and that $f(x + t\theta) = e^{\varphi(t)}$. We conclude from (12) that $f(0) \ge e^{-n} f(x) > e^{\alpha n} f(x - n^{-4\alpha}\theta)$, and (11) is proved. This completes the proof of (10).

For $x \in \mathbb{R}^n$ and $\delta > 0$ denote $B(x, \delta) = \{y \in \mathbb{R}^n ; |y - x| \le \delta\}$. Fix $x \in K_0$ such that $B(x, n^{-3\alpha}) \subset K_0$. Then for any $y \in B(x, n^{-10\alpha})$ we have $y \notin K$ and hence $|\nabla\psi(y)| \le n^{5\alpha}$, by (10). Consequently,

$$|\psi(y) - \psi(x)| \le n^{5\alpha} |x - y| \le n^{-5\alpha} \quad \text{for all } \ y \in B(x, n^{-10\alpha}).$$

Recalling that $f = e^{\psi}$, we obtain

$$|f(y) - f(x)| \le 2n^{-5\alpha} f(x) \quad \text{for all } \ y \in B(x, n^{-10\alpha}). \tag{13}$$

We will also make use of the crude estimate

$$\int_{\mathbb{R}^n \setminus B(0, n^{-10\alpha})} \gamma_{n, n^{-30\alpha}}(x) dx \le 2\exp(-n^{4\alpha}/10) \le e^{-20\alpha n}, \tag{14}$$

that follows, for example, from Lemma 2.1(i) as $\sqrt{\int_{\mathbb{R}^n} |x|^2 \gamma_{n, n^{-30\alpha}}(x) dx} = n^{1/2 - 15\alpha}$. According to [14, Theorem 4],

$$\sup f \le e^n f(0) \le e^{(\alpha+1)n} f(x), \tag{15}$$

since $x \in K_0$. Recall that $g = f * \gamma_{n, n^{-30\alpha}}$. We use (13), (14) and (15) to conclude that

$$|g(x) - f(x)| \le \int_{\mathbb{R}^n} \gamma_{n, n^{-30\alpha}}(x - y) \left| f(y) - f(x) \right| dy \tag{16}$$

$$\le 2n^{-5\alpha} f(x) + 2 \sup f \cdot \int_{\mathbb{R}^n \setminus B(x, n^{-10\alpha})} \gamma_{n, n^{-30\alpha}}(x - y) dy \le \frac{C}{n^{5\alpha}} f(x).$$

Denote $T = \{x \in K_0; B(x, n^{-3\alpha}) \subset K_0\}$. We have shown that (16) holds for any $x \in T$. Thus,

$$\int_T |g(x) - f(x)|dx \leq \frac{C}{n^{5\alpha}} \int_T f(x)dx \leq \frac{C}{n^{5\alpha}}. \tag{17}$$

Note that $\mathbb{R}^n \setminus T \subset (\mathbb{R}^n \setminus K_0) \cup \{x \in \mathbb{R}^n; \exists y \notin K_0, |x - y| \leq n^{-3\alpha}\}$. Corollary 5.3 and Lemma 5.5 show that

$$\int_T f(x)dx = 1 - \int_{\mathbb{R}^n \setminus T} f(x)dx \geq 1 - e^{-\alpha n/8} - \frac{C}{n^\alpha} \geq 1 - \frac{C'}{n^{\alpha/10}}. \tag{18}$$

By (17) and (18),

$$\int_T g(x)dx \geq \int_T f(x)dx - \int_T |g(x) - f(x)|dx \geq 1 - \frac{\tilde{C}}{n^{\alpha/10}}. \tag{19}$$

Since $\int f = \int g = 1$, then according to (18) and (19),

$$\int_{\mathbb{R}^n \setminus T} |g(x) - f(x)|dx \leq \int_{\mathbb{R}^n \setminus T} [g(x) + f(x)]\, dx \leq \hat{C} n^{-\alpha/10}. \tag{20}$$

The lemma follows by adding inequalities (17) and (20). $\square$

Lemma 5.1 allows us to convolve our log-concave function with a small gaussian. The proof of the next lemma is the most straightforward adaptation of the proof of Lemma 4.2. We sketch the main points of difference between the proofs.

Lemma 5.6 *Let $n \geq 2$ be an integer, let $\alpha \geq 10$, and let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, log-concave function. Assume that*

$$\sup_{\theta \in S^{n-1}} M_f(\theta, t) \leq e^{-5\alpha n \log n} + \inf_{\theta \in S^{n-1}} M_f(\theta, t) \quad \textit{for all} \;\; t \in \mathbb{R}. \tag{21}$$

Denote $g = f * \gamma_{n,n^{-\alpha}}$, where $*$ stands for convolution. Then,

$$\sup_{\theta \in S^{n-1}} g(t\theta) \leq e^{-\alpha n \log n} + \inf_{\theta \in S^{n-1}} g(t\theta) \quad \textit{for all} \;\; t \geq 0.$$

Sketch of proof: For $\xi_1, \xi_2 \in \mathbb{R}^n$ with $|\xi_1| = |\xi_2| = r$,

$$\left|\hat{f}(\xi_1) - \hat{f}(\xi_2)\right| \leq 2\pi r \int_{-\infty}^{\infty} \left| M_f\left(\frac{\xi_1}{|\xi_1|}, t\right) - M_f\left(\frac{\xi_2}{|\xi_2|}, t\right)\right| dt$$

and consequently $\left|\hat{f}(\xi_1) - \hat{f}(\xi_2)\right| \leq r e^{-2\alpha n \log n}$, by (21) and Lemma 2.2. Note that $\hat{g}(\xi) = \hat{f}(\xi) \cdot \exp(-2\pi^2 n^{-\alpha}|\xi|^2)$ (see, e.g., [49, page 6]). Therefore

$$|\hat{g}(\xi_1) - \hat{g}(\xi_2)| \leq r e^{-2\pi^2 n^{-\alpha} r^2} e^{-2\alpha n \log n} \quad \text{when} \;\; |\xi_1| = |\xi_2| = r. \tag{22}$$

Let $x \in \mathbb{R}^n$ and $U \in O(n)$. From (22),

$$\left| \int_{\mathbb{R}^n} (\hat{g}(\xi) - \hat{g}(U\xi))\, e^{2\pi i \langle x, \xi \rangle} d\xi \right| \leq e^{-2\alpha n \log n} \int_{\mathbb{R}^n} |\xi| e^{-2\pi^2 n^{-\alpha} |\xi|^2} d\xi$$
$$= e^{-2\alpha n \log n} n^{\frac{\alpha(n+1)}{2}} \int_{\mathbb{R}^n} |\xi| e^{-2\pi^2 |\xi|^2} d\xi \leq e^{-\alpha n \log n}. \tag{23}$$

Since $x \in \mathbb{R}^n$ and $U \in O(n)$ are arbitrary, the lemma follows from (23) by the Fourier inversion formula. □

Later, we will combine the following proposition with Lemma 3.2 in order to show that a typical marginal is very close, in the total-variation metric, to a spherically-symmetric concentrated distribution. A random vector X in $\mathbb{R}^n$ has a spherically-symmetric distribution if $Prob\{X \in U(A)\} = Prob\{X \in A\}$ for any measurable set $A \subset \mathbb{R}^n$ and an orthogonal transformation $U \in O(n)$.

Proposition 5.7 *There exist universal constants $C_1, c, C > 0$ for which the following holds: Let $n \geq 2$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, log-concave function. Let X be a random vector in $\mathbb{R}^n$ with density f. Assume that*

$$\sup_{\theta \in S^{n-1}} M_f(\theta, t) \leq e^{-C_1 n \log n} + \inf_{\theta \in S^{n-1}} M_f(\theta, t) \quad \textit{for all} \;\; t \in \mathbb{R}. \tag{24}$$

Then there exists a random vector Y in $\mathbb{R}^n$ such that

(i) $d_{TV}(X, Y) \leq C/n^{10}$.

(ii) *Y has a spherically-symmetric distribution.*

(iii) $Prob\left\{ \left| \, |Y| - \sqrt{n} \, \right| \geq \varepsilon \sqrt{n} \right\} \leq C e^{-c\varepsilon^2 n}$ *for any* $0 \leq \varepsilon \leq 1$.

Proof: Recall that

$$Vol(\sqrt{n} D^n) \leq \hat{C}^n \tag{25}$$

for some universal constant $\hat{C} > 1$. We will define two universal constants:

$$\alpha_0 = 10^4 [\log(\hat{C}) + 1] \quad \text{and} \quad C_1 = \max\{5\alpha_0, 2C_0\}$$

where C_0 is the constant from Proposition 4.1 and $\hat{C}$ is the constant from (25). Throughout this proof, α_0, C_0, C_1 and $\hat{C}$ will stand for the universal constants just mentioned. We assume that inequality (24) – the main assumption of this proposition – holds, with the constant C_1 as was just defined. We may apply Proposition 4.1, based on (24), since $C_0 n \leq C_1 n \log n$. By the conclusion of that proposition,

$$Prob\left\{ \left| \, |X| - \sqrt{n} \, \right| \geq \varepsilon \sqrt{n} \right\} \leq C e^{-c\varepsilon^2 n} \;\; (0 \leq \varepsilon \leq 1). \tag{26}$$

Let Z' be a gaussian random vector in $\mathbb{R}^n$, independent of X, with $\mathbb{E}Z' = 0$ and $Cov(Z') = n^{-\alpha_0} Id$. Then $\mathbb{E}|Z'|^2 = n^{1-\alpha_0}$, and, for example, by Lemma 2.1(i), we know that

$$Prob\left\{|Z'| \geq 1\right\} \leq Prob\left\{|Z'| \geq 20n \cdot \sqrt{n^{1-\alpha_0}}\right\} \leq e^{-n}.$$

Consequently, the event $-1 \leq |X + Z'| - |X| \leq 1$ holds with probability greater than $1 - e^{-n}$. By applying (26) we obtain that for $0 \leq \varepsilon \leq 1$,

$$\begin{aligned} & Prob\left\{\big|\, |X + Z'| - \sqrt{n}\,\big| \geq \varepsilon\sqrt{n}\right\} \qquad (27) \\ & \leq e^{-n} + Prob\left\{\big|\, |X| - \sqrt{n}\,\big| \geq \left(\varepsilon - \frac{1}{\sqrt{n}}\right)\sqrt{n}\right\} \leq C' e^{-c'\varepsilon^2 n} \end{aligned}$$

(in obtaining the last inequality in (27), one needs to consider separately the cases $\varepsilon < 2/\sqrt{n}$ and $\varepsilon \geq 2/\sqrt{n}$).

The density of Z' is $\gamma_{n,n^{-\alpha_0}}$. Denote by $g = f * \gamma_{n,n^{-\alpha_0}}$ the density of the random vector $X + Z'$. Since $C_1 \geq 5\alpha_0$ and $\alpha_0 \geq 10$, then (24) implies the main assumption of Lemma 5.6 for $\alpha = \alpha_0$. By the conclusion of that lemma, for all $\theta_1, \theta_2 \in S^{n-1}$ and $r \geq 0$,

$$|g(r\theta_1) - g(r\theta_2)| \leq e^{-\alpha_0 n \log n}. \qquad (28)$$

Denote, for $x \in \mathbb{R}^n$,

$$\tilde{g}(x) = \int_{S^{n-1}} g(|x|\theta) d\sigma_{n-1}(\theta),$$

the spherical average of g. The function $\tilde{g}$ is a spherically-symmetric function with $\int \tilde{g} = 1$, and from (28),

$$|\tilde{g}(x) - g(x)| \leq e^{-\alpha_0 n \log n} \quad \text{for all} \quad x \in \mathbb{R}^n. \qquad (29)$$

According to (29) and the case $\varepsilon = 1$ in (27),

$$\begin{aligned} \|\, \tilde{g} \,-\, g \,\|_{L^1(\mathbb{R}^n)} &\leq \int_{|x| \leq 2\sqrt{n}} |\tilde{g}(x) - g(x)| dx + 2\int_{|x| \geq 2\sqrt{n}} g(x) dx \\ &\leq Vol(2\sqrt{n} D^n) e^{-\alpha_0 n \log n} + 2C' e^{-c' n} \leq C'' e^{-c'' n}, \qquad (30) \end{aligned}$$

by the definition of α_0, where $\|F\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |F(x)| dx$ for any measurable function $F : \mathbb{R}^n \to \mathbb{R}$.

Let Y be a random variable that is distributed according to the density $\tilde{g}$. Then Y satisfies the conclusion (ii) of the present proposition, since $\tilde{g}$ is a radial function. Additionally, (27) shows that Y satisfies (iii), since the random variables $|Y|$ and $|X + Z'|$ have the same distribution. It remains to prove (i). To that end, we employ Lemma 5.1. The assumptions of Lemma

5.1 are satisfied for $\alpha = \alpha_0/30$, since $\alpha_0 \geq 300$. We use (30) and the conclusion of Lemma 5.1 to obtain

$$d_{TV}(X,Y) = \| f - \tilde{g} \|_{L^1(\mathbb{R}^n)} \leq \| \tilde{g} - g \|_{L^1(\mathbb{R}^n)} + \| g - f \|_{L^1(\mathbb{R}^n)}$$
$$\leq C'' e^{-c''n} + Cn^{-\alpha_0/300} \leq \tilde{C} n^{-10},$$

as $\alpha_0 \geq 3000$. This completes the proof of (i). $\square$

Lemma 5.8 *Let $1 \leq k \leq n$ be integers, let $1 \leq r \leq n$, let $\alpha, \beta > 0$ and let X be a random vector in $\mathbb{R}^n$ with a spherically-symmetric distribution. Suppose $E \subset \mathbb{R}^n$ is a k-dimensional subspace. Assume that for $0 \leq \varepsilon \leq 1$,*

$$Prob\left\{ \left| \, |X| - \sqrt{n} \, \right| \geq \varepsilon \sqrt{n} \right\} \leq \beta e^{-\alpha \varepsilon^2 r}. \tag{31}$$

Then,

$$d_{TV}\left(Proj_E(X) \, , \, Z_E \right) \leq C \frac{\sqrt{k}}{\sqrt{r}}$$

where Z_E is a standard gaussian random vector in E, and $c, C > 0$ are constants depending only on α and β.

Proof: In this proof we write $c, C, C', \tilde{C}$ etc. to denote various positive constants depending only on α and β. We may clearly assume that $n \geq 5$ and $k \leq n-4$, as otherwise the result of the lemma is trivial with $C \geq 2$. Let Y be a random vector, independent of X, that is distributed uniformly in S^{n-1}. Let Z_E be a standard gaussian vector in E, independent of X and Y. We will use a quantitative estimate for Maxwell's principle by Diaconis and Freedman [12]. According to their bound,

$$d_{TV}\left(Proj_E(tY) \, , \, \frac{t}{\sqrt{n}} Z_E \right) \leq 2(k+3)/(n-k-3),$$

for any $t \geq 0$. Since X is independent of Y and Z_E, then also

$$d_{TV}\left(Proj_E(|X|Y) \, , \, \frac{|X|}{\sqrt{n}} Z_E \right) \leq 2(k+3)/(n-k-3). \tag{32}$$

For $t \geq 0$, the density of tZ_E is the function $x \mapsto \gamma_{k,t^2}(x)$ $(x \in E)$. Lemma 4.9 implies that $d_{TV}(tZ_E, Z_E) \leq C\sqrt{k}|t^2 - 1|$, for some universal constant $C \geq 1$. Hence,

$$d_{TV}\left(\frac{|X|}{\sqrt{n}} Z_E \, , \, Z_E \right) \leq \mathbb{E}_X \min\left\{ C\sqrt{k} \left| \frac{|X|^2}{n} - 1 \right|, 2 \right\} \tag{33}$$
$$= \int_0^2 Prob\left\{ C\sqrt{k} \left| \frac{|X|^2}{n} - 1 \right| \geq t \right\} dt \leq \int_0^2 C' e^{-c' r t^2 / k} dt \leq \tilde{C} \sqrt{\frac{k}{r}},$$

where we used (31). Note that the random vectors X and $|X|Y$ have the same distribution, since the distribution of X is spherically-symmetric. By combining (32) and (33),

$$d_{TV}\left(\,Proj_E(X)\,,\,Z_E\,\right) \leq 2\frac{k+3}{n-k-3} + \tilde{C}\sqrt{\frac{k}{r}} \leq \bar{C}\sqrt{\frac{k}{r}}$$

because $r \leq n$. This completes the proof. $\square$

We are now in a position to prove Theorem 1.3. Theorem 1.3 is directly equivalent to the following result.

Theorem 5.9 *Let $n \geq 1$ and $1 \leq k \leq c\frac{\log n}{\log\log n}$ be integers, and let X be a random vector in $\mathbb{R}^n$ with an isotropic, log-concave density. Then there exists a subset $\mathcal{E} \subset G_{n,k}$ with $\sigma_{n,k}(\mathcal{E}) \geq 1 - e^{-cn^{0.99}}$ such that for any $E \in \mathcal{E}$,*

$$d_{TV}\left(\,Proj_E(X)\,,\,Z_E\,\right) \leq C\sqrt{k}\cdot\sqrt{\frac{\log\log n}{\log n}},$$

where Z_E is a standard gaussian random vector in E, and $c, C > 0$ are universal constants.

Proof: We use the constant C_1 from Proposition 5.7, and the constant c from Lemma 3.2. We begin as in the proof of Theorem 1.4. Denote the density of X by f. Set

$$\ell = \left\lfloor \frac{c}{100C_1}\frac{\log n}{\log\log n} \right\rfloor.$$

We may assume that n exceeds a certain universal constant, hence $\ell \geq 1$. Fix a subspace $E \in G_{n,\ell}$ that satisfies

$$\sup_{\theta \in S^{n-1}\cap E} M_f(\theta, t) \leq e^{-C_1 \ell \log \ell} + \inf_{\theta \in S^{n-1}\cap E} M_f(\theta, t) \quad \text{for all } t \in \mathbb{R}. \tag{34}$$

Denote $g = \pi_E(f)$. Then g is log-concave and isotropic, and by combining (34) with (2) from Section 2,

$$\sup_{\theta \in S^{n-1}\cap E} M_g(\theta, t) \leq e^{-C_1 \ell \log \ell} + \inf_{\theta \in S^{n-1}\cap E} M_g(\theta, t) \quad \text{for all } t \in \mathbb{R}. \tag{35}$$

We invoke Proposition 5.7, for ℓ and g, based on (35). Recall that g is the density of $Proj_E(X)$. By the conclusion of Proposition 5.7, there exists a random vector Y in E, with a spherically-symmetric distribution, such that

$$d_{TV}\left(\,Proj_E(X)\,,\,Y\,\right) \leq \frac{C}{\ell^{10}} \tag{36}$$

and

$$Prob\left\{\left|\,|Y| - \sqrt{\ell}\,\right| \geq \varepsilon\sqrt{\ell}\right\} \leq C' e^{-c'\varepsilon^2 \ell} \quad \text{for } 0 \leq \varepsilon \leq 1. \tag{37}$$

Fix $1 \leq k \leq \ell$, and let $F \subset E$ be a k-dimensional subspace. Since the distribution of Y is spherically-symmetric, we may apply Lemma 5.8 for $n = \ell$ and $r = \ell$, based on (37). By the conclusion of that lemma,

$$d_{TV}\left(Proj_F(Y)\, ,\, Z_F \right) \leq C'' \frac{\sqrt{k}}{\sqrt{\ell}},$$

where Z_F is a standard gaussian random vector in F. We combine the above with (36), and obtain

$$d_{TV}\left(Proj_F(X)\, ,\, Z_F \right) \leq C'' \frac{\sqrt{k}}{\sqrt{\ell}} + \frac{C}{\ell^{10}} \leq \tilde{C} \frac{\sqrt{k}}{\sqrt{\ell}}. \tag{38}$$

(Note that $d_{TV}(Proj_F(X), Proj_F(Y)) \leq d_{TV}(Proj_E(X), Y)$.) In summary, we have proved that whenever E is an ℓ-dimensional subspace that satisfies (34), then all the k-dimensional subspaces $F \subset E$ satisfy (38).

Suppose that $E \in G_{n,\ell}$ is a random ℓ-dimensional subspace. We will use Lemma 3.2, for $A = C_1 \log \ell$ and $\delta = 1/100$. Note that $\ell \leq \log n$, hence $\ell \leq c\delta A^{-1} \log n$, by the definition of ℓ above. Therefore we may safely apply Lemma 3.2, and conclude that with probability greater than $1 - e^{-cn^{0.99}}$, the subspace E satisfies (34). Therefore, with probability greater than $1 - e^{-cn^{0.99}}$ of selecting E, all k-dimensional subspaces $F \subset E$ satisfy (38).

Next, we select a random subspace F inside the random subspace E. That is, fix $k \leq \ell - 4$, and suppose that $F \subset E$ is a random subspace, that is distributed uniformly over the grassmannian of k-dimensional subspaces of E. Since E is distributed uniformly over $G_{n,\ell}$, it follows that F is distributed uniformly over $G_{n,k}$. We thus conclude that F – which is a random, uniformly distributed, k-dimensional subspace in $\mathbb{R}^n$ – satisfies (38) with probability greater than $1 - e^{-cn^{0.99}}$. Recall that $\ell > \bar{c}(\log n)/\log\log n$ for a universal constant $\bar{c} > 0$, and that our only assumption about k was that $1 \leq k \leq \ell$. The theorem is therefore proved. $\square$

Proof of Theorem 1.3: Observe that

$$\frac{1}{\sqrt{c}} \cdot \sqrt{k} \cdot \sqrt{\frac{\log\log n}{\log n}} \leq \varepsilon,$$

under the assumptions of Theorem 1.3. The theorem thus follows from Theorem 5.9, for an appropriate choice of a universal constant $c > 0$. $\square$

Proof of Theorem 1.1: Substitute $k = 1$ and $\varepsilon = \sqrt{\frac{\log\log n}{c \log n}}$ in Theorem 1.3, for c being the constant from Theorem 1.3. $\square$

An additional notion of distance between multi-dimensional measures is known in the literature under the name of "T-distance" (see, e.g., [30],

[36]). For two random vectors X and Y in a subspace $E \subset \mathbb{R}^n$, their T-distance is defined as

$$T(X,Y) = \sup_{\theta \in S^{n-1}, t \in \mathbb{R}} \left| \, Prob \left\{ \langle X, \theta \rangle \leq t \right\} - Prob \left\{ \langle Y, \theta \rangle \leq t \right\} \right| .$$

The T-distance between X and Y compares only one-dimensional marginals of X and Y, hence it is weaker than the total-variation distance. The following proposition is proved by directly adapting the arguments of Naor and Romik [36].

Proposition 5.10 *Let $\varepsilon > 0$, and assume that $n > \exp(C/\varepsilon^2)$ is an integer. Suppose that X is a random vector in $\mathbb{R}^n$ with an isotropic, log-concave density. Let $1 \leq k \leq c\varepsilon^2 n$ be an integer, and let $E \in G_{n,k}$ be a random k-dimensional subspace. Then, with probability greater than $1 - e^{-c\varepsilon^2 n}$ of choosing E,*

$$T \left(Proj_E(X), Z_E \right) \leq \varepsilon,$$

where Z_E is a standard gaussian random vector in the subspace E. Here, $c, C > 0$ are universal constants.

Sketch of Proof: Let $g(x) = \int_{S^{n-1}} f(|x|\theta) d\sigma_{n-1}(\theta)$ $(x \in \mathbb{R}^n)$ be the spherical average of f. For $0 \leq \delta \leq 1$, set $A_\delta = \{ x \in \mathbb{R}^n ; \left| \, |x|/\sqrt{n} - 1 \, \right| \geq \delta \}$. According to Theorem 1.4,

$$\int_{A_\delta} g(x) dx = \int_{A_\delta} f(x) dx \leq C' n^{-c'\delta^2} \quad \text{for } 0 \leq \delta \leq 1. \tag{39}$$

Denote $\varPhi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-s^2/2} ds$ $(t \in \mathbb{R})$ and fix $\theta_0 \in S^{n-1}$. We apply Lemma 5.8 (for $r = \log n$ and $k = 1$) based on (39), to obtain the inequality

$$\left| \int_{S^{n-1}} M_f(\theta, t) d\sigma_{n-1}(\theta) - \varPhi(t) \right| = |M_g(\theta_0, t) - \varPhi(t)| \leq \frac{C''}{\sqrt{\log n}}, \tag{40}$$

valid for any $t \in \mathbb{R}$. Let us fix $t \in \mathbb{R}$. By Proposition 2.3, the function $\theta \mapsto M_f(\theta, t)$ $(\theta \in S^{n-1})$ is $\hat{C}$-Lipshitz. We apply Proposition 3.1 for $L = \hat{C}$ and then we use (40) to conclude that with probability greater than $1 - e^{-\bar{c}\varepsilon^2 n}$ of selecting E,

$$|M_f(\theta, t) - \varPhi(t)| \leq \varepsilon + \frac{C}{\sqrt{\log n}} \leq \tilde{C}\varepsilon \quad \text{for all } \theta \in S^{n-1} \cap E. \tag{41}$$

Here we used the fact that $k \leq c\varepsilon^2 n$. Recall that $t \in \mathbb{R}$ is arbitrary. Let $t_i = \varPhi^{-1}(\varepsilon \cdot i)$ for $i = 1, ..., \lfloor 1/\varepsilon \rfloor$, where $\varPhi^{-1}$ is the inverse function to $\varPhi$. Then, with probability greater than $1 - e^{-c'\varepsilon^2 n}$ of selecting E, the estimate

(41) holds for all $t = t_i$ $(i = 1, ..., \lfloor 1/\varepsilon \rfloor)$. By using, e.g., [36, Lemma 6] we see that with probability greater than $1 - e^{-c'\varepsilon^2 n}$ of selecting E,

$$|M_f(\theta, t) - \Phi(t)| < \bar{C}\varepsilon \quad \forall \theta \in S^{n-1} \cap E,\ t \in \mathbb{R}. \tag{42}$$

The proposition follows from (42) and the definition of the T-distance. □

Remark. At first glance, the estimates in Proposition 5.10 seem surprisingly good: Marginals of almost-proportional dimension are allegedly close to gaussian. The problem with Proposition 5.10 hides, first, in the requirement that $\varepsilon > C/\sqrt{\log n}$, and second, in the use of the rather weak T-distance.

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