Isothermal coordinates comparison and Moderately varying Gauss curvature

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Motivation

Let (M, g) be a C^2-smooth Riemannian surface.

Classical Theorem

The Gauss curvature function $K : M \to \mathbb{R}$ vanishes everywhere if and only if the surface M is locally isometric to the Euclidean plane.

- A quantitative version: Express in terms of different coordinate charts on a Riemannian disc $B := B_M(p_0, \delta)$ with

$$\delta^2 \sup_B |K| < \varepsilon. \tag{1}$$

- For example: In a polar normal coordinate chart (around p_0)

$$g = dr^2 + \phi^2 d\theta^2.$$

A standard comparison argument shows that if (1) holds, then

$$\frac{\sin(\sqrt{\varepsilon})}{\sqrt{\varepsilon}} \leq \frac{\phi(r,\theta)}{r} \leq \frac{\sinh(\sqrt{\varepsilon})}{\sqrt{\varepsilon}}.$$

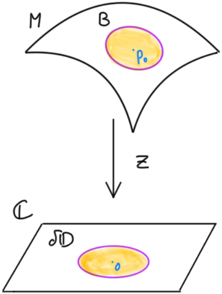
Isothermal coordinates comparison

Our main result is a quantitative version with respect to an isothermal coordinate chart

$$z : \overline{B_M(p_0, \delta)} \to \overline{B_{\mathbb{C}}(0, \delta)}$$

such that

$$z(p_0) = 0 \quad \text{and} \quad z(\partial B_M(p_0, \delta)) = \partial B_{\mathbb{C}}(0, \delta).$$



- The metric is given by $g = \varphi \cdot |dz|^2$.
- φ is the *conformal factor* corresponding to z.
- Exists due to the uniformization theorem.
- Liouville's formula (Δ - flat Laplacian):

$$K = -\frac{\Delta(\log \varphi)}{2\varphi}.$$

Isothermal coordinates comparison

Theorem 1.1

Let (M, g) be a C^2-smooth Riemannian surface, fix $p_0 \in M$ and let $\delta > 0$. Suppose that:

- The injectivity radius of all points of $B := B_M(p_0, \delta)$ is at least 2δ.
- The Gauss curvature function $K : M \to \mathbb{R}$ satisfies $-\kappa \le K\big|_B \le \kappa$, for some constant $\kappa > 0$ such that $\delta^2\kappa < \pi^2/4$.

Let $z : \overline{B} \to \delta\overline{\mathbb{D}}$ be an isothermal coordinate chart such that $z(p_0) = 0$ and $z(\partial B) = \delta\mathbb{S}^1$, and whose conformal factor is φ. Then

$$\sup_B |\log\varphi| \le \frac{\delta^2\kappa}{2}\left[1 + \frac{\pi^2}{4}\cdot\left(\frac{\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{\delta^2\kappa}\right)^2\right].$$

Note: Due to the Whitehead theorem, B is strongly convex.

Immediate Corollaries

Corollary 1.3

Under the assumptions of the theorem together with the additional assumption that $\delta^2\kappa < \pi^2/8$, we have that

$$\sup_B |\log \varphi| \leq 8\delta^2\kappa.$$

Corollary 1.4 - The coordinate map is bi-Lipschitz

Under the assumptions of the theorem together with the additional assumption that $\delta^2\kappa < \pi^2/8$, for any $p \neq q \in B$

$$\exp(-4\delta^2\kappa) \leq \frac{d_g(p,q)}{|z(p)-z(q)|} \leq \exp(4\delta^2\kappa)$$

Since $a \leq \frac{d_g(p,q)}{|z(p)-z(q)|} \leq b \quad \forall p \neq q \in B \iff a^2 \leq \varphi \leq b^2 \quad \forall p \in B$

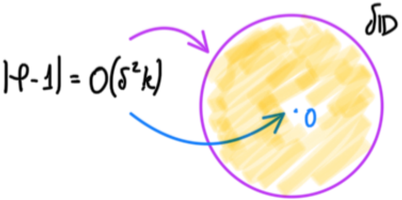
Proof of Theorem 1.1 - Reduction

Proposition 3.1 (Boundary and origin estimates)

Under the assumptions of Theorem 1.1

$$\frac{4\tanh^2(\delta\sqrt{\kappa}/2)}{\delta^2\kappa} \le \varphi(p_0) \le \frac{4\tan^2(\delta\sqrt{\kappa}/2)}{\delta^2\kappa}$$

$$\exp\left(-\delta^2\kappa/2\right) \le \frac{\sin^2(\delta\sqrt{\kappa})}{\delta^2\kappa} \le \varphi\big|_{\partial B} \le \frac{\sinh^2(\delta\sqrt{\kappa})}{\delta^2\kappa} \le \exp\left(\delta^2\kappa/2\right)$$

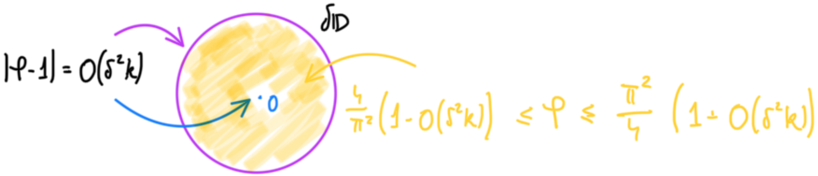


Proof of Theorem 1.1 - Reduction

Proposition 3.3 (Uniform $\pi/2$ estimates)

Under the assumptions of Theorem 1.1, for any $p \neq q \in B$

$$\frac{2}{\pi} \cdot \frac{\sin(\delta\sqrt{\kappa})}{\delta\sqrt{\kappa}\cosh(\delta\sqrt{\kappa})} \leq \frac{d_g(p,q)}{|z(p)-z(q)|} \leq \frac{\pi}{2} \cdot \frac{\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{\delta^2\kappa}.$$



The maximum principle

The main tool we use is the maximum (resp. minimum) principle for subharmonic (resp. superharmonic) functions.

The maximum (resp. minimum) principle

Let Ω be a bounded domain in $\mathbb{C}$ and let $u(z)$ be a subharmonic (resp. superharmonic) function on Ω. If $\limsup u(z) \leq 0$ (resp. $\liminf u(z) \geq 0$) as $z \to \partial\Omega$, then $u(z) \leq 0$ (resp. $u(z) \geq 0$) for all $z \in \Omega$.

Proof of Theorem 1.1 - Reduction

Given Propositions 3.1 and 3.3, we are able to prove Theorem 1.1. The only missing ingredient is the following elliptic-regularity lemma:

Lemma 3.4

Suppose that $u \in C^2(\delta\mathbb{D}) \cap C^0(\delta\overline{\mathbb{D}})$ satisfies $\Delta u = f$ on $\delta\mathbb{D}$. Then

$$\sup_{\delta\mathbb{D}} |u| \leq \sup_{\delta\mathbb{S}^1} |u| + (\delta^2/4) \cdot \sup_{\delta\mathbb{D}} |f| =: a(u, f).$$

Proof.

Define $v(z) = \sup_{\delta\mathbb{S}^1} |u| + \frac{\delta^2 - |z|^2}{4} \sup_{\delta\mathbb{D}} |f|$. Then $\Delta v = -\sup_{\delta\mathbb{D}} |f|$

$$\Delta(v - u) \leq 0 \qquad \text{and} \qquad (v - u)|_{\delta\mathbb{S}^1} \geq 0.$$

By the min. principle $u \leq v \leq a(u, f)$. Similarly $-u \leq a(u, f)$. □

Proof of Theorem 1.1 - Reduction

Proof of Theorem 1.1.

$$\begin{aligned}
\sup_B |\log\varphi| &\le \sup_{\partial B} |\log\varphi| + (\delta^2/4)\cdot \sup_B |\Delta \log\varphi| && \text{(Lemma 3.4)}\\
&= \sup_{\partial B} |\log\varphi| + (\delta^2/2)\cdot \sup_B |K\cdot\varphi| && (\Delta\log\varphi = -2K\varphi)\\
&\le \frac{\delta^2\kappa}{2} + \frac{\delta^2\kappa}{2}\cdot \sup_B |\varphi| && \text{(Prop. 3.1)}\\
&\le \frac{\delta^2\kappa}{2}\left[1 + \frac{\pi^2}{4}\left(\frac{\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{\delta^2\kappa}\right)^2\right] && \text{(Prop. 3.3)}
\end{aligned}$$

□

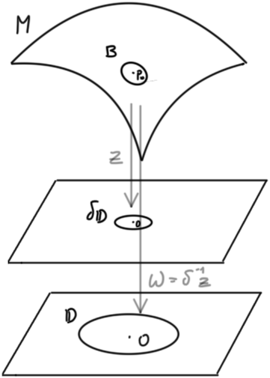
Notation

We write $w = \delta^{-1}z : \overline{B} \to \overline{\mathbb{D}}$, so that w is an isothermal coordinate chart with

$$w(p_0) = 0 \qquad \text{and} \qquad w(\partial B) = \mathbb{S}^1.$$

The corresponding conformal factor is given by

$$\lambda(w) = \delta^2\varphi(z).$$



Green's function

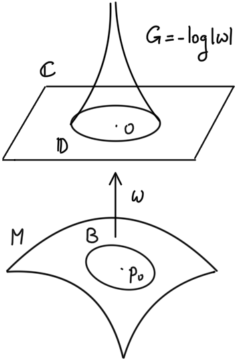
Given such an isothermal coordinate chart $w : \overline{B} \to \overline{\mathbb{D}}$, we let

$$G = -\log|w|.$$

Then:

- $G > 0$ on B.
- $G(p) \xrightarrow{p \to \partial B} 0$.
- $G(p) \xrightarrow{p \to p_0} \infty$.
- G is harmonic on $B \setminus \{p_0\}$ with a logarithmic pole at p_0.
- For any $p \in \overline{B} \setminus \{p_0\}$ we have

$$\lambda(p) \cdot \|\nabla G(p)\|_g^2 \cdot |w(p)|^2 = 1.$$



The harmonic barriers

For $p \in \overline{B}$ (The "base-point"), define

$$G_p^{(s)}(q) = -\log\left(\tan\left(d_g(q,p)\sqrt{\kappa}/2\right)\right) \quad \text{(The "spherical barrier")}$$

$$G_p^{(h)}(q) = -\log\left(\tanh\left(d_g(q,p)\sqrt{\kappa}/2\right)\right) \quad \text{(The "hyperbolic barrier")}$$

Laplacian comparison: Under our assumptions

$$\sqrt{\kappa}\cot(\sqrt{\kappa}\cdot d_g(p,\cdot)) \leq \Delta_M d_g(p,\cdot) \leq \sqrt{\kappa}\coth(\sqrt{\kappa}\cdot d_g(p,\cdot))$$

$$\Downarrow$$

$$\Delta_M G_p^{(s)} \leq 0 \leq \Delta_M G_p^{(h)}$$

The harmonic barriers

Hence if $p \in B$ and $w = \delta^{-1}z : \overline{B} \to \overline{\mathbb{D}}$ is an isothermal coordinate chart with $w(p_0) = 0$ and $w(\partial B) = \mathbb{S}^1$, by the max. (resp. min.) principle

$$\begin{aligned} & G_p^{(s)}(q) + C \geq -\log|w(p) - w(q)| && \forall q \in \partial B \\ \implies & G_p^{(s)}(q) + C \geq -\log|w(p) - w(q)| && \forall q \in B \setminus \{p\} \end{aligned}$$

and

$$\begin{aligned} & G_p^{(h)}(q) + C \leq -\log|w(p) - w(q)| && \forall q \in \partial B \\ \implies & G_p^{(h)}(q) + C \leq -\log|w(p) - w(q)| && \forall q \in B \setminus \{p\} \end{aligned}$$

Proof of Proposition 3.1

Proposition 3.1 (Boundary and origin estimates)

Under the assumptions of Theorem 1.1

$$\frac{4\tanh^2(\delta\sqrt{\kappa}/2)}{\delta^2\kappa} \leq \varphi(p_0) \leq \frac{4\tan^2(\delta\sqrt{\kappa}/2)}{\delta^2\kappa}$$

$$\frac{\sin^2(\delta\sqrt{\kappa})}{\delta^2\kappa} \leq \varphi\big|_{\partial B} \leq \frac{\sinh^2(\delta\sqrt{\kappa})}{\delta^2\kappa}$$

Let $w = \delta^{-1}z : \overline{B} \to \overline{\mathbb{D}}$, so that w is an isothermal coordinate chart with $w(p_0) = 0$ and $w(\partial B) = \mathbb{S}^1$. Write λ for the corresponding conformal factor. Since $\varphi(z) = \delta^{-2}\lambda(w)$, it suffices to show that

$$\frac{4\tanh^2(\delta\sqrt{\kappa}/2)}{\kappa} \leq \lambda(p_0) \leq \frac{4\tan^2(\delta\sqrt{\kappa}/2)}{\kappa}$$

$$\frac{\sin^2(\delta\sqrt{\kappa})}{\kappa} \leq \lambda\big|_{\partial B} \leq \frac{\sinh^2(\delta\sqrt{\kappa})}{\kappa}$$

Proof of Proposition 3.1

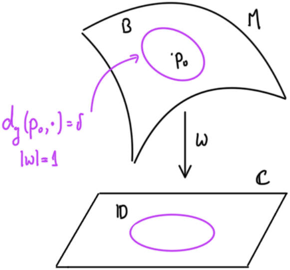
Write $G = -\log|w|$, and let

$$G^{(s)} := G^{(s)}_{p_0} - \log(\cot(\delta\sqrt{\kappa}/2))$$
$$= -\log(\tan(d_g(p_0,\cdot))\sqrt{\kappa}/2) - \log(\cot(\delta\sqrt{\kappa}/2))$$

$$G^{(h)} := G^{(h)}_{p_0} - \log(\coth(\delta\sqrt{\kappa}/2))$$
$$= -\log(\tanh(d_g(p_0,\cdot))\sqrt{\kappa}/2) - \log(\coth(\delta\sqrt{\kappa}/2))$$

For any $q \in \partial B$

$$G^{(s)}(q) = G^{(h)}(q) = G(q) = 0$$



Proof of Proposition 3.1

Thus by the max. (resp. min.) principle, for any $p \in B \setminus \{p_0\}$

$$G^{(h)}(p) \leq G(p) \leq G^{(s)}(p).$$

Since $\lim_{x\to 0} \tan(x)/x = \lim_{x\to 0} \tanh(x)/x = 1$, writing $r(p) = d_g(p_0, p)$,

$$\frac{2\tanh(\delta\sqrt{\kappa}/2)}{\sqrt{\kappa}} \leq \lim_{p\to p_0} \frac{r(p)}{|w(p)|} \leq \frac{2\tan(\delta\sqrt{\kappa}/2)}{\sqrt{\kappa}}.$$

Hence

$$\frac{4\tanh^2(\delta\sqrt{\kappa}/2)}{\kappa} \leq \lambda(p_0) \leq \frac{4\tan^2(\delta\sqrt{\kappa}/2)}{\kappa},$$

proving the estimates at the origin.

Proof of Proposition 3.1

Since

$$G^{(s)} = G^{(h)} = G = 0 \qquad \text{on } \partial B = \{r(q) = \delta\}$$

and

$$G^{(h)} \leq G \leq G^{(s)} \qquad \text{on } B,$$

for any $q \in \partial B$ we have $\|\nabla G(q)\|_g = -\partial_r G(q)$ and

$$\frac{\sqrt{\kappa}}{\sinh(\delta\sqrt{\kappa})} = -\partial_r G^{(h)}_{p_0}(q) \leq \|\nabla G(q)\|_g \leq -\partial_r G^{(s)}_{p_0}(q) = \frac{\sqrt{\kappa}}{\sin(\delta\sqrt{\kappa})}.$$

Since $\lambda(q) = \|\nabla G(q)\|_g^{-2}$, we obtain

$$\frac{\sin^2(\delta\sqrt{\kappa})}{\kappa} \leq \lambda\big|_{\partial B} \leq \frac{\sinh^2(\delta\sqrt{\kappa})}{\kappa},$$

proving the estimates on the boundary.

Proof of Proposition 3.3

Proposition 3.3 (Uniform $\pi/2$ estimates)

Under the assumptions of Theorem 1.1, for any $p \neq q \in B$

$$\frac{2}{\pi} \cdot \frac{\sin(\delta\sqrt{\kappa})}{\delta\sqrt{\kappa}\cosh(\delta\sqrt{\kappa})} \leq \frac{d_g(p,q)}{|z(p)-z(q)|} \leq \frac{\pi}{2} \cdot \frac{\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{\delta^2\kappa}.$$

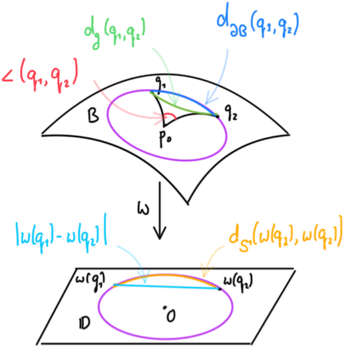
Strategy:

- Bound the distances-ratio between any two boundary points.
- Use the harmonic barriers to deduce bounds the distances-ratio between two interior points (which proves the proposition).

Proof of Proposition 3.3 - boundary distances-ratio

Fix $q_1 \neq q_2 \in \partial B$. As before, let $w = \delta^{-1} z : \overline{B} \to \overline{\mathbb{D}}$, so that w is an isothermal coordinate chart with $w(p_0) = 0$ and $w(\partial B) = \mathbb{S}^1$. We will consider several distance metrics:

- Riemannian distance $d_g(q_1, q_2)$
- Euclidean distance $|w(q_1) - w(q_2)|$
- Riemannian arc-distance $d_{\partial B}(q_1, q_2)$
- Euclidean arc-distance $d_{\mathbb{S}^1}(w(q_1), w(q_2))$
- Riemannian angle $\angle(q_1, q_2)$ at p_0



Proof of Proposition 3.3 - boundary distances-ratio

For the upper bound we use:

$$\frac{d_g(q_1,q_2)}{|w(q_1)-w(q_2)|} = \underbrace{\frac{d_g(q_1,q_2)}{d_{\partial B}(q_1,q_2)}}_{\leq 1} \cdot \frac{d_{\partial B}(q_1,q_2)}{d_{\mathbb{S}^1}(w(q_1),w(q_2))} \cdot \underbrace{\frac{d_{\mathbb{S}^1}(w(q_1),w(q_2))}{|w(q_1)-w(q_2)|}}_{\leq \frac{\pi}{2}}.$$

Proposition 3.1 provides estimates on the conformal factor on the boundary. Hence

$$\frac{\sin(\delta\sqrt{\kappa})}{\sqrt{\kappa}} \leq \frac{d_{\partial B}(q_1,q_2)}{d_{\mathbb{S}^1}(w(q_1),w(q_2))} \leq \frac{\sinh(\delta\sqrt{\kappa})}{\sqrt{\kappa}},$$

and we obtain

$$\frac{d_g(q_1,q_2)}{|w(q_1)-w(q_2)|} \leq \frac{\pi}{2} \cdot \frac{\sinh(\delta\sqrt{\kappa})}{\sqrt{\kappa}}.$$

Proof of Proposition 3.3 - boundary distances-ratio

For the lower bound we use

$$\frac{d_g(q_1,q_2)}{|w(q_1)-w(q_2)|} = \frac{d_g(q_1,q_2)}{\angle(q_1,q_2)} \cdot \frac{\angle(q_1,q_2)}{d_{\partial B}(q_1,q_2)} \cdot \underbrace{\frac{d_{\partial B}(q_1,q_2)}{d_{\mathbb{S}^1}(w(q_1),w(q_2))}}_{\geq \sin(\delta\sqrt{\kappa})/\sqrt{\kappa}} \cdot \underbrace{\frac{d_{\mathbb{S}^1}(w(q_1),w(q_2))}{|w(q_1)-w(q_2)|}}_{\geq 1}$$

In polar normal coordinates $g = dr^2 + \phi^2 d\theta^2$. By a comparison argument for the Jacobi equation, under our assumptions

$$\frac{\sin(\sqrt{\kappa}\cdot r)}{\sqrt{\kappa}} \leq \phi(r,\theta) \leq \frac{\sinh(\sqrt{\kappa}\cdot r)}{\sqrt{\kappa}}.$$

Assuming $\theta(q_1) = 0$ and $0 \leq \theta(q_2) = \angle(q_1,q_2) \leq \pi$, we have

$$d_{\partial B(q_1,q_2)} \leq \int_0^{\theta(q_2)} \|\partial_\theta(\delta,t)\| dt = \int_0^{\theta(q_2)} \phi(\delta,t) dt \leq \frac{\sinh(\sqrt{\kappa}\cdot\delta)}{\sqrt{\kappa}} \cdot \theta(q_2).$$

Proof of Proposition 3.3 - boundary distances-ratio

For the lower bound we use

$$\frac{d_g(q_1,q_2)}{|w(q_1)-w(q_2)|} = \frac{d_g(q_1,q_2)}{\angle(q_1,q_2)} \cdot \underbrace{\frac{\angle(q_1,q_2)}{d_{\partial B}(q_1,q_2)}}_{\geq \sqrt{\kappa}/\sinh(\delta\sqrt{\kappa})} \cdot \underbrace{\frac{d_{\partial B}(q_1,q_2)}{d_{\mathbb{S}^1}(w(q_1),w(q_2))}}_{\geq \sin(\delta\sqrt{\kappa})/\sqrt{\kappa}} \cdot \underbrace{\frac{d_{\mathbb{S}^1}(w(q_1),w(q_2))}{|w(q_1)-w(q_2)|}}_{\geq 1}$$

By Toponogov's theorem and the spherical law of cosines,

$$\frac{d_g(q_1,q_2)}{\angle(q_1,q_2)} \geq \frac{\cos^{-1}(\cos^2(\delta\sqrt{\kappa}) + \sin^2(\delta\sqrt{\kappa})\cos(\angle(q_1,q_2)))}{\sqrt{\kappa}\cdot\angle(q_1,q_2)}.$$

The infimum of the RHS is obtained for $\angle(q_1,q_2)=\pi$, thus

$$\frac{d_g(q_1,q_2)}{\angle(q_1,q_2)} \geq \frac{2\delta}{\pi}.$$

Hence

$$\frac{d_g(q_1,q_2)}{|w(q_1)-w(q_2)|} \geq \frac{2}{\pi}\cdot\frac{\delta\sin(\delta\sqrt{\kappa})}{\sinh(\delta\sqrt{\kappa})}$$

Proof of Proposition 3.3 - boundary distances-ratio

Combining all the inequalities above, we have

Corollary 3.13

Under the assumptions of Theorem 1.1, write $w = \delta^{-1} z : \overline{B} \to \overline{\mathbb{D}}$. Then for any $q_1 \neq q_2 \in \partial B$

$$\frac{2}{\pi} \cdot \frac{\delta \sin(\delta\sqrt{\kappa})}{\sinh(\delta\sqrt{\kappa})} \leq \frac{d_g(q_1, q_2)}{|w(q_1) - w(q_2)|} \leq \frac{\pi}{2} \cdot \frac{\sinh(\delta\sqrt{\kappa})}{\sqrt{\kappa}}$$

Proof of Proposition 3.3

Using Corollary 3.13 and the fact that $x/\tanh(x)$ is increasing and $x/\tan(x)$ is decreasing, we obtain

Lemma 3.14

Under the assumptions of Theorem 1.1, write $w = \delta^{-1}z : \overline{B} \to \overline{\mathbb{D}}$ and fix $q_0 \in \partial B$. Then for any $q \in \partial B \setminus \{q_0\}$

$$G_{q_0}^{(h)}(q) + C^{(h)} \leq -\log|w(q) - w(q_0)|$$

$$G_{q_0}^{(s)}(q) + C^{(s)} \geq -\log|w(q) - w(q_0)|$$

where

$$C^{(h)} = \log\left(\frac{\sin(\delta\sqrt{\kappa})}{\pi\cosh(\delta\sqrt{\kappa})}\right), \quad C^{(s)} = \log\left(\frac{\pi\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{4\delta\sqrt{\kappa}}\right)$$

Proof of Proposition 3.3

Sketch of the proof of Lemma 3.14: For any $q \in \partial B \setminus \{q_0\}$

$$\begin{aligned} G_{q_0}^{(h)}(q) &= -\log(\tanh(r(q)\sqrt{\kappa}/2)) \\ &\leq -\log(r(q)) - \log(\tanh(\delta\sqrt{\kappa})) + \log(2\delta) \\ &\qquad\qquad (x/\tanh(x) \text{ increasing, } r(q) \leq 2\delta) \\ &\leq -C^{(h)} - \log|w(q) - w(q_0)| \qquad \text{(Corollary 3.13)} \end{aligned}$$

and similarly

$$\begin{aligned} G_{q_0}^{(s)}(q) &= -\log(\tan(r(q)\sqrt{\kappa}/2)) \\ &\geq -\log(r(q)) - \log(\tan(\delta\sqrt{\kappa})) + \log(2\delta) \\ &\qquad\qquad (x/\tan(x) \text{ decreasing, } r(q) \leq 2\delta) \\ &\geq -C^{(s)} - \log|w(q) - w(q_0)| \qquad \text{(Corollary 3.13)} \end{aligned}$$

Proof of Proposition 3.3

Fix $q_0 \in \partial B$. By Lemma 3.14

$$G_{q_0}^{(h)}(q) + C^{(h)} \leq -\log|w(q) - w(q_0)| \qquad \forall q \in \partial B \setminus \{q_0\}$$

$$G_{q_0}^{(s)}(q) + C^{(s)} \geq -\log|w(q) - w(q_0)| \qquad \forall q \in \partial B \setminus \{q_0\}$$

By the max. (resp. min.) principle

$$G_{q_0}^{(h)}(q) + C^{(h)} \leq -\log|w(q) - w(q_0)| \qquad \forall q \in B$$

$$G_{q_0}^{(s)}(q) + C^{(s)} \geq -\log|w(q) - w(q_0)| \qquad \forall q \in B$$

Proof of Proposition 3.3

Fix $p \in B$. By the symmetry $G_q^{(h)}(p) = G_p^{(h)}(q)$, $G_q^{(s)}(p) = G_p^{(s)}(q)$,

$$G_p^{(h)}(q) + C^{(h)} \leq -\log|w(p) - w(q)| \qquad \forall q \in \partial B$$

$$G_p^{(s)}(q) + C^{(s)} \geq -\log|w(p) - w(q)| \qquad \forall q \in \partial B$$

By the max. (resp. min.) principle

$$G_p^{(h)}(q) + C^{(h)} \leq -\log|w(p) - w(q)| \qquad \forall q \in B$$

$$G_p^{(s)}(q) + C^{(s)} \geq -\log|w(p) - w(q)| \qquad \forall q \in B$$

These inequalities together with $\tanh(x) < x < \tan(x)$ translate to

$$\frac{2}{\pi} \cdot \frac{\sin(\delta\sqrt{\kappa})}{\sqrt{\kappa}\cosh(\delta\sqrt{\kappa})} \leq \frac{d_g(p,q)}{|w(p) - w(q)|} \leq \frac{\pi}{2} \cdot \frac{\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{\delta\kappa}.$$

Since $z = \delta w$, the proof is completed.