מכון ויצמן למדע
WEIZMANN INSTITUTE OF SCIENCE

Thesis for the degree
Master of Science

עבודת גמר (תזה) לתואר
מוסמך למדעים

Submitted to the Scientific Council of the
Weizmann Institute of Science
Rehovot, Israel

מוגשת למועצה המדעית של
מכון ויצמן למדע
רחובות, ישראל

By
Matan Eilat

מאת
מתן אילת

קואורדינטות איזותרמיות ועקמומיות גאוס המשתנה במתינות
Isothermal coordinates comparison and Moderately varying Gauss curvature

Advisor: Prof. Bo’az Klartag

מנחה: פרופ׳ בועז קלרטג

December 2020

טבת תשפ״א

Isothermal coordinates comparison and Moderately varying Gauss curvature

Matan Eilat

Abstract

Let (M, g) be a C^2-smooth Riemannian surface. A well-known theorem states that the Gauss curvature function $K : M \to \mathbb{R}$ vanishes everywhere if and only if the surface is locally isometric to the Euclidean plane. The first main result of our study is a quantitative version of this theorem with respect to an isothermal coordinate chart. We essentially show that if B is a Riemannian disc of radius $\delta > 0$ with $\delta^2 \sup_B |K| < \varepsilon$, for say $0 < \varepsilon < 1$, then there is an isothermal coordinate map from B onto an Euclidean disc of radius δ which is bi-Lipschitz with constant $\exp(4\varepsilon)$.

We then use this result to study surfaces with moderately varying Gauss curvature, i.e. surfaces whose Gauss curvature function is Hölder continuous with exponent $0 < \alpha < 1$. For this purpose we define the notion of an α-almost-flat radius at a given point. We introduce two definitions, one via the Gauss curvature function and the other via a coordinate chart. Our second main result is the equivalence of these definitions up to a constant depending on α. The use of an isothermal coordinate chart in our proof implies a quantitative statement on the superior regularity of the isothermal coordinate chart, somewhat similar to the optimality shown by Deturck and Kazdan [1].

1 Introduction

Let (M, g) be a C^2-smooth Riemannian surface, i.e. a C^2-smooth two-dimensional Riemannian manifold. A well-known theorem states that the Gauss curvature function $K : M \to \mathbb{R}$ is zero if and only if the metric is locally flat, i.e. around every point there is a neighborhood in which the metric is Euclidean. A quantitative version of this theorem might investigate the behavior of different coordinate charts in case the Gauss curvature is not identically zero, but say satisfies $|K| < \varepsilon$ for a small $\varepsilon > 0$.

For example, fix a point $p_0 \in M$ and write $B_M(p_0, 1)$ for the Riemannian disc of radius 1 around p_0. In a normal polar coordinate chart (r, θ), the metric takes the form

$$g = dr^2 + \phi^2(r, \theta) d\theta^2,$$

for some function ϕ. In case $\phi = r$, the metric is Euclidean. A simple comparison argument (see, e.g., Proposition 3.8) states that if $|K| < \varepsilon$, for a small enough $\varepsilon > 0$, then

$$\frac{\sin(r\sqrt{\varepsilon})}{r\sqrt{\varepsilon}} \leq \frac{\phi(r, \theta)}{r} \leq \frac{\sinh(r\sqrt{\varepsilon})}{r\sqrt{\varepsilon}},$$

which implies that $\phi/r \longrightarrow 1$ as $\varepsilon \to 0$, and yields information on the rate of this convergence.

The geodesic normal coordinate chart arises naturally in the study of Riemannian manifolds, and its properties are relatively well-known, e.g. the comparison argument above. However, the isothermal coordinate chart we investigate has several advantages over other coordinate charts, including the geodesic normal coordinate chart. The isothermal coordinate chart is a two-dimensional special case of a harmonic coordinate chart, where the coordinate functions are conjugate harmonic. As shown by Deturck and Kazdan [1], a harmonic coordinate chart satisfies the optimal regularity properties, while changing to normal coordinates may involve loss of two derivatives.

The first main result of our study is a quantitative version of the theorem stated above, with respect to an isothermal coordinate chart $z : \overline{B_M(p_0, \delta)} \to \overline{B_{\mathbb{C}}(0, \delta)}$, where we write $B_M(p_0, \delta) \subset M$ for the Riemannian disc of radius $\delta > 0$ around $p_0 \in M$, and $B_{\mathbb{C}}(0, \delta) \subset \mathbb{C}$ for the Euclidean disc of radius δ around the origin. The metric in such an isothermal coordinate chart is given by

$$g = \varphi \cdot |dz|^2,$$

where φ is a positive function, to which we refer as the conformal factor. Our theorem implies that there is no disadvantage in considering this isothermal coordinate chart in the aspect of comparison theorems such as the one discussed. We discuss the isothermal coordinate chart and the conformal factor in greater detail in upcoming sections.

The formulation of the theorem is followed by several remarks and corollaries, which hopefully shed some light on the behavior of the bounding constant, and explain in what sense the estimates are sharp. Note that in most cases, when given such an isothermal coordinate chart z, we will not distinguish between the conformal factor φ as a function on $\overline{B_M(p_0, \delta)}$ and as a function on $\overline{B_{\mathbb{C}}(0, \delta)}$. For the Euclidean disc of radius $\delta > 0$ around the origin, its closure and its boundary, we will use the abbreviations

$$\delta\mathbb{D} := B_{\mathbb{C}}(0, \delta), \qquad \delta\overline{\mathbb{D}} := \overline{B_{\mathbb{C}}(0, \delta)} \qquad \text{and} \qquad \delta\mathbb{S}^1 := \partial B_{\mathbb{C}}(0, \delta).$$

Theorem 1.1. *Let (M, g) be a C^2-smooth Riemannian surface, fix $p_0 \in M$ and let $\delta > 0$. Suppose that the injectivity radius of all points of $B := B_M(p_0, \delta)$ is at least 2δ, and that the Gauss curvature function $K : M \to \mathbb{R}$ satisfies $-\kappa \leq K\big|_B \leq \kappa$, for some constant $\kappa > 0$ such that $\delta^2\kappa < \pi^2/4$. Let $z : \overline{B} \to \delta\overline{\mathbb{D}}$ be an isothermal coordinate chart such that $z(p_0) = 0$ and $z(\partial B) = \delta\mathbb{S}^1$, and whose conformal factor is φ. Then*

$$\sup_B |\log \varphi| \leq \frac{\delta^2\kappa}{2}\left[1 + \frac{\pi^2}{4} \cdot \left(\frac{\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{\delta^2\kappa}\right)^2\right].$$

Remark 1.2. There are various ways to express the bound for the conformal factor φ from the theorem above. The important thing to notice is that the term in the inner brackets is increasing as a function of $\delta^2\kappa$, tends to 1 as $\delta^2\kappa \searrow 0$, and tends to infinity as $\delta^2\kappa \nearrow \pi^2/4$. In case $\delta^2\kappa$ is assumed to be bounded away from $\pi^2/4$, the estimate may be expressed as $C \cdot \delta^2\kappa$ for some universal constant C. We thus formulate the following corollary, for simplicity and ease of use.

Corollary 1.3. *Under the assumptions of the theorem together with the additional assumption that $\delta^2\kappa < \pi^2/8$, we have that*

$$\sup_B |\log \varphi| \leq 8\delta^2\kappa.$$

Such a uniform bound on the conformal factor immediately yields a uniform bound on the bi-Lipschitz constant of the coordinate map, as mentioned below, namely in (8). Writing $d_g : M \times M \to \mathbb{R}$ for the Riemannian distance function on M induced by the metric g, and $|\cdot|$ for the Euclidean norm on $\mathbb{C}$, we may thus formulate the following corollary.

Corollary 1.4. *Under the assumptions of the theorem together with the additional assumption that $\delta^2\kappa < \pi^2/8$, for any $p \neq q \in B$,*

$$\exp(-4\delta^2\kappa) \leq \frac{d_g(p,q)}{|z(p) - z(q)|} \leq \exp(4\delta^2\kappa).$$

Remark 1.5. As a part of the proof of Theorem 1.1, we will produce bounds for the conformal factor on the boundary, as expressed in Proposition 3.1. These bounds originate in the extremal cases of the spherical and hyperbolic metrics, which implies that they are sharp. In light of these bounds, together with the fact that $\log(\sinh^2(x)/x^2) = \Theta(x^2)$ and $\log(\sin^2(x)/x^2) = \Theta(-x^2)$ as $x \searrow 0$, we see that in the case where $\delta^2\kappa$ is bounded away from $\pi^2/4$, as in Corollary 1.3 for example, the estimates are sharp up to some multiplicative universal constant.

The first part of our study is dedicated to the proof of Theorem 1.1. Afterwards, we apply the theorem to study surfaces with moderately varying Gauss curvature.

The simplest geometric spaces in two dimensions are probably the simply-connected manifolds of constant Gauss curvature: The sphere, the hyperbolic space and the Euclidean space. We believe that next in line should be surfaces whose Gauss curvature is not constant, yet it does not vary too wildly. This motivates us to study surfaces whose Gauss curvature function is Hölder continuous with exponent $0 < \alpha < 1$. For this purpose we define the notion of a surface being α-almost-flat at a given point to a certain distance.

Our first definition, namely Definition 1.6, involves conditions on the C^α-norm of the Gauss curvature function. The second definition, namely Definition 1.7, relies on the existence of a coordinate chart in which the metric coefficients satisfy bounds on their $C^{2,\alpha}$-norms. To each of these definitions we add a definition of a corresponding α-almost-flat radius, dependent on the point and on α, which is the supremum of all the distances to which the surface is α-almost-flat. In Theorem 1.8 we show the equivalence of these two radius definitions up to a constant depending on α. The Hölder norms $|\cdot|'_{k,\alpha}$ we use in our formulations, together with the proof of Theorem 1.8, are presented in the last section.

Let M be a C^2-smooth Riemannian surface, let $0 < \alpha < 1$ and write $K : M \to \mathbb{R}$ for the Gauss curvature function.

Definition 1.6. *We say that M is α-almost-flat via curvature at a point $p_0 \in M$ to distance $\delta > 0$ if the injectivity radius of all points of $B := B_M(p_0, \delta)$ is at least 2δ, and*

$$\delta^2 \cdot |K|'_{0,\alpha;B} \leq 1.$$

We also define the α-almost-flat radius via curvature of M at a point $p_0 \in M$ to be the supremum of all such δ, and denote it by $\rho_{curvature} = \rho_{curvature}(p_0, \alpha)$.

Definition 1.7. *We say that M is α-almost-flat via a coordinate chart at a point $p_0 \in M$ to distance $\delta > 0$ if the injectivity radius of all points of $B := B_M(p_0, \delta)$ is at least 2δ, and B is isometric to a bounded open set $U \subset \mathbb{C}$ endowed with the Riemannian tensor*

$$g = \sum_{i,j=1}^{2} g_{ij} dx^i dx^j,$$

such that $g_{ij} : U \to \mathbb{R}$ satisfy

$$g_{ij}(x_0) = \delta_{ij} \qquad \text{and} \qquad |g_{ij} - \delta_{ij}|'_{2,\alpha;U} \leq 1 \qquad \text{for all } \ i,j \in \{1,2\},$$

where $x_0 \in U$ is the image of p_0 under the implied isometry, and δ_{ij} is the Kronecker delta function. We also define the α-almost-flat radius via a coordinate chart of M at a point $p_0 \in M$ to be the supremum of all such δ, and denote it by $\rho_{chart} = \rho_{chart}(p_0, \alpha)$.

Theorem 1.8. *Let M be a C^2-smooth Riemannian surface, let $p_0 \in M$ and $0 < \alpha < 1$. Then*

$$\rho_{curvature} \geq C_1 \cdot \rho_{chart} \tag{1}$$

and

$$\rho_{chart} \geq C_2 \cdot \rho_{curvature}, \tag{2}$$

for some constants $C_1, C_2 > 0$ depending on α.

The proofs of the two claims in Theorem 1.8, namely (1) and (2), are given in the last section. For the proof of (2) we use the isothermal coordinate chart from Theorem 1.1. Therefore, by combining Claims (1) and (2) from Theorem 1.8, we see that if the metric satisfies the $C^{2,\alpha}$-estimates in some coordinate chart on a Riemannian disc of radius $\delta > 0$, then it satisfies these $C^{2,\alpha}$-estimates in an isothermal coordinate chart on a disc of radius $C\delta$, where $0 < C < 1$ is a constant depending on α. This fact may be thought of as a quantitative evidence of the superiority of the isothermal coordinate chart in regularity aspects. As mentioned before, the fact that harmonic coordinates have optimal regularity was shown by Deturck and Kazdan [1].

In the n-dimensional case, there is a notion of a similar flavour to ours, called the harmonic radius. Loosely speaking, the $C^{k,\alpha}$-harmonic radius at $p_0 \in M$ is the largest radius $\delta > 0$ such that there exists a harmonic coordinate chart on the geodesic ball $B_M(p_0, \delta)$ in which the metric tensor is $C^{k,\alpha}$-controlled. Lower bounds on the $C^{1,\alpha}$-harmonic radius in terms of the injectivity radius and the Ricci curvature were given by Anderson in the context of Gromov-Hausdorff convergence of manifolds [15]. For other estimates and more details, we refer to [16] and [17].

Acknowledgements. I would like to express my deep gratitude to my advisor, Prof. Bo'az Klartag, for his continuous guidance, assistance and encouragement.

2 Background on Riemannian Geometry and the Uniformization Theorem

An n-dimensional Riemannian manifold M is C^k-smooth if each point in M is covered by some C^k-smooth coordinate chart, i.e. in this chart the metric tensor takes the form

$$g = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j,$$

where g_{ij} are C^k-smooth functions. One similarly defines a C^∞-smooth or a $C^{k,\alpha}$-smooth Riemannian manifold for some $0 < \alpha < 1$. Any isometry between C^k-smooth metrics is itself C^{k+1}-smooth, as shown by Calabi and Hartman [2]. A two-dimensional Riemannian manifold is referred to as a Riemannian surface.

When a Riemannian manifold M is C^2-smooth, we may speak of its exponential map and curvature tensor. We write T_pM for the tangent space of the manifold M at the point $p \in M$, and $\exp_p : T_pM \to M$ for the exponential map. The injectivity radius of the manifold M at a point $p \in M$ is at least $\delta > 0$ if and only if the exponential map $\exp_p$ is a diffeomorphism from $B_{T_pM}(0, \delta)$ onto its image, where $B_{T_pM}(0, \delta)$ is the ball of radius δ around the origin in T_pM. We write $d_g : M \times M \to \mathbb{R}$ for the Riemannian distance function, and $B_M(p, \delta)$ for the open Riemannian ball of radius $\delta > 0$ around a point $p \in M$. If the injectivity radius at $p \in M$ is at least $\delta > 0$, then the distance function from $p \in M$ is smooth on $B_M(p, \delta) \setminus \{p\}$, and every value in $(0, \delta) \subset \mathbb{R}$ is a regular value of this function.

The curvature tensor of a C^2-smooth Riemannian surface M is determined by the Gauss curvature function $K : M \to \mathbb{R}$. Due to Gauss' Theorema Egregium, the Gauss curvature of a surface is invariant under isometries, making it an intrinsic measure of the curvature, depending only on distances that are measured on the surface. The Killing-Hopf theorem [3, 4] states that a simply-connected geodesically-complete Riemannian manifold of constant sectional curvature is isometric to either the sphere, the hyperbolic space or the Euclidean space, depending on the sign of the curvature. In a local coordinate chart $(x_1, x_2) : M \to \mathbb{R}^2$, the Gauss curvature function may be expressed in terms of the metric tensor as follows (see, e.g., [5])

$$K = -\frac{1}{g_{11}} \left(\frac{\partial}{\partial x_1} \Gamma_{12}^2 - \frac{\partial}{\partial x_2} \Gamma_{11}^2 + \Gamma_{12}^1 \Gamma_{11}^2 - \Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2 \right), \tag{3}$$

where Γ_{ij}^k are the Christoffel symbols of the second kind.

A set $U \subset M$ is called strongly convex if its closure $\overline{U}$ has the property that for any $p, q \in \overline{U}$ there is a unique minimizing geodesic in M from p to q, and the interior of this geodesic is contained in U. In the case where M is two-dimensional, assuming that the injectivity radius at all points of the Riemannian disc $B_M(p_0, \delta)$ is at least 2δ and furthermore

$$\delta^2 \cdot \sup_{p \in B_M(p_0, \delta)} K(p) < \pi^2/4,$$

where $K : M \to \mathbb{R}$ is the Gauss curvature function, the Riemannian disc $B_M(p_0, \delta)$ is strongly convex, according to the Whitehead theorem [6] (see also [7] for more details).

From now on in this section and throughout the first part of our study, we will consider the setup from Theorem 1.1. We let (M, g) be a C^2-smooth Riemannian surface, fix a point $p_0 \in M$ and $\delta > 0$, and consider the Riemannian disc $B := B_M(p_0, \delta)$. We assume that the injectivity radius of all points of B is at least 2δ, and that the Gauss curvature function $K : M \to \mathbb{R}$ satisfies $-\kappa \le K\big|_B \le \kappa$, for some constant $\kappa > 0$ such that $\delta^2\kappa < \pi^2/4$. In light of the discussion above, we see that B is always strongly convex.

Toponogov's theorem is a triangle comparison theorem, one of a family of theorems that quantify the assertion that a pair of geodesics emanating from a point spread apart more slowly in a region of high curvature than they would in a region of low curvature. It is a powerful global generalization of the first Rauch comparison theorem. The classical formulation of Toponogov's theorem considers the case where the curvature is bounded from below (see, e.g., [7]). However, under some conditions, a symmetric assertion holds when the curvature is bounded from above, as we discuss below.

Denote by M_κ the simply-connected two-dimensional model space of constant sectional curvature κ, and let d_κ denote the induced metric on M_κ. For $p, q, r \in \overline{B}$ we write $\Delta = \Delta(p, q, r) \subset \overline{B}$ for the associated geodesic triangle, meaning that its sides are the minimizing geodesic segments $[p, q], [q, r], [r, p]$ connecting their corresponding endpoints. We let $L(\Delta) = d_g(p, q) + d_g(q, r) + d_g(p, r)$ denote its perimeter. Since B is strongly convex, and the Gauss curvature function satisfies $K\big|_B \le \kappa$, it follows that B is a CAT(κ) space. By one of several equivalent definitions of a CAT(κ) space, one obtains the following lemma (see, e.g., [13] and [14] for more details).

Lemma 2.1. *For any geodesic triangle $\Delta = \Delta(p, q, r) \subset \overline{B}$ with perimeter $L(\Delta) < 2\pi/\sqrt{\kappa}$, with $p \ne q$ and $p \ne r$, if θ denotes the angle between $[p, q]$ and $[p, r]$ at p, and if $\Delta_0 = \Delta(p_0, q_0, r_0)$ is a geodesic triangle in M_κ with $d_\kappa(p_0, q_0) = d_g(p, q)$, $d_\kappa(p_0, r_0) = d_g(p, r)$, and angle at p_0 equal to θ, then $d_g(q, r) \ge d_\kappa(q_0, r_0)$.*

Since (M, g) is a C^2-smooth Riemannian surface, around any point $p \in M$ there exists an isothermal coordinate chart (see, e.g., [1]), i.e. there is neighborhood U containing p and a coordinate chart $w = x + iy : U \to \mathbb{C}$, such that the metric in these coordinates is of the form

$$g = \lambda |dw|^2 = \lambda(dx^2 + dy^2),$$

where λ is a positive function, to which we refer as the conformal factor. By virtue of Deturck and Kazdan [1], we know that λ is a C^2-smooth function. The transition maps between these isothermal coordinate charts are holomorphic functions, hence the surface M admits a complex structure, making it a Riemann surface, i.e. a one-dimensional complex manifold. Note that every open connected subset of a Riemann surface, e.g. the Riemannian disc $B \subset M$, is a Riemann surface on its own right.

An upper semi-continuous function $u : M \to \mathbb{R} \cup \{-\infty\}$ is said to be subharmonic, if each $p \in M$ belongs to an isothermal coordinate chart z, such that u is subharmonic as a function of z. The function u is called superharmonic if $-u$ is subharmonic. A function which is both subharmonic and superharmonic is called harmonic. An important theorem in the study of subharmonic (resp. superharmonic) functions is the maximum (resp. minimum) principle (see, e.g., [9]). Note that when we refer to a set as a domain, we always mean a connected open set.

Theorem 2.2 (The maximum (resp. minimum) principle for subharmonic (resp. superharmonic) functions)**.** *Let Ω be a bounded domain in $\mathbb{C}$ and let $u(z)$ be a subharmonic (resp. superharmonic) function on Ω. If $\limsup u(z) \leq 0$ (resp. $\liminf u(z) \geq 0$) as $z \to \partial\Omega$, then $u(z) \leq 0$ (resp. $u(z) \geq 0$) for all $z \in \Omega$.*

Two Riemann surfaces are said to be conformally equivalent if there is a one-to-one conformal mapping of one onto the other. A fundamental theorem in the study of Riemann surfaces is the uniformization theorem.

Theorem 2.3 (The Uniformization Theorem)**.** *Every simply-connected Riemann surface is conformally equivalent to a disc, to the complex plane, or to the Riemann sphere.*

As a result of the uniformization theorem, a non-compact Riemann surface must be conformally equivalent to either the unit disc $\mathbb{D} \subset \mathbb{C}$, in which case it is called hyperbolic, or the complex plane $\mathbb{C}$, in which case it is called parabolic. (see [8] for more details on the uniformization theorem).

If (and only if) a Riemann surface R is hyperbolic, for any point $p_0 \in R$ there exists a function $G : R \setminus \{p_0\} \to \mathbb{R}$, called Green's function with a pole at p_0, which is a generalization of Green's function on the plane - the fundamental solution of the Laplace operator. Green's function can be used to construct the conformal equivalence $w : R \to \mathbb{D}$, and their relation is given by $|w| = e^{-G}$. Our approach to Green's function will be "reversed", meaning that under our assumptions, as we will explain shortly, the Riemannian disc B is hyperbolic, i.e. conformally equivalent to the Euclidean unit disc. Given the conformal equivalence $w : B \to \mathbb{D}$ with $w(p_0) = 0$, we merely use $G = -\log |w|$. This way, the properties of Green's function are evident. We see that

$$G > 0, \qquad G(p) \xrightarrow{p \to \partial B} 0 \qquad \text{and} \qquad G(p) \xrightarrow{p \to p_0} \infty.$$

Moreover, the function G is harmonic on $B \setminus \{p_0\}$, with a logarithmic pole at p_0. The relation between the conformal factor, i.e. the function λ for which the metric takes the form $g = \lambda |dw|^2$, and the gradient of Green's function may be expressed by

$$\lambda(w(p)) \cdot \|\nabla G(p)\|_g^2 \cdot |w(p)|^2 = 1 \qquad \text{for any } p \in \overline{B} \setminus \{p_0\}, \tag{4}$$

where $\|\cdot\|_g$ denotes the Riemannian norm on the appropriate tangent space.

As mentioned above, it follows from the uniformization theorem that B is hyperbolic. To see this, consider the distance function $r : B \to \mathbb{R}$ from the center of the disc $p_0 \in B$. It follows from our assumptions that the function r is subharmonic (This is an immediate consequence of Proposition 3.8), and it is clearly non-constant and bounded from above. By Liouville's theorem, such a function does not exist on a parabolic surface. The conformal equivalence between B and $\mathbb{D}$ may be given by an isothermal coordinate chart

$$w = x + iy : \overline{B} \to \overline{\mathbb{D}} \qquad \text{such that} \quad w(p_0) = 0 \quad \text{and} \quad w(\partial B) = \mathbb{S}^1.$$

To explain the boundary behavior, consider a slightly larger Riemannian disc $B_\varepsilon := B_M(p_0, \delta + \varepsilon)$. For a sufficiently small $\varepsilon > 0$, the Riemannian disc B_ε will also be conformally equivalent to the unit disc $\mathbb{D}$ via some mapping $w_\varepsilon : B_\varepsilon \to \mathbb{D}$ with $w(p_0) = 0$. The set $U := w_\varepsilon(B) \subset \mathbb{D}$ is a simply-connected open set, hence by the Riemann mapping theorem, there exists a conformal mapping $f : U \to \mathbb{D}$ with $f(0) = 0$. Moreover, the set U has a C^∞-smooth boundary, being the level set of the regular value δ of the Riemannian distance function from 0. Hence the map f extends to a diffeomorphism between $\overline{U}$ and $\overline{\mathbb{D}}$, due to Painlevé [10] and Kellogg [11]. Thus $w := f \circ w_\varepsilon|_B$ extends continuously to a function on the closure $\overline{B}$. Note that the conformal factor λ associated with the coordinate chart w also extends continuously to a function on $\overline{\mathbb{D}}$. Indeed, if we consider the conformal factor λ_ε associated with w_ε, we see that $\lambda = |(f^{-1})'|^2 \cdot (\lambda_\varepsilon|_U \circ f^{-1})$, and thus extends continuously to a function on $\overline{\mathbb{D}}$.

Under the isothermal coordinate chart $w : \overline{B} \to \overline{\mathbb{D}}$ with the conformal factor λ, the Gauss curvature function $K : B \to \mathbb{R}$ is given by Liouville's equation

$$K = -\frac{\Delta (\log \lambda)}{2\lambda}, \tag{5}$$

where Δ is the flat Laplace operator with respect to the coordinate map $w = x + iy$, i.e.

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For the Laplace-Beltrami operator, we use the notation Δ_M. Under the isothermal coordinate chart, we have $\Delta_M = \lambda^{-1}\Delta$. Since $\lambda > 0$, we see that a smooth function $u : M \to \mathbb{R}$ is harmonic (resp. subharmonic, superharmonic) if and only if $\Delta_M u = 0$ (resp. $\Delta_M u \geq 0$, $\Delta_M u \leq 0$).

To avoid confusion, we will stick to the following notation throughout: The coordinate map $z : \overline{B} \to \delta\overline{\mathbb{D}}$ will be used for isothermal coordinates with $z(p_0) = 0$ and $z(\partial B) = \delta\mathbb{S}^1$, and the corresponding conformal factor will be denoted by φ. The coordinate map $w = \delta^{-1} z : \overline{B} \to \overline{\mathbb{D}}$ will be used for isothermal coordinates with $w(p_0) = 0$ and $w(\partial B) = \mathbb{S}^1$, and the corresponding conformal factor will be denoted by λ. We thus have the relations

$$z = \delta w \qquad \text{and} \qquad \varphi(z) = \delta^{-2}\lambda(w). \tag{6}$$

Theorem 1.1 provides bounds on the conformal factor φ associated with the isothermal coordinate chart $z : \overline{B} \to \delta\overline{\mathbb{D}}$. Since the metric in these coordinates is given by $g = \varphi|dz|^2$, the infinitesimal ratio between the Riemannian distance d_g and the Euclidean distance under the coordinate chart z is given by $\sqrt{\varphi}$, i.e.

$$\lim_{\overline{B}\setminus\{p\}\ni q \to p} \frac{d_g(p,q)}{|z(p) - z(q)|} = \sqrt{\varphi(p)} \qquad \text{for any } p \in \overline{B}. \tag{7}$$

Moreover, in case φ is uniformly bounded, the ratio between the distances is also uniformly bounded by the square-roots of the corresponding bounds, i.e.

$$a \leq \frac{d_g(p,q)}{|z(p)-z(q)|} \leq b \quad \forall p \neq q \in B \iff a^2 \leq \varphi(p) \leq b^2 \quad \forall p \in B. \tag{8}$$

The following section is dedicated to the proof of Theorem 1.1, and the final section to the proof of Theorem 1.8.

3 Proof of Theorem 1.1

3.1 Reduction to Propositions 3.1 and 3.3

We split the proof of Theorem 1.1 into two intermediate steps. The first key step is producing bounds for the conformal factor on the boundary which originate in the extremal cases of the spherical and hyperbolic metrics, as expressed in the following proposition.

Proposition 3.1. *Under the assumptions of Theorem 1.1, we have that*

$$\frac{4\tanh^2(\delta\sqrt{\kappa}/2)}{\delta^2\kappa} \leq \varphi(p_0) \leq \frac{4\tan^2(\delta\sqrt{\kappa}/2)}{\delta^2\kappa},$$

and

$$\frac{\sin^2(\delta\sqrt{\kappa})}{\delta^2\kappa} \leq \varphi\big|_{\partial B} \leq \frac{\sinh^2(\delta\sqrt{\kappa})}{\delta^2\kappa}.$$

In order to simplify our estimates on the boundary, we use the fact that for $0 < x < \pi/2$ we have $\log(\sin^2(x)/x^2) > -x^2/2$ and $\log(\sinh^2(x)/x^2) < x^2/2$ (see Lemma 5.2 from the appendix) to obtain the following corollary.

Corollary 3.2. *Under the assumptions of the proposition we have*

$$-\frac{\delta^2\kappa}{2} \leq \log(\varphi\big|_{\partial B}) \leq \frac{\delta^2\kappa}{2}.$$

The second central proposition we will prove yields, together with (8), a bound for the conformal factor which is asymptotically $(\pi/2)^2$. This $\pi/2$ factor is in a sense a "by-product" of our proof strategy, which relies on the ratio between the chordal-distance and the arc-distance.

Proposition 3.3. *Under the assumptions of Theorem 1.1, for any $p \neq q \in B$*

$$\frac{4}{\pi^2\cosh(\pi/2)} < \frac{2}{\pi}\cdot\frac{\sin(\delta\sqrt{\kappa})}{\delta\sqrt{\kappa}\cosh(\delta\sqrt{\kappa})} \leq \frac{d_g(p,q)}{|z(p)-z(q)|} \leq \frac{\pi}{2}\cdot\frac{\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{\delta^2\kappa}.$$

Given Propositions 3.1 and 3.3, we are able to prove Theorem 1.1. The only missing ingredient is the following lemma, which is a simple application of the minimum principle for superharmonic functions. Recall that we abbreviate $\delta\mathbb{D} := B_{\mathbb{C}}(0,\delta)$, $\delta\overline{\mathbb{D}} := \overline{B_{\mathbb{C}}(0,\delta)}$ and $\delta\mathbb{S}^1 := \partial B_{\mathbb{C}}(0,\delta)$ for the Euclidean disc of radius $\delta > 0$ around the origin, its closure and its boundary, respectively.

Lemma 3.4. *Suppose that $u \in C^2(\delta\mathbb{D}) \cap C^0(\delta\overline{\mathbb{D}})$ satisfies $\Delta u = f$ on $\delta\mathbb{D}$. Then*

$$\sup_{\delta\mathbb{D}} |u| \leq \sup_{\delta\mathbb{S}^1} |u| + (\delta^2/4) \cdot \sup_{\delta\mathbb{D}} |f|.$$

Proof. In case $\sup_{\delta\mathbb{D}} |f|$ is infinite, the inequality is trivial. Otherwise, define $v : \delta\overline{\mathbb{D}} \to \mathbb{R}$ by

$$v(z) = \sup_{\delta\mathbb{S}^1} |u| + \frac{\delta^2 - |z|^2}{4} \sup_{\delta\mathbb{D}} |f|.$$

Since $\Delta|z|^2 = 4$, we have that $\Delta v = -\sup_{\delta\mathbb{D}} |f|$. Therefore, for any $z \in \delta\mathbb{D}$ we have that

$$\Delta(v-u)(z) = -\sup_{\delta\mathbb{D}} |f| - f(z) \leq 0,$$

which means that $v - u$ is superharmonic on $\delta\mathbb{D}$. Moreover, for any $\zeta \in \delta\mathbb{S}^1$ we have that

$$(v-u)(\zeta) = \sup_{\delta\mathbb{S}^1} |u| - u(\zeta) \geq 0.$$

Thus by the minimum principle for superharmonic functions, i.e. Theorem 2.2, for any $z \in \delta\mathbb{D}$ we have

$$u(z) \leq v(z) \leq \sup_{\delta\mathbb{S}^1} |u| + (\delta^2/4) \sup_{\delta\mathbb{D}} |f|.$$

Replacing u by $-u$, for any $z \in \delta\mathbb{D}$ we also have that

$$-u(z) \leq \sup_{\delta\mathbb{S}^1} |u| + (\delta^2/4) \sup_{\delta\mathbb{D}} |f|,$$

thus completing the proof. □

Proof of Theorem 1.1. Using (5) we see that

$$\Delta(\log\varphi) = -2K\varphi.$$

By Proposition 3.3 and (8), we have that

$$\varphi \leq \frac{\pi^2}{4} \cdot \left(\frac{\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{\delta^2\kappa}\right)^2.$$

By Lemma 3.4 and Corollary 3.2, we obtain that

$$\sup_B |\log\varphi| \leq \sup_{\partial B} |\log\varphi| + (\delta^2/2) \cdot \sup_B |K \cdot \varphi| \leq \frac{\delta^2\kappa}{2}\left[1 + \frac{\pi^2}{4} \cdot \left(\frac{\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{\delta^2\kappa}\right)^2\right],$$

and the theorem is proved. □

The following sections are dedicated to the proof of Propositions 3.1 and 3.3.

3.2 The Harmonic Barriers

Our strategy will be to use the maximum (resp. minimum) principle, as stated in Theorem 2.2, to produce bounds involving the Euclidean distance under the coordinate chart $w : \overline{B} \to \overline{\mathbb{D}}$. Our functions will be defined on a punctured domain, and will have a logarithmic pole at the point in which they are not defined. To ensure that the singularity is removable, we use the following lemma (see, e.g., [18] for more details on subharmonic functions and removable singularities).

Lemma 3.5. *Assume that Ω is a bounded domain in $\mathbb{C}$ and let $z_0 \in \Omega$. If $u : \overline{\Omega} \setminus \{z_0\} \to \mathbb{R}$ is a continuous function, subharmonic (resp. superharmonic) on $\Omega \setminus \{z_0\}$ such that*

$$\limsup_{z\to z_0}[u(z) + \log|z - z_0|] < \infty \qquad (\textit{resp.} \quad \liminf_{z\to z_0}[u(z) + \log|z - z_0|] > -\infty),$$

Then $u(z) + \log|z - z_0|$ extends (uniquely) to a subharmonic (resp. superharmonic) function on Ω.

We now formulate the main proposition we will use to produce the bounds on the distances ratio.

Proposition 3.6. *Under the assumptions of Theorem 1.1, write $w = \delta^{-1}z : \overline{B} \to \overline{\mathbb{D}}$, so that w is an isothermal coordinate chart with $w(p_0) = 0$ and $w(\partial B) = \mathbb{S}^1$. Let $p \in B$ and suppose that $u : \overline{B} \setminus \{p\} \to \mathbb{R}$ is a continuous function satisfying:*

- *(i) For any $q \in \partial B$ we have $u(q) \le -\log|w(p) - w(q)|$ (resp. $u(q) \ge -\log|w(p) - w(q)|$),*
- *(ii) The function u is subharmonic (resp. superharmonic) on $B \setminus \{p\}$, and*
- *(iii) $\limsup_{q\to p}(u(q) + \log|w(p) - w(q)|) < \infty$ (resp. $\liminf_{q\to p}(u(q) + \log|w(p) - w(q)|) > -\infty$).*

Then $u(q) \le -\log|w(p) - w(q)|$ (resp. $u(q) \ge -\log|w(p) - w(q)|$) for any $q \in B \setminus \{p\}$.

Proof. Consider the function $\psi = u \circ w^{-1} : \overline{\mathbb{D}} \setminus \{w(p)\} \to \mathbb{R}$ so that

$$u(q) = \psi(w(q)) \qquad \text{for any } q \in \overline{B} \setminus \{p\}.$$

The function ψ is continuous on $\overline{\mathbb{D}} \setminus \{w(p)\}$, and by (ii) it is subharmonic (resp. superharmonic) on $\mathbb{D} \setminus \{w(p)\}$. Using (iii) we see that

$$\limsup_{\omega\to w(p)}(\psi(\omega) + \log|\omega - w(p)|) < \infty \qquad (\text{resp.} \quad \liminf_{\omega\to w(p)}(\psi(\omega) + \log|\omega - w(p)|) > -\infty).$$

By Lemma 3.5, the function $\psi(\omega) + \log|\omega - w(p)|$ thus extends to a subharmonic (resp. superharmonic) function Φ on $\mathbb{D}$. Using (i) we see that

$$\Phi(\omega) \le 0 \qquad (\text{resp.} \quad \Phi(\omega) \ge 0) \qquad \text{for any } \omega \in \mathbb{S}^1.$$

By the maximum (resp. minimum) principle, i.e Theorem 2.2, we obtain that

$$\Phi(\omega) \le 0 \qquad (\text{resp.} \quad \Phi(\omega) \ge 0) \qquad \text{for any } \omega \in \mathbb{D}.$$

Thus for any $q \in B \setminus \{p\}$ we have that

$$u(q) + \log|w(q) - w(p)| = \Phi(w(q)) \leq 0 \qquad (\text{resp.} \quad u(q) + \log|w(q) - w(p)| \geq 0),$$

and the proof is completed. ☐

Our choice of harmonic barriers is motivated by Green's functions in the spherical and hyperbolic cases. Let us discuss some of their properties. Consider the Riemannian surfaces $(\mathbb{D}, g_s)$ and $(\mathbb{D}, g_h)$, where $\mathbb{D} \subset \mathbb{C}$ is the unit disc, and the metrics are given by

$$g_s = \frac{4|dz|^2}{\kappa \cdot (1 + |z|^2)^2} \qquad \text{and} \qquad g_h = \frac{4|dz|^2}{\kappa \cdot (1 - |z|^2)^2},$$

for some constant $\kappa > 0$. The Riemannian surface $(\mathbb{D}, g_s)$ is of constant curvature κ, while the Riemannian surface $(\mathbb{D}, g_h)$ is of constant curvature $-\kappa$. Fix $\delta > 0$ such that $\delta^2\kappa < \pi^2/4$, and abbreviate

$$B_s := B_{(\mathbb{D},g_s)}(0, \delta) \qquad \text{and} \qquad B_h := B_{(\mathbb{D},g_h)}(0, \delta), \tag{9}$$

for the respective Riemannian discs of radius δ around the origin. Writing $r_s(z) = d_{g_s}(z, 0)$ and $r_h(z) = d_{g_h}(z, 0)$, we have that

$$|z| = \tan\left(r_s(z)\sqrt{\kappa}/2\right) = \tanh\left(r_h(z)\sqrt{\kappa}/2\right).$$

Thus an isothermal coordinate chart w_s which maps $\overline{B_s}$ onto $\overline{\mathbb{D}}$ with $w_s(0) = 0$ and $w_s(\partial B_s) = \mathbb{S}^1$ may be given by

$$w_s(z) = \cot(\delta\sqrt{\kappa}/2) \cdot z,$$

and an isothermal coordinate chart w_h which maps $\overline{B_h}$ onto $\overline{\mathbb{D}}$ with $w_h(0) = 0$ and $w_h(\partial B_h) = \mathbb{S}^1$ may be given by

$$w_h(z) = \coth(\delta\sqrt{\kappa}/2) \cdot z.$$

In particular, the corresponding Green functions are given by

$$G_s(z) = -\log|w_s(z)| = -\log\left(\cot\left(\delta\sqrt{\kappa}/2\right)\right) - \log\left(\tan\left(r_s(z)\sqrt{\kappa}/2\right)\right),$$

and

$$G_h(z) = -\log|w_h(z)| = -\log\left(\coth\left(\delta\sqrt{\kappa}/2\right)\right) - \log\left(\tanh\left(r_h(z)\sqrt{\kappa}/2\right)\right).$$

Since these are radial functions, their gradients satisfy

$$\|\nabla G_s\|_{g_s} = |\partial_{r_s} G_s| = \sqrt{\kappa}/\sin(r_s\sqrt{\kappa}) \qquad \text{and} \qquad \|\nabla G_h\|_{g_h} = |\partial_{r_h} G_h| = \sqrt{\kappa}/\sinh(r_h\sqrt{\kappa}).$$

By (4), the conformal factors may be expressed by

$$\lambda_s(z) = \|\nabla G_s(z)\|^{-2} \cdot |w_s(z)|^{-2} = (4/\kappa) \cdot \tan^2(\delta\sqrt{\kappa}/2)\cos^4(r_s(z)\sqrt{\kappa}/2), \tag{10}$$

and

$$\lambda_h(z) = \|\nabla G_h(z)\|^{-2} \cdot |w_h(z)|^{-2} = (4/\kappa) \cdot \tanh^2(\delta\sqrt{\kappa}/2)\cosh^4(r_h(z)\sqrt{\kappa}/2). \tag{11}$$

Our harmonic barriers will have a varying "base-point", which will be denoted in the subscript, and might be in the interior of B or on the boundary ∂B. For $p \in \overline{B}$ we define $G_p^{(s)} : \overline{B} \setminus \{p\} \to \mathbb{R}$ and $G_p^{(h)} : \overline{B} \setminus \{p\} \to \mathbb{R}$ by

$$G_p^{(s)}(q) = -\log\left(\tan\left(d_g(q,p)\sqrt{\kappa}/2\right)\right), \tag{12}$$

and

$$G_p^{(h)}(q) = -\log\left(\tanh\left(d_g(q,p)\sqrt{\kappa}/2\right)\right). \tag{13}$$

Note that these functions are well-defined as long as $\delta^2\kappa < \pi^2/4$. The relevant properties of $G_p^{(s)}$ and $G_p^{(h)}$ are summarized in the following proposition.

Proposition 3.7. *Under the assumptions of Theorem 1.1, write $w = \delta^{-1}z : \overline{B} \to \overline{\mathbb{D}}$, so that w is an isothermal coordinate chart with $w(p_0) = 0$ and $w(\partial B) = \mathbb{S}^1$, write λ for the conformal factor corresponding to w, and let $p \in \overline{B}$. Then*

(i) The function $G_p^{(s)}$ is superharmonic on $B \setminus \{p\}$,

(ii) The function $G_p^{(h)}$ is subharmonic on $B \setminus \{p\}$,

(iii) Both functions are monotonically decreasing with the distance from p, i.e.

$$0 < d(p,q_1) \le d(p,q_2) \implies G_p^{(s)}(q_1) \ge G_p^{(s)}(q_2) \quad \textit{and} \quad G_p^{(h)}(q_1) \ge G_p^{(h)}(q_2),$$

(iv) Both functions have a logarithmic pole at p, and in fact

$$\begin{aligned}\lim_{\overline{B}\setminus\{p\}\ni q\to p}(G_p^{(s)}(q) + \log|w(p)-w(q)|) &= \lim_{\overline{B}\setminus\{p\}\ni q\to p}(G_p^{(h)}(q) + \log|w(p)-w(q)|) \\ &= -\log(\sqrt{\kappa\cdot\lambda(p)}/2).\end{aligned}$$

In the proof of Proposition 3.7, and specifically in the harmonicity claims, we will use a comparison argument involving the Laplacian of the distance function. For the statement of this argument we use normal polar coordinates. Fix $p \in \overline{B}$ and consider a normal polar coordinate chart $(r,\theta) : \overline{B} \setminus \{p\} \to (0, 2\delta] \times [0, 2\pi)$. The metric in these coordinates is given by

$$g = dr^2 + \phi^2(r,\theta)d\theta^2,$$

for some positive function ϕ. If we fix $\theta \in [0, 2\pi)$ and define $\phi_\theta(r) = \phi(\theta, r)$, then

$$\lim_{r\to 0}\phi_\theta(r) = 0, \qquad \lim_{r\to 0}\phi'_\theta(r) = 1, \qquad \text{and} \qquad \phi''_\theta(r) + K(r,\theta)\cdot\phi_\theta(r) = 0,$$

where K is the Gauss curvature function. Moreover, the Laplacian of the distance function r from p is given by

$$\Delta_M r(\rho,\theta) = \frac{\phi'_\theta(\rho)}{\phi_\theta(\rho)}.$$

From the properties above one may obtain the following well-known comparison argument (see, e.g., [12]).

Proposition 3.8. *Under the assumptions of Theorem 1.1 and the setup above, we have that*

$$\frac{\sin(\sqrt{\kappa}\cdot r)}{\sqrt{\kappa}} \leq \phi(r,\theta) \leq \frac{\sinh(\sqrt{\kappa}\cdot r)}{\sqrt{\kappa}},$$

and

$$\sqrt{\kappa}\cot(\sqrt{\kappa}\cdot r) \leq \Delta_M r \leq \sqrt{\kappa}\coth(\sqrt{\kappa}\cdot r).$$

Proof of Proposition 3.7. Write $r : \overline{B} \to \mathbb{R}$ for the distance function from p. Define $\varphi_s : (0, 2\delta] \to \mathbb{R}$ and $\varphi_h : (0, 2\delta] \to \mathbb{R}$ by

$$\varphi_s(x) = -\log\left(\tan\left(x\sqrt{\kappa}/2\right)\right) \qquad \text{and} \qquad \varphi_h(x) = -\log\left(\tanh\left(x\sqrt{\kappa}/2\right)\right),$$

so that

$$G_p^{(s)} = \varphi_s \circ r \qquad \text{and} \qquad G_p^{(h)} = \varphi_h \circ r.$$

Differentiating φ_s and φ_h, we obtain that

$$\varphi_s'(x) = -\frac{\sqrt{\kappa}}{\sin(x\sqrt{\kappa})} \qquad \text{and} \qquad \varphi_h'(x) = -\frac{\sqrt{\kappa}}{\sinh(x\sqrt{\kappa})}.$$

Since $0 \leq r \leq 2\delta$ and $\delta\sqrt{\kappa} < \pi/2$, we see that $\varphi_s' < 0$ and $\varphi_h' < 0$, thus proving (iii). Differentiating once more, we have that

$$\varphi_s''(x) = \frac{\kappa\cot(x\sqrt{\kappa})}{\sin(x\sqrt{\kappa})} \qquad \text{and} \qquad \varphi_h''(x) = \frac{\kappa\coth(x\sqrt{\kappa})}{\sinh(x\sqrt{\kappa})}.$$

Using the chain rule, we see that

$$\Delta_M G_p^{(s)} = (\varphi_s'' \circ r) + (\varphi_s' \circ r)\cdot\Delta_M r = \frac{\kappa\cot(r\sqrt{\kappa})}{\sin(r\sqrt{\kappa})} - \frac{\sqrt{\kappa}\cdot\Delta_M r}{\sin(r\sqrt{\kappa})},$$

and

$$\Delta_M G_p^{(h)} = (\varphi_h'' \circ r) + (\varphi_h' \circ r)\cdot\Delta_M r = \frac{\kappa\coth(r\sqrt{\kappa})}{\sinh(r\sqrt{\kappa})} - \frac{\sqrt{\kappa}\cdot\Delta_M r}{\sinh(r\sqrt{\kappa})}.$$

By Proposition 3.8, we know that $\sqrt{\kappa}\cot(\sqrt{\kappa}\cdot r) \leq \Delta_M r \leq \sqrt{\kappa}\coth(\sqrt{\kappa}\cdot r)$, which implies that

$$\Delta_M G_p^{(s)} \leq 0 \qquad \text{and} \qquad \Delta_M G_p^{(h)} \geq 0,$$

thus proving (i) and (ii). Since $\lim_{x\to 0}\tan(ax)/x = \lim_{x\to 0}\tanh(ax)/x = a$, we see that

$$\lim_{\overline{B}\setminus\{p\}\ni q\to p}(G_p^{(s)}(q) + \log(r(q))) = \lim_{\overline{B}\setminus\{p\}\ni q\to p}(G_p^{(h)}(q) + \log(r(q))) = -\log(\sqrt{\kappa}/2).$$

By (7), we see that

$$\lim_{\overline{B}\setminus\{p\}\ni q\to p}(\log|w(p) - w(q)| - \log(r(q))) = -\log(\sqrt{\lambda(p)}),$$

and the proof of (iv) is completed. □

3.3 Bounding the conformal factor on the boundary and at the origin - Proof of Proposition 3.1

The bounds for the conformal factor at the origin and on the boundary are given by the corresponding values of the spherical and hyperbolic conformal factors λ_s and λ_h, as described in (10) and (11), respectively. This is shown in the following proposition. In the formulation we use the abbreviations B_s and B_h as in (9).

Proposition 3.9. *Under the assumptions of Theorem 1.1, write $w = \delta^{-1}z : \overline{B} \to \overline{\mathbb{D}}$, so that w is an isothermal coordinate chart with $w(p_0) = 0$ and $w(\partial B) = \mathbb{S}^1$, and write λ for the conformal factor corresponding to w. Then*

$$(4/\kappa)\tanh^2(\delta\sqrt{\kappa}/2) = \lambda_h(0) \leq \lambda(p_0) \leq \lambda_s(0) = (4/\kappa)\tan^2(\delta\sqrt{\kappa}/2), \tag{14}$$

and

$$\sin^2(\delta\sqrt{\kappa})/\kappa = \lambda_s\big|_{\partial B_s} \leq \lambda\big|_{\partial B} \leq \lambda_h\big|_{\partial B_h} = \sinh^2(\delta\sqrt{\kappa})/\kappa. \tag{15}$$

Proof. Write $G = -\log|w|$ for the associated Green's function. Then

$$G^{(s)}_{p_0}(q) - \log(\cot(\delta\sqrt{\kappa}/2)) = G^{(h)}_{p_0}(q) - \log(\coth(\delta\sqrt{\kappa}/2)) = G(q) = 0 \quad \text{for all } q \in \partial B. \tag{16}$$

By Propositions 3.6 and 3.7, we have that

$$G^{(h)}_{p_0}(p) - \log(\coth(\delta\sqrt{k}/2)) \leq G(p) \leq G^{(s)}_{p_0}(p) - \log(\cot(\delta\sqrt{k}/2)) \qquad \text{for all } p \in B. \tag{17}$$

Define $r : \overline{B} \to \mathbb{R}$ by $r(p) = d_g(p_0, p)$. By (17) we see that for any $p \in B$ we have

$$\cot\left(\delta\sqrt{\kappa}/2\right)\tan\left(r(p)\sqrt{\kappa}/2\right) \leq |w(p)| \leq \coth\left(\delta\sqrt{\kappa}/2\right)\tanh\left(r(p)\sqrt{\kappa}/2\right).$$

Using the fact that $\lim_{x\to 0}\tan(x)/x = \lim_{x\to 0}\tanh(x)/x = 1$, we obtain that

$$(2/\sqrt{\kappa})\tanh(\delta\sqrt{\kappa}/2) \leq \lim_{p\to p_0}\frac{r(p)}{|w(p)|} \leq (2/\sqrt{\kappa})\tan(\delta\sqrt{\kappa}/2),$$

and (7) completes the proof of (14). By (16) and (17) we see that for any $q \in \partial B$ we have

$$\sqrt{\kappa}/\sinh(\delta\sqrt{\kappa}) = -\partial_r G^{(h)}_{p_0}(q) \leq -\partial_r G(q) \leq -\partial_r G^{(s)}_{p_0}(q) = \sqrt{\kappa}/\sin(\delta\sqrt{\kappa}).$$

Moreover, for $q \in \partial B$ we have that $\|\nabla G(q)\|_g = -\partial_r G(q)$ and $\lambda(q) = \|\nabla G(q)\|_g^{-2}$ by (4). Thus

$$\sin^2(\delta\sqrt{\kappa})/\kappa \leq \lambda(q) \leq \sinh^2(\delta\sqrt{\kappa})/\kappa \qquad \text{for all } q \in \partial B,$$

completing the proof of (15). ∎

Remark 3.10. Note that the conformal factor λ at the origin is bounded from above by the spherical conformal factor at the origin, i.e. $\lambda_s(0)$, and below by the hyperbolic one, i.e. $\lambda_h(0)$. However, on the boundary the bounds are reversed, meaning that λ on the boundary is bounded from above by the hyperbolic one, and from below by the spherical one. This might indicate why the bounds on the entire disc are not trivially given by the extremal cases.

Proof of Proposition 3.1. Recall from (6) that the relation between the conformal factor φ of the coordinate chart z and the conformal factor λ of the coordinate chart w is given by $\varphi(z) = \delta^{-2}\lambda(w)$. Proposition 3.9 thus translates to the desired inequalities. $\square$

In the following section we obtain estimates on the distances ratio for boundary points. We will use these estimates in the proof of Proposition 3.3.

3.4 Bounding the distances-ratio on the boundary

Given two points $q_1 \neq q_2$ on the boundary ∂B, there are two paths joining them in ∂B. The length of the shorter path gives us the arc-distance, which we denote by $d_{\partial B}(q_1, q_2)$. When $w : \overline{B} \to \overline{\mathbb{D}}$ is an isothermal coordinate chart with $w(p_0) = 0$ and $w(\partial B) = \mathbb{S}^1$, we denote the arc-distance between $w(q_1)$ and $w(q_2)$ in $\mathbb{S}^1$ by $d_{\mathbb{S}^1}(w(q_1), w(q_2))$. Using Proposition 3.9, while recalling that the metric in these coordinates is given by $\lambda|dw|^2$, we see that

$$\frac{\sin(\delta\sqrt{\kappa})}{\sqrt{\kappa}} \leq \frac{d_{\partial B}(q_1, q_2)}{d_{\mathbb{S}^1}(w(q_1), w(q_2))} \leq \frac{\sinh(\delta\sqrt{\kappa})}{\sqrt{\kappa}}. \tag{18}$$

Given $q_1 \neq q_2 \in \partial B$ we may also consider the angle between the geodesics connecting each of them with $p_0 \in B$. We denote this angle by $\angle(q_1, q_2)$. In the Euclidean plane, the arc-distance and the angle coincide on the unit disc. For a general surface, this is not necessarily the case. However, assuming that the curvature is bounded does yield a bound on the ratio between them.

Lemma 3.11. *Under the assumptions of Theorem 1.1, for any $q_1 \neq q_2 \in \partial B$*

$$\frac{d_{\partial B}(q_1, q_2)}{\angle(q_1, q_2)} \leq \frac{\sinh(\delta\sqrt{\kappa})}{\sqrt{\kappa}}.$$

Proof. Let $q_1 \neq q_2 \in \partial B$. Consider a normal polar coordinate chart $(r, \theta) : \overline{B} \setminus \{p_0\} \to (0, \delta] \times [0, 2\pi)$ such that $\theta(q_1) = 0$ and $0 < \theta(q_2) \leq \pi$. The angle is then given by

$$\angle(q_1, q_2) = \theta(q_2).$$

The metric in these coordinates is given by

$$g = dr^2 + \phi^2(r, \theta)d\theta^2,$$

for some function $\phi : (0,\delta] \times [0,2\pi) \to \mathbb{R}$. By Proposition 3.8, we see that

$$\frac{\sin(\sqrt{\kappa}\cdot r)}{\sqrt{\kappa}} \leq \phi(r,\theta) \leq \frac{\sinh(\sqrt{\kappa}\cdot r)}{\sqrt{\kappa}}.$$

The arc-distance between q_1 and q_2 thus satisfies

$$d_{\partial B}(q_1,q_2) \leq \int_0^{\theta(q_2)} \sqrt{g(\partial_\theta,\partial_\theta)}(\delta,t)dt = \int_0^{\theta(q_2)} \phi(\delta,t)dt \leq \frac{\sinh(\delta\sqrt{\kappa})}{\sqrt{\kappa}}\cdot\theta(q_2),$$

and the proof is completed. □

In the case of the Euclidean unit disc, we have the following inequalities bounding the ratio between the Euclidean distance and the arc-distance of two points $w(q_1) \neq w(q_2) \in \mathbb{S}^1$,

$$1 \leq \frac{d_{\mathbb{S}^1}(w(q_1),w(q_2))}{|w(q_1)-w(q_2)|} \leq \frac{\pi}{2}. \tag{19}$$

A similar inequality also holds in the more general case of a surface that satisfies our assumptions, as formulated in the following lemma. The qualitative statement of the lemma is that the maximum of the ratio between the angle at the origin and the Riemannian distance of two points on the boundary is attained for antipodal points, i.e. points for which the angle is π. Our proof relies on Toponogov's theorem, or more precisely Lemma 2.1, together with Lemma 5.4 from the appendix. We believe that a simpler argument can be found for the proof of the lemma, since the statement does not involve the Gauss curvature bounds.

Lemma 3.12. *Under the assumptions of Theorem 1.1, for any* $q_1 \neq q_2 \in \partial B$

$$\frac{\angle(q_1,q_2)}{d_g(q_1,q_2)} \leq \frac{\pi}{2\delta}.$$

Proof. Consider the sphere of constant curvature κ, and a geodesic triangle on this sphere with edges a,b,c and opposite angles A,B,C, respectively. By the spherical law of cosines (see, e.g., [14]), we know that

$$\cos(c\sqrt{\kappa}) = \cos(a\sqrt{\kappa})\cos(b\sqrt{\kappa}) + \sin(a\sqrt{\kappa})\sin(b\sqrt{\kappa})\cos(C).$$

Since $d_g(p_0,q_1) = d_g(p_0,q_2) = \delta$, we see that the perimeter of the geodesic triangle $\Delta(p_0,q_1,q_2)$ is at most 4δ, which is smaller than $2\pi/\sqrt{\kappa}$ by our assumptions. By Lemma 2.1 we thus obtain that

$$d_g(q_1,q_2) \geq \frac{1}{\sqrt{\kappa}}\cos^{-1}(\cos^2(\delta\sqrt{\kappa}) + \sin^2(\delta\sqrt{\kappa})\cos(\angle(q_1,q_2))).$$

Dividing both sides by $\angle(q_1, q_2)$, we obtain that

$$\frac{d_g(q_1, q_2)}{\angle(q_1, q_2)} \geq \frac{\cos^{-1}(\cos^2(\delta\sqrt{\kappa}) + \sin^2(\delta\sqrt{\kappa})\cos(\angle(q_1, q_2)))}{\sqrt{\kappa} \cdot \angle(q_1, q_2)}.$$

By Lemma 5.4 from the appendix, the infimum of the right-hand side is attained when $\angle(q_1, q_2) = \pi$. Moreover, since $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ and $\delta\sqrt{\kappa} < \pi/2$, we see that

$$\frac{d_g(q_1, q_2)}{\angle(q_1, q_2)} \geq \frac{2\delta\sqrt{\kappa}}{\sqrt{\kappa} \cdot \pi} = \frac{2\delta}{\pi},$$

and the proof is completed. □

Combining all of the inequalities above, we obtain the following corollary.

Corollary 3.13. *Under the assumptions of Theorem 1.1, write $w = \delta^{-1}z : \overline{B} \to \overline{\mathbb{D}}$, so that w is an isothermal coordinate chart with $w(p_0) = 0$ and $w(\partial B) = \mathbb{S}^1$. Then for any $q_1 \neq q_2 \in \partial B$*

$$\frac{2\delta \sin(\delta\sqrt{\kappa})}{\pi \sinh(\delta\sqrt{\kappa})} \leq \frac{d_g(q_1, q_2)}{|w(q_1) - w(q_2))|} \leq \frac{\pi \sinh(\delta\sqrt{\kappa})}{2\sqrt{\kappa}}.$$

Proof. Combining (18) with (19) we see that

$$\frac{\sin(\delta\sqrt{\kappa})}{\sqrt{\kappa}} \leq \frac{d_{\partial B}(q_1, q_2)}{|w(q_1) - w(q_2)|} \leq \frac{\pi \sinh(\delta\sqrt{\kappa})}{2\sqrt{\kappa}}. \tag{20}$$

By Lemma 3.11 and Lemma 3.12 we have that

$$1 \leq \frac{d_{\partial B}(q_1, q_2)}{d_g(q_1, q_2)} = \frac{d_{\partial B}(q_1, q_2)}{\angle(q_1, q_2)} \cdot \frac{\angle(q_1, q_2)}{d_g(q_1, q_2)} \leq \frac{\sinh(\delta\sqrt{\kappa})}{\sqrt{\kappa}} \cdot \frac{\pi}{2\delta}. \tag{21}$$

The desired inequalities follow from (20) and (21). □

3.5 Bounding the distances-ratio - Proof of Proposition 3.3

In order to bound the distances-ratio between any two points, we will successively use the barriers $G_p^{(s)}$ and $G_p^{(h)}$ with respect to a different point $p \in \overline{B}$. The boundary inequalities required by Proposition 3.6 will be met by varying these barriers by additive constants, whose values are given by virtue of Corollary 3.13.

Lemma 3.14. *Under the assumptions of Theorem 1.1, write $w = \delta^{-1}z : \overline{B} \to \overline{\mathbb{D}}$, so that w is an isothermal coordinate chart with $w(p_0) = 0$ and $w(\partial B) = \mathbb{S}^1$, and fix $q_0 \in \partial B$. Then for any $q \in \partial B \setminus \{q_0\}$*

$$G_{q_0}^{(h)}(q) + C^{(h)} \leq -\log|w(q) - w(q_0)|,$$

and

$$G^{(s)}_{q_0}(q) + C^{(s)} \geq -\log|w(q) - w(q_0)|,$$

where

$$C^{(h)} = \log\left(\frac{\sin(\delta\sqrt{\kappa})}{\pi\cosh(\delta\sqrt{\kappa})}\right) \qquad \textit{and} \qquad C^{(s)} = \log\left(\frac{\pi\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{4\delta\sqrt{\kappa}}\right).$$

Proof. Define $r(p) := d_g(p, q_0)$. By Corollary 3.13, for any $q \in \partial B \setminus \{q_0\}$

$$\log\left(\frac{2\delta\sin(\delta\sqrt{\kappa})}{\pi\sinh(\delta\sqrt{\kappa})}\right) \leq \log(r(q)) - \log|w(q) - w(q_0)| \leq \log\left(\frac{\pi\sinh(\delta\sqrt{\kappa})}{2\sqrt{\kappa}}\right). \tag{22}$$

For any $a > 0$, the function $x/\tanh(ax)$ is increasing on $(0, \infty)$ (see Lemma 5.3 from the appendix). Since $0 < r(q) \leq 2\delta$, we have that

$$\log(r(q)) - \log\left(\tanh\left(r(q)\sqrt{\kappa}/2\right)\right) \leq \log(2\delta) - \log(\tanh(\delta\sqrt{\kappa})).$$

Thus by (22), for any $q \in \partial B \setminus \{q_0\}$ we have that

$$\begin{aligned} G^{(h)}_{q_0}(q) &= -\log\left(\tanh\left(r(q)\sqrt{\kappa}/2\right)\right) \leq -\log(r(q)) - \log(\tanh(\delta\sqrt{\kappa})) + \log(2\delta) \\ &\leq -C^{(h)} - \log|w(q) - w(q_0)|. \end{aligned}$$

For any $a > 0$, the function $x/\tan(ax)$ is decreasing on $(0, \frac{\pi}{2a})$ (see Lemma 5.3 from the appendix). Since $0 < r(q) \leq 2\delta < \pi/\sqrt{\kappa}$, we have that

$$\log(r(q)) - \log\left(\tan\left(r(q)\sqrt{\kappa}/2\right)\right) \geq \log(2\delta) - \log(\tan(\delta\sqrt{\kappa})).$$

Thus by (22), for any $q \in \partial B \setminus \{q_0\}$ we have that

$$\begin{aligned} G^{(s)}_{q_0}(q) &= -\log\left(\tan\left(r(q)\sqrt{\kappa}/2\right)\right) \geq -\log(r(q)) - \log(\tan(\delta\sqrt{\kappa})) + \log(2\delta) \\ &\geq -C^{(s)} - \log|w(q) - w(q_0)|, \end{aligned}$$

and the proof is completed. □

Proof of Proposition 3.3. Write $w = \delta^{-1}z : \overline{B} \to \overline{\mathbb{D}}$, so that w is an isothermal coordinate chart with $w(p_0) = 0$ and $w(\partial B) = \mathbb{S}^1$. Fix $q_0 \in \partial B$ and consider the functions $u, v : \overline{B} \setminus \{q_0\} \to \mathbb{R}$ given by

$$u(p) = G^{(h)}_{q_0}(p) + \log|w(p) - w(q_0)| + C^{(h)} \quad \text{and} \quad v(p) = G^{(s)}_{q_0}(p) + \log|w(p) - w(q_0)| + C^{(s)},$$

where $C^{(h)}$ and $C^{(s)}$ are as defined in Lemma 3.14. By this lemma, we know that

$$u(q) \leq 0 \quad \text{and} \quad v(q) \geq 0 \qquad \text{for all } q \in \partial B \setminus \{q_0\}.$$

Moreover, by Proposition 3.7 we see that u is subharmonic on B, that v is superharmonic on B, and that both functions are well-defined and continuous at q_0. Hence by the maximum (resp. minimum) principle, i.e. Theorem 2.2, for any $p \in B$ we have that

$$G_{q_0}^{(h)}(p) + C^{(h)} \leq -\log|w(p) - w(q_0)| \quad \text{and} \quad G_{q_0}^{(s)}(p) + C^{(s)} \geq -\log|w(p) - w(q_0)|. \quad (23)$$

For every $p \neq q \in \overline{B}$, we have the symmetry

$$G_q^{(h)}(p) = G_p^{(h)}(q) \qquad \text{and} \qquad G_q^{(s)}(p) = G_p^{(s)}(q). \quad (24)$$

Fix $p_1 \in B$. Using (23) and (24), and since $q_0 \in \partial B$ was chosen arbitrarily, for any $q \in \partial B$

$$G_{p_1}^{(h)}(q) + C^{(h)} \leq -\log|w(p_1) - w(q)| \quad \text{and} \quad G_{p_1}^{(s)}(q) + C^{(s)} \geq -\log|w(p_1) - w(q)|.$$

Invoking Propositions 3.6 and 3.7 one last time, for any $p_2 \in B \setminus \{p_1\}$ we have

$$G_{p_1}^{(h)}(p_2) + C^{(h)} \leq -\log|w(p_1) - w(p_2)| \quad \text{and} \quad G_{p_1}^{(s)}(p_2) + C^{(s)} \geq -\log|w(p_1) - w(p_2)|.$$

Since $\tanh(x) < x < \tan(x)$ for $0 < x < \pi/2$, for any $p \neq q \in B$ we thus obtain that

$$-\log(d_g(p,q)\sqrt{\kappa}/2) + \log\left(\frac{\sin(\delta\sqrt{\kappa})}{\pi\cosh(\delta\sqrt{\kappa})}\right) < G_p^{(h)}(q) + C^{(h)} \leq -\log|w(p) - w(q)|,$$

and

$$-\log(d_g(p,q)\sqrt{\kappa}/2) + \log\left(\frac{\pi\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{4\delta\sqrt{\kappa}}\right) > G_p^{(s)}(q) + C^{(s)} \geq -\log|w(p) - w(q)|.$$

Therefore for any $p \neq q \in B$ we have that

$$\frac{2\sin(\delta\sqrt{\kappa})}{\pi\sqrt{\kappa}\cosh(\delta\sqrt{\kappa})} \leq \frac{d_g(p,q)}{|w(p) - w(q)|} \leq \frac{\pi\sinh(\delta\sqrt{\kappa})\tan(\delta\sqrt{\kappa})}{2\delta\kappa}.$$

Recalling that the coordinate chart $z : \overline{B} \to \delta\overline{\mathbb{D}}$ satisfies $z = \delta w$, the result is obtained. □

This completes the proof of Theorem 1.1. In the following section we elaborate on Definitions 1.6 and 1.7, and prove Theorem 1.8.

4 The almost-flat radius - Proof of Theorem 1.8

4.1 Hölder norms

In this section we discuss the Hölder norms used in Definitions 1.6 and 1.7. We follow the definitions and notation given in [19].

For a subset D of a metric space (X, d), a function $f : D \to \mathbb{R}$ and $0 < \alpha < 1$, we write

$$|f|_{0;D} = \sup_{x \in D} |f(x)| \qquad \text{and} \qquad [f]_{\alpha;D} = \sup_{x \neq y \in D} \frac{|f(x) - f(y)|}{d^\alpha(x, y)}.$$

If the quantity $[f]_{\alpha;D}$ is finite, the function f is said to be Hölder continuous with exponent α in D. The function f is said to be locally Hölder continuous with exponent α in D if f is Hölder continuous with exponent α on compact subsets of D. The diameter of D is given by

$$\text{diam}(D) := \inf\{r > 0 : d(x, y) \leq r \text{ for all } x, y \in D\},$$

and when it is finite, the set D is said to be bounded. For a bounded set D with $d := \text{diam}(D)$, we define

$$|f|'_{0,\alpha;D} = |f|_{0;D} + d^\alpha [f]_{\alpha;D}.$$

For an open set Ω in $\mathbb{R}^n$ and a non-negative integer k, the Hölder spaces $C^{k,\alpha}(\overline{\Omega})$ (resp. $C^{k,\alpha}(\Omega)$) are defined as the subspaces of $C^k(\overline{\Omega})$ (resp. $C^k(\Omega)$) consisting of functions whose k-th order partial derivatives are Hölder continuous (resp. locally Hölder continuous) with exponent α in Ω. For simplicity we write $C^\alpha(\Omega) = C^{0,\alpha}(\Omega)$ and $C^\alpha(\overline{\Omega}) = C^{0,\alpha}(\overline{\Omega})$. For $f \in C^k(\Omega)$ we set

$$[f]_{k,0;\Omega} = |D^k f|_{0;\Omega} = \sup_{\Omega} \sup_{|\beta|=k} |D^\beta f| \quad \text{and} \quad [f]_{k,\alpha;\Omega} = [D^k f]_{\alpha;\Omega} = \sup_{|\beta|=k} [D^\beta f]_{\alpha;\Omega}, \tag{25}$$

where β is an n-dimensional multi-index. With these seminorms, assuming that $\Omega \subset \mathbb{R}^n$ is a bounded open set with $d := \text{diam}(\Omega)$, one can define the following "non-dimensional" norms on $C^k(\overline{\Omega})$ and $C^{k,\alpha}(\overline{\Omega})$ respectively,

$$\|f\|'_{C^k(\overline{\Omega})} = |f|'_{k;\Omega} = \sum_{j=0}^{k} d^j |D^j f|_{0;\Omega} \quad \text{and} \quad \|f\|'_{C^{k,\alpha}(\overline{\Omega})} = |f|'_{k,\alpha;\Omega} = |f|'_{k;\Omega} + d^{k+\alpha} [D^k f]_{\alpha;\Omega}.$$

When the set Ω is fixed, we sometimes omit it from our notation. Let us mention several properties of these norms which we will use later. Fix $0 < \alpha < 1$ and a bounded open set Ω in $\mathbb{R}^n$, with $d := \text{diam}(\Omega)$. By the triangle inequality, it is easy to verify that $|\cdot|'_{0,\alpha}$ is sub-multiplicative, i.e. for $f, g \in C^\alpha(\overline{\Omega})$ we have that $fg \in C^\alpha(\overline{\Omega})$, and

$$|fg|'_{0,\alpha} \leq |f|'_{0,\alpha} \cdot |g|'_{0,\alpha}. \tag{26}$$

For the reciprocal of a non-vanishing f we have the following inequality

$$|1/f|'_{0,\alpha} \leq |1/f|_0 + |f|'_{0,\alpha} \cdot (|1/f|_0)^2. \tag{27}$$

In case Ω is convex, the mean value theorem implies that for $f \in C^2(\Omega)$ we have

$$|f|'_{0,\alpha} \leq \sqrt{n} \cdot |f|'_1 \leq \sqrt{n} \cdot |f|'_{2,\alpha} \qquad \text{and} \qquad d|D^1 f|'_{0,\alpha} \leq \sqrt{n} \cdot |f|'_2 \leq \sqrt{n} \cdot |f|'_{2,\alpha}, \tag{28}$$

where $|D^1 f|'_{0,\alpha} = \sup_{|\beta|=1} |D^\beta f|'_{0,\alpha}$.

4.2 Scaling Definition 1.7

The first step in the proof of Claim (1) from Theorem 1.8 will be to allow ourselves to work with metrics whose coefficients' $C^{2,\alpha}$-distances from δ_{ij} are bounded by a constant much smaller than 1. For this purpose we let $0 < \varepsilon \leq 1$, and introduce the following definition, which depends on ε. Note that we will not use this definition with arbitrary values of ε, but rather with a constant value which is small enough for our requirements. The lemma following the definition shows that in order to prove (1), it is enough to consider a metric which satisfies the conditions of Definition 4.1, for some fixed $0 < \varepsilon \leq 1$. From now on, the notation δ_{ij} will always be used for the Kronecker delta function.

Definition 4.1. *We say that M is α-almost-flat via an ε-restricted coordinate chart at a point $p_0 \in M$ to distance $\delta > 0$ if the injectivity radius of all points of $B := B_M(p_0, \delta)$ is at least 2δ, and B is isometric to a bounded open set $U \subset \mathbb{C}$ endowed with the Riemannian tensor*

$$g = \sum_{i,j=1}^{2} g_{ij} dx^i dx^j,$$

such that $g_{ij} : U \to \mathbb{R}$ satisfy

$$g_{ij}(x_0) = \delta_{ij} \qquad \text{and} \qquad |g_{ij} - \delta_{ij}|'_{2,\alpha;U} \leq \varepsilon \qquad \text{for all } \ i, j \in \{1, 2\},$$

where $x_0 \in U$ is the image of p_0 under the implied isometry. We also define the α-almost-flat radius via an ε-restricted coordinate chart of M at a point $p_0 \in M$ to be the supremum over all such δ, and denote it by $\rho_{\varepsilon-chart} = \rho_{\varepsilon-chart}(p_0, \alpha)$.

Lemma 4.2. *Let M be a C^2-smooth Riemannian surface, let $p_0 \in M$, $0 < \alpha < 1$ and $0 < \varepsilon \leq 1$. Then*

$$\rho_{\varepsilon-chart} \geq C \cdot \rho_{chart},$$

for some constant $C > 0$ depending on ε.

Remark 4.3. Since we assume that $0 < \varepsilon \leq 1$, the reversed inequality to the one in Lemma 4.2 clearly holds with constant 1, i.e.

$$\rho_{\text{chart}} \geq \rho_{\varepsilon-\text{chart}}. \tag{29}$$

Note that in case ε is taken to be sufficiently small, it is also possible to omit the requirement $g_{ij}(x_0) = \delta_{ij}$ from Definition 4.1, and still obtain (29). In this case, one merely "fixes the origin" by considering the pull-back metric under the linear transformation $(G(x_0)^{-1})^{1/2}$, where $G(x_0) = [g_{ij}(x_0)]$ is the matrix of metric coefficients evaluated at x_0. Since ε is assumed to be small, one would obtain that $G(x_0) = I_2 + E$, where $I_2 = [\delta_{ij}]$ is the 2×2 identity matrix and $E \approx 0$. It is thus possible to show that the $C^{2,\alpha}$-distances of the pull-back metric's coefficients from δ_{ij} are bounded by 1. We will use a similar argument later on, in the special case where $g = \varphi \cdot (dx^2 + dy^2)$, so that $G(x_0)$ is a scalar matrix.

Before we prove Lemma 4.2, let us discuss a few estimates which we will also use later on. From now on, we write

$$g_0 = \delta_{ij} dx^i dx^j$$

for the Euclidean metric on $\mathbb{C}$. Suppose that $g = g_{ij} dx^i dx^j$ is another metric on an open set $U \subset \mathbb{C}$. We write $\|\cdot\|_{g_0} = \sqrt{g_0(\cdot,\cdot)}$ for the Euclidean norm on the tangent space, and $\|\cdot\|_g = \sqrt{g(\cdot,\cdot)}$ for the norm with respect to the metric g. For two points $x, y \in U$, we denote their Euclidean distance by $|x-y|$, and their distance with respect to the metric g by $d_g(x,y)$. For a path $\gamma \subset U$, we denote its Euclidean length by $L_{g_0}(\gamma)$, and its length with respect to the metric g by $L_g(\gamma)$.

Let $x \in U$, and suppose that $\max_{i,j\in\{1,2\}} |g_{ij}(x) - \delta_{ij}| \leq c$, for some $c > 0$. Then for any two vectors $u = u^k \partial_k, v = v^k \partial_k \in T_x U$ we have that

$$|g(v,u) - g_0(v,u)| \leq c \cdot (|v^1| + |v^2|)(|u^1| + |u^2|) \leq 2c \|v\|_{g_0} \|u\|_{g_0}, \tag{30}$$

where all the expressions are evaluated at x. In particular, we see that

$$(1-2c)\|v\|_{g_0}^2 \leq \|v\|_g^2 \leq (1+2c)\|v\|_{g_0}^2. \tag{31}$$

Hence for any path $\gamma \subset U$ such that $\max_{i,j\in\{1,2\}} |g_{ij}(x) - \delta_{ij}| \leq c$ for any $x \in \gamma$, we have that

$$L_g(\gamma) \leq \sqrt{1+2c} \cdot L_{g_0}(\gamma). \tag{32}$$

In case $c \leq 1/2$, we also have

$$L_g(\gamma) \geq \sqrt{1-2c} \cdot L_{g_0}(\gamma). \tag{33}$$

The following lemma exploits these estimates in order to obtain relations between the radius of the Riemannian disc and the Euclidean diameter of the isometric set.

Lemma 4.4. *Let M be a C^2-smooth Riemannian surface, fix $p_0 \in M$ and let $\delta > 0$ be such that the injectivity radius of all points in the Riemannian disc $B := B_M(p_0, \delta)$ is at least 2δ. Suppose that B is isometric to a bounded open set $U \subset \mathbb{C}$ endowed with the Riemannian tensor $g = g_{ij} dx^i dx^j$ such that $g_{ij} : U \to \mathbb{R}$ satisfy $|g_{ij} - \delta_{ij}|_{0;U} \leq c$ for any $i, j \in \{1,2\}$. Then*

$$\delta \leq \sqrt{1+2c} \cdot d,$$

where d is the Euclidean diameter of U. Furthermore, if $c < 1/2$, then

$$d \leq 2\delta/\sqrt{1-2c}.$$

Proof. Write $x_0 \in U$ for the image of $p_0 \in B$ under the implied isometry, and let $q \in \partial U$ be such that

$$|x_0 - q| = \inf\{|x_0 - p| : p \in \partial U\}.$$

The line $\gamma(t) = (1-t)x_0 + tq$ connecting x_0 and q is contained in U, i.e. $\gamma([0,1)) \subset U$. Since $d_g(x_0, q) = \delta$, by (32) we obtain that

$$d \geq |x_0 - q| = L_{g_0}(\gamma) \geq L_g(\gamma)/\sqrt{1+2c} \geq \delta/\sqrt{1+2c}.$$

Now suppose that $c < 1/2$, and let $x, y \in U$. Since the injectivity radius of p_0 is at least 2δ, there exist minimizing geodesics $\gamma^{(x)}, \gamma^{(y)} \subset U$ with respect to g, such that $\gamma^{(x)}$ connects x_0 to x, while $\gamma^{(y)}$ connects x_0 to y. Then by (33) we obtain that

$$|x - y| \leq |x - x_0| + |x_0 - y| \leq L_{g_0}(\gamma^{(x)}) + L_{g_0}(\gamma^{(y)}) \leq \frac{L_g(\gamma^{(x)}) + L_g(\gamma^{(y)})}{\sqrt{1-2c}} \leq \frac{2\delta}{\sqrt{1-2c}},$$

which implies that $d \leq 2\delta/\sqrt{1-2c}$, thus completing the proof. □

Proof of Lemma 4.2. Suppose that M is α-almost-flat via a coordinate chart at $p_0 \in M$ to distance $\delta > 0$, i.e. the injectivity radius of all points of $B := B_M(p_0, \delta)$ is at least 2δ, and B is isometric to a bounded open set $U \subset \mathbb{C}$ endowed with the Riemannian tensor $g = g_{ij}dx^i dx^j$, such that $g_{ij} : U \to \mathbb{R}$ satisfy

$$g_{ij}(x_0) = \delta_{ij} \qquad \text{and} \qquad |g_{ij} - \delta_{ij}|'_{2,\alpha;U} \leq 1 \qquad \text{for all } \ i, j \in \{1, 2\},$$

where $x_0 \in U$ is the image of p_0 under the implied isometry.

Let $q \in \partial U$. Since the injectivity radius of p_0 is at least 2δ, the exists a minimizing geodesic $\gamma^{(q)} \subset U$ with respect to g connecting x_0 and q, so that $L_g(\gamma^{(q)}) = \delta$. Then by (32) we have

$$L_{g_0}(\gamma^{(q)}) \geq \frac{\delta}{\sqrt{3}}. \tag{34}$$

For every $p \in \gamma^{(q)}$ write $\gamma_p^{(q)} \subset \gamma^{(q)}$ for the portion of $\gamma^{(q)}$ connecting x_0 and p. From (34) we deduce that there exists $y = y(q) \in \gamma^{(q)}$ such that

$$L_{g_0}(\gamma_y^{(q)}) = \frac{\varepsilon\delta}{8\sqrt{3}}. \tag{35}$$

Let d be the Euclidean diameter of U. By Lemma 4.4 we see that $\delta \leq d\sqrt{3}$, and together with (35) we see that

$$L_{g_0}(\gamma_y^{(q)}) \leq \frac{\varepsilon d}{8}. \tag{36}$$

For any $i, j \in \{1, 2\}$ we have that

$$g_{ij}(x_0) = \delta_{ij} \qquad \text{and} \qquad |D^1 g_{ij}|_{0;U} = \sup_{|\beta|=1} |D^\beta g_{ij}|_{0;U} \leq d^{-1}.$$

Hence by the mean value theorem together with (36), we see that for any $i, j \in \{1, 2\}$ and $x \in \gamma_y^{(q)}$ we have

$$|g_{ij}(x) - \delta_{ij}| \leq \sqrt{2} d^{-1} \cdot \frac{\varepsilon d}{8} < \frac{\varepsilon}{4} \leq \frac{1}{4}. \tag{37}$$

Thus by (33) and (35), we obtain that

$$L_g(\gamma_y^{(q)}) \geq \sqrt{1/2} \cdot L_{g_0}(\gamma_y^{(q)}) = \frac{\varepsilon\delta}{8\sqrt{6}}.$$

The Riemannian disc $B_M(p_0, \frac{\varepsilon\delta}{8\sqrt{6}})$ is thus isometric to a set $V \subset \bigcup_{q\in\partial U} \gamma_{y(q)}^{(q)}$ endowed with the restricted metric $g\big|_V$. For any $q \in \partial U$, by (36) we have

$$|y(q) - x_0| \leq L_{g_0}(\gamma_{y(q)}^{(q)}) \leq \frac{\varepsilon d}{8},$$

which implies that $d_V \leq \varepsilon d/4$, where d_V is the Euclidean diameter of V. Thus for any $i, j \in \{1, 2\}$ we have

$$d_V^k |D^k g_{ij}|_{0;V} \leq (\varepsilon/4)^k \quad \text{for any } k \in \{1,2\} \quad \text{and} \quad d_V^{2+\alpha}[D^2 g_{ij}]_{\alpha;V} \leq (\varepsilon/4)^{2+\alpha}.$$

By (37) we see that $\max_{i,j\in\{1,2\}} |g_{ij} - \delta_{ij}|_{0;V} \leq \varepsilon/4$. Thus for any $i, j \in \{1, 2\}$ we have

$$|g_{ij} - \delta_{ij}|'_{2,\alpha;V} = \left(\sum_{k=0}^{2} d_V^k |D^k(g_{ij} - \delta_{ij})|_{0;V}\right) + d_V^{2+\alpha}[D^2 g_{ij}]_{\alpha;V} \leq \epsilon,$$

and the proof is completed, with the constant being $\frac{\varepsilon}{8\sqrt{6}}$.

□

4.3 Bounding the curvature via metric coefficients - Proof of (1)

Throughout this section, we fix $\varepsilon = 1/100$. In light of Lemma 4.2, we see that showing Claim (1) from Theorem 1.8 reduces to the following proposition.

Proposition 4.5. *Let M be a C^2-smooth Riemannian surface, let $p_0 \in M$ and $0 < \alpha < 1$. Then*

$$\rho_{\mathit{curvature}} \geq C \cdot \rho_{\varepsilon-\mathit{chart}},$$

for some constant $C > 0$ depending on α.

This section is thus dedicated to the proof of Proposition 4.5. Throughout this section, we let M be a C^2-smooth Riemannian surface, we fix a point $p_0 \in M$ and let $\delta > 0$ be such that the injectivity radius of all points in the Riemannian disc $B := B_M(p_0, \delta)$ is at least 2δ. In our formulations we use the Einstein summation convention.

We use formula (3) in order to deduce bounds on the Gauss curvature function from the information we have on the metric's coefficients. For the evaluation of the Christoffel symbols we use the following formula

$$\Gamma_{ij}^k = \frac{1}{2} g^{lk}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ji}), \tag{38}$$

where $[g^{ij}]$ denotes the inverse matrix of $[g_{ij}]$. Differentiating (38) yields

$$\partial_m \Gamma_{ij}^k = \frac{1}{2}[\partial_m g^{lk}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ji}) + g^{lk}(\partial_m \partial_j g_{il} + \partial_m \partial_i g_{jl} - \partial_m \partial_l g_{ji})]. \tag{39}$$

In Definition 4.1 we introduce a bounded open set $U \subset \mathbb{C}$, which is the image of the Riemannian disc B under the implied isometry. Since B is a Riemannian disc of radius δ, and the given metric's coefficients are "C^2-close" to the Euclidean metric's coefficients, it makes sense to expect that the set U will be somehow "geometrically-close" to an Euclidean disc of radius δ. We will only show that U is convex in the Euclidean sense, i.e. as a subset of $\mathbb{C}$ with respect to the Euclidean metric. This will ensure that in our case, the inclusion of Hölder spaces $C^{k+1}(\overline{U}) \subset C^{k,\alpha}(\overline{U})$ holds. In particular, the inequalities given in (28) will be available to us. Note that this inclusion does not hold in general, without any assumption on the geometry of U. In order to show that U is convex, we first show that under the assumptions on the metric's coefficients, the quantity $\delta^2|K|_0$ is bounded by a small universal constant. Note that according to the Whitehead theorem [6], as mentioned before, this would imply that B is strongly convex.

From now on we use the notation from (25) for the derivatives' norms, meaning that whenever we write $|D^j f|'_{k,\alpha}$ for some integer $j \geq 0$, it should be understood as $\sup_{|\beta|=j} |D^\beta f|'_{k,\alpha}$, where β is a 2-dimensional multi-index.

Lemma 4.6. *Suppose that B is isometric to a bounded open set $U \subset \mathbb{C}$ endowed with the metric $g = g_{ij}dx^i dx^j$, such that $g_{ij} : U \to \mathbb{R}$ satisfy $|g_{ij} - \delta_{ij}|'_{2;U} \leq \varepsilon = 1/100$ for any $i, j \in \{1,2\}$. Then $\delta^2|K|_{0;B} < 1/10$.*

Proof. The norms in this proof are implicitly with respect to the set U. When we refer to K as a function on U, it should be understood as the composition of K with the implied isometry. By definition, we have that

$$d^k|D^k(g_{ij} - \delta_{ij})|_0 \leq \varepsilon \qquad \text{for any } i, j \in \{1,2\} \text{ and } k \in \{0,1,2\}, \tag{40}$$

where d is the Euclidean diameter of U. For the following inequalities, we implicitly use the triangle inequality and the fact that $|\cdot|_0$ is sub-multiplicative. All indices are implicitly taken from the set $\{1,2\}$. Since $\det g = g_{11}g_{22} - g_{12}g_{21} \geq 1 - 2\varepsilon$, we see that

$$\max_{i,j} |g^{ij}|_0 \leq \max_{i,j} |g_{ij}|_0 \cdot \left|\frac{1}{\det g}\right|_0 \leq \frac{1+\varepsilon}{1-2\varepsilon}. \tag{41}$$

By formula (38) for the Christoffel symbols, together with (40) and (41), we obtain that

$$\max_{i,j,k} |\Gamma_{ij}^k|_0 \leq \max_{i,j} |g^{ij}|_0 \cdot 3 \max_{i,j} |D^1 g_{ij}|_0 \leq \frac{1+\varepsilon}{1-2\varepsilon} \cdot 3\varepsilon d^{-1} \leq \frac{d^{-1}}{32}. \tag{42}$$

Since $\partial_l g^{ij} = -g^{ia}g^{jb}\partial_l g_{ab}$, using (40) and (41), we see that

$$\max_{i,j} |D^1 g^{ij}|_0 \leq 4 \cdot \left(\max_{i,j} |g^{ij}|_0\right)^2 \cdot \max_{i,j} |D^1 g_{ij}|_0 \leq 4\left(\frac{1+\varepsilon}{1-2\varepsilon}\right)^2 \varepsilon d^{-1}. \tag{43}$$

Differentiating the Christoffel symbols formula (38) yields (39), and by (40), (41) and (43), we have that

$$\begin{aligned}\max_{i,j,k}|D^1\Gamma^k_{ij}|_0 &\le \max_{i,j}|D^1 g^{ij}|_0\cdot 3\max_{i,j}|D^1 g_{ij}|_0+\max_{i,j}|g^{ij}|_0\cdot 3\max_{i,j}|D^2 g_{ij}|_0\\ &\le d^{-2}\cdot\frac{\varepsilon(1+\varepsilon)}{1-2\varepsilon}\cdot\left[12\varepsilon\left(\frac{1+\varepsilon}{1-2\varepsilon}\right)+3\right]\le\frac{d^{-2}}{31}. \qquad (44)\end{aligned}$$

By formula (3), together with (42), (44), and the fact that $1/g_{11}\le 1/(1-\varepsilon)$, we obtain that

$$d^2|K|_0\le\left|\frac{1}{g_{11}}\right|_0\cdot\left[2\cdot d^2\max_{i,j,k}|D^1\Gamma^k_{ij}|_0+4\cdot\left(d\max_{i,j,k}|\Gamma^k_{ij}|_0\right)^2\right]\le\frac{100}{99}\cdot\left(\frac{2}{31}+\frac{4}{32^2}\right)<\frac{1}{14}.$$

By Lemma 4.4 we have $\delta\le\sqrt{1+2\varepsilon}\cdot d$, which implies that $\delta^2|K|_0<1/10$. □

Lemma 4.7. *Suppose that B is isometric to a bounded open set $U\subset\mathbb{C}$ endowed with the metric $g=g_{ij}dx^idx^j$, such that $g_{ij}:U\to\mathbb{R}$ satisfy $|g_{ij}-\delta_{ij}|'_{2;U}\le\varepsilon=1/100$ for any $i,j\in\{1,2\}$. Then the set U is convex in the Euclidean sense.*

Proof. We show that U is convex by showing that the boundary of U is a convex curve. Since the C^2-norms of the functions g_{ij} are bounded, we may extend the functions g_{ij} and their derivatives continuously to the closure of U. Hence given a point $q\in\partial U$, we may speak of the inner product $g(\cdot,\cdot):T_q\mathbb{C}\times T_q\mathbb{C}\to\mathbb{R}$ and the Christoffel symbols Γ^k_{ij} at the point q.

Fix $q\in\partial U$ and let $\gamma:[-\rho,\rho]\to\partial U$ be a unit speed curve with respect to g, such that $\gamma(0)=q$. Let $N\in T_q\mathbb{C}$ be the inward unit normal at q with respect to g, i.e.

$$\|N\|_g=1,\qquad g(N,\dot\gamma)=0,\qquad\text{and}\qquad\{\exp_q(tN):t\in(0,t_0]\}\subset U\text{ for some }t_0>0,$$

where all the expressions are evaluated at q. Since B is a Riemannian disc, we have that $N=-\partial_r$, where r is the Riemannian distance function from the center of the disc p_0. Similarly, let N_0 be the inward unit normal at q with respect to g_0. The set $T_q\mathbb{C}\setminus\{t\cdot\dot\gamma(0):t\in\mathbb{R}\}$ consists of two connected components, denote them by I_{in} and I_{out}. One of these connected components, say the one we denoted by I_{in}, may be characterized by

$$I_{\text{in}}=\{u\in T_q\mathbb{C}:g(u,N)>0\}=\{u\in T_q\mathbb{C}:g_0(u,N_0)>0\}.$$

In particular, we have that

$$g(N,N_0)>0\qquad\text{and}\qquad g_0(N,N_0)>0. \qquad (45)$$

Write D_vu for the covariant derivative with respect to g_0, and ∇_vu for the covariant derivative with respect to g. In order to show that U is convex, it suffices to show that the normal component of the Euclidean acceleration $D_{\dot\gamma}\dot\gamma$ points in the direction of the inward normal N_0, i.e.

$$g_0(D_{\dot\gamma}\dot\gamma,N_0)>0. \qquad (46)$$

Since B is a Riemannian disc, and γ is a unit speed curve with respect to g, the Riemannian acceleration $\nabla_{\dot\gamma}\dot\gamma$ is given by the Laplacian of the distance function r from the center of the disc p_0, i.e.

$$\nabla_{\dot\gamma}\dot\gamma(q) = \Delta_M r(q) \cdot N.$$

Hence by the triangle inequality, we have that

$$|g_0(D_{\dot\gamma}\dot\gamma, N_0) - \Delta_M r| \le |g_0(D_{\dot\gamma}\dot\gamma - \nabla_{\dot\gamma}\dot\gamma, N_0)| + \Delta_M r \, |g_0(N, N_0) - 1| \,, \tag{47}$$

where all the expressions are evaluated at q. Since γ is a unit speed curve with respect to g, by (31) we see that $\|\dot\gamma\|_{g_0}^2 \le 1/(1-2\varepsilon)$. Using the bound we obtained for the supremum norm of the Christoffel symbols in (42) together with Lemma 4.4, we see that for any $k \in \{1,2\}$ we have that $\max_{i,j} |\Gamma_{ij}^k|_0 \le \delta^{-1}/31$. Thus for any $k \in \{1,2\}$ we have

$$\left| \sum_{i,j} \dot\gamma^i \dot\gamma^j \Gamma_{ij}^k \right| \le \max_{i,j} |\Gamma_{ij}^k|_0 \cdot (|\dot\gamma^1| + |\dot\gamma^2|)^2 \le 2 \cdot \max_{i,j} |\Gamma_{ij}^k|_0 \cdot \|\dot\gamma\|_{g_0}^2 \le \frac{\delta^{-1}}{15}.$$

Since $\nabla_{\dot\gamma}\dot\gamma - D_{\dot\gamma}\dot\gamma = (\dot\gamma^i\dot\gamma^j\Gamma_{ij}^k)\partial_k$, we obtain that

$$\|\nabla_{\dot\gamma}\dot\gamma - D_{\dot\gamma}\dot\gamma\|_{g_0} \le \frac{\delta^{-1}\sqrt{2}}{15}. \tag{48}$$

Writing $\kappa := |K|_{0;B}$, by Lemma 4.6 we have that $\delta\sqrt{\kappa} < 1/3$. By Proposition 3.8, we thus obtain that

$$(\delta \cdot \Delta_M r(q))^{-1} \le \frac{\tan(\delta\sqrt{\kappa})}{\delta\sqrt{\kappa}} \le 2. \tag{49}$$

By the Cauchy-Schwartz inequality, together with (48) and (49), we obtain that

$$(\Delta_M r)^{-1} |g_0(\nabla_{\dot\gamma}\dot\gamma - D_{\dot\gamma}\dot\gamma, N_0)| \le 1/5. \tag{50}$$

Let $v := \dot\gamma / \|\dot\gamma\|_{g_0} \in T_q\mathbb{C}$. Using (30) we see that

$$|g_0(N, v)| = |g_0(N, v) - g(N, v)| \le \frac{2\varepsilon}{\sqrt{1-2\varepsilon}}.$$

Moreover, since v and N_0 are orthonormal with respect to g_0, we have that

$$\frac{97}{100} < \frac{1}{1+2\varepsilon} - \frac{4\varepsilon^2}{1-2\varepsilon} \le \|N\|_{g_0}^2 - |g_0(N, v)|^2 = |g_0(N, N_0)|^2 \le \|N\|_{g_0}^2 \le \frac{1}{1-2\varepsilon} = \frac{100}{98}.$$

Since $g_0(N, N_0) > 0$ by (45), we see that $g_0(N, N_0)$ is very close to 1. Using these estimates together with (50), by (47) we see that

$$|(\Delta_M r)^{-1} g_0(D_{\dot\gamma}\dot\gamma, N_0) - 1| < \frac{1}{5} + \frac{3}{100} < 1,$$

which implies that $g_0(D_{\dot\gamma}\dot\gamma, N_0) > 0$, thus proving (46). □

From now on, we will say that $x \lesssim y$ if there exists a constant $C \geq 1$ depending only on α, such that $x \leq C \cdot y$. In case $x \lesssim y$ and $y \lesssim x$, we write $x \approx y$.

Lemma 4.8. *Let $0 < \alpha < 1$, and suppose that B is isometric to a bounded open set $U \subset \mathbb{C}$ endowed with the metric $g = g_{ij}dx^i dx^j$, such that $g_{ij} : U \to \mathbb{R}$ satisfy $|g_{ij} - \delta_{ij}|'_{2,\alpha;U} \leq \varepsilon = 1/100$ for any $i, j \in \{1, 2\}$. Then $\delta^2|K|'_{0,\alpha;B} \lesssim 1$.*

Proof. As a result of Lemma 4.6 and the Whitehead theorem [6], we see that B is strongly convex. Moreover, by Lemma 4.7, we have that U is convex in the Euclidean sense. Hence the assumption that $\max_{i,j\in\{1,2\}} |g_{ij} - \delta_{ij}|_{0;U} \leq 1/100$ together with (32) and (33), imply that for any $p, q \in U$ we have

$$0.98 \cdot |p - q| \leq d_g(p, q) \leq 1.01 \cdot |p - q|.$$

In particular, writing d for the Euclidean diameter of U, we see that

$$0.49d \leq \delta \leq 0.51d.$$

Hence if we wish to refer to $|K|'_{0,\alpha;U}$ in the Euclidean sense, i.e. where distances are with respect to the Euclidean distance, we only lose a constant factor depending on α. In other words,

$$(2\delta)^\alpha [K]_{\alpha;B} = (2\delta)^\alpha \sup_{p\neq q\in B} \frac{|K(p) - K(q)|}{d_g^\alpha(p, q)} \approx d^\alpha \sup_{x\neq y\in U} \frac{|K(x) - K(y)|}{|x - y|^\alpha} = d^\alpha [K]_{\alpha;U}.$$

Therefore all of the norms in this proof are implicitly with respect to U, where distances are with respect to the Euclidean metric, and our goal is to show that

$$d^2|K|'_{0,\alpha} \lesssim 1. \tag{51}$$

The assumption that $|g_{ij} - \delta_{ij}|'_{2,\alpha} \leq \varepsilon$ for any $i, j \in \{1, 2\}$ implies that

$$|g_{ij} - \delta_{ij}|_0 \leq \varepsilon, \qquad d|D^1 g_{ij}|_0 \leq \varepsilon, \qquad \text{and} \qquad d^2|D^2 g_{ij}|'_{0,\alpha} \leq \varepsilon. \tag{52}$$

Furthermore, since U is convex by Lemma 4.7, from (28) we see that that for any $i, j \in \{1, 2\}$

$$|g_{ij}|'_{0,\alpha} \leq \delta_{ij} + \sqrt{2}\varepsilon \qquad \text{and} \qquad d|D^1 g_{ij}|'_{0,\alpha} \leq \sqrt{2}\varepsilon. \tag{53}$$

For the following inequalities, we implicitly use the triangle inequality and (26), i.e. the fact that $|\cdot|'_{0,\alpha}$ is sub-multiplicative. All indices are implicitly taken from the set $\{1, 2\}$. From (53) and (27) it follows that

$$\max_{i,j} |g^{ij}|'_{0,\alpha} \leq \max_{i,j} |g_{ij}|'_{0,\alpha} \cdot |1/\det g|'_{0,\alpha} \lesssim 1. \tag{54}$$

For the Christoffel symbols we have formula (38). Thus by (53) and (54), we have that

$$d \max_{i,j,k} |\Gamma^k_{ij}|'_{0,\alpha} \lesssim \max_{i,j} |g^{ij}|'_{0,\alpha} \cdot d \max_{i,j} |D^1 g_{ij}|'_{0,\alpha} \lesssim 1. \tag{55}$$

Since $\partial_l g^{ij} = -g^{ia}g^{jb}\partial_l g_{ab}$, using (53) together with (54), we obtain

$$d \max_{i,j} |D^1 g^{ij}|'_{0,\alpha} \lesssim \left(\max_{i,j} |g^{ij}|'_{0,\alpha}\right)^2 \cdot d \max_{i,j} |D^1 g_{ij}|'_{0,\alpha} \lesssim 1. \tag{56}$$

Differentiating the Christoffel symbols formula (38) yields (39). Thus by (52), (53), (54) and (56), we see that

$$\begin{aligned} d^2 \max_{i,j,k} |D^1 \Gamma^k_{ij}|'_{0,\alpha} &\lesssim d \max_{i,j} |D^1 g^{ij}|'_{0,\alpha} \cdot d \max_{i,j} |D^1 g_{ij}|'_{0,\alpha} + \max_{i,j} |g^{ij}|'_{0,\alpha} \cdot d^2 \max_{i,j} |D^2 g_{ij}|'_{0,\alpha} \\ &\lesssim 1. \end{aligned} \tag{57}$$

By applying (27) and using (52) and (53), we see that $|1/g_{11}|'_{0,\alpha} \lesssim 1$. Hence by formula (3), together with (55) and (57), we see that

$$d^2 |K|'_{0,\alpha} \lesssim \left| \frac{1}{g_{11}} \right|'_{0,\alpha} \cdot \left[d^2 \max_{i,j,k} |D^1 \Gamma^k_{ij}|'_{0,\alpha} + \left(d \max_{i,j,k} |\Gamma^k_{ij}|'_{0,\alpha} \right)^2 \right] \lesssim 1.$$

Thus (51) holds and the proof is completed. □

Proof of Proposition 4.5. Suppose that M is α-almost-flat via an ε-restricted coordinate chart at $p_0 \in M$ to distance $\delta > 0$, i.e. the injectivity radius of all points of $B := B_M(p_0, \delta)$ is at least 2δ, and B is isometric to a bounded open set $U \subset \mathbb{C}$ endowed with the Riemannian tensor $g = g_{ij} dx^i dx^j$, such that $g_{ij} : U \to \mathbb{R}$ satisfy

$$|g_{ij} - \delta_{ij}|'_{2,\alpha;U} \leq \varepsilon = 1/100 \qquad \text{for all } \ i, j \in \{1, 2\}.$$

By Lemma 4.8, there exists a constant $C \geq 1$ depending on α such that $\delta^2 |K|'_{0,\alpha;B} \leq C$. Set $C_0 := 1/\sqrt{C}$, and define $\delta_0 := C_0 \delta$ and $B_0 := B_M(p_0, \delta_0)$. Then $B_0 \subset B$, and we have

$$\delta_0^2 |K|'_{0,\alpha;B_0} = (\delta^2/C) \cdot |K|'_{0,\alpha;B_0} \leq (1/C) \cdot \delta^2 |K|'_{0,\alpha;B} \leq 1.$$

Hence we have that $\rho_{\text{curvature}} \geq C_0 \cdot \rho_{\text{chart}}$.

□

Remark 4.9. Note that in the proof of Proposition 4.5 we did not use the fact that $g_{ij}(x_0) = \delta_{ij}$, where $x_0 \in U$ is the image of $p_0 \in B$ under the implied isometry.

4.4 Apriori bounds on a Poisson's equation solution - Proof of (2)

In order to prove Claim (2) from Theorem 1.8, we must find a coordinate chart which behaves nicely with respect to the Gauss curvature. For this purpose we use the isothermal coordinate chart discussed in earlier sections. The Gauss curvature function and the conformal factor in these coordinates are related by Liouville's equation (5), which is a form of Poisson's equation. We may thus use the Schauder estimates to relate the C^α-norm of the Gauss curvature and the $C^{2,\alpha}$-distance between the conformal factor and the constant function 1. In the proof we use Corollary 1.3 to obtain a bound for the conformal factor involving the supremum norm of the Gauss curvature. The definitions, notation, and general propositions (namely 4.10, 4.11 and 4.12) in this section are taken from [19].

Let Ω be a domain (i.e. a connected open set) in $\mathbb{R}^n$. A function $u : \Omega \to \mathbb{R}$ satisfies Laplace's equation if $\Delta u = 0$. In this case, as mentioned before, the function is called harmonic. A more general equation is Poisson's equation, in which $\Delta u = f$ for some function $f : \Omega \to \mathbb{R}$. The properties of harmonic functions, and in particular the maximum principle, provide apriori bounds for a solution to Poisson's equation, e.g. Lemma 3.4. Another example of such a bound is given by the following proposition.

Fix a bounded domain Ω in $\mathbb{R}^n$. For $x \in \Omega$, we write $d_x := \text{dist}(x, \partial\Omega)$ for the distance between x and the boundary of Ω.

Proposition 4.10. *Let $u \in C^2(\Omega)$ satisfy Poisson's equation, $\Delta u = f$, in Ω. Then*

$$\sup_{\Omega} d_x |\nabla u(x)| \leq C(\sup_{\Omega} |u| + \sup_{\Omega} d_x^2 |f(x)|),$$

where $C = C(n)$ is a constant depending on n.

Note that if the function u satisfies Poisson's equation $\Delta u = f$, and $\tau_\rho(x) = \rho x$ is a scaling function, for some $\rho \in \mathbb{R}$, then $\Delta(u \circ \tau_\rho) = \rho^2(f \circ \tau_\rho)$. This phenomena is evident in the formulation of Lemma 3.4 and of Proposition 4.10, as well as in more general estimates, commonly referred to as the Schauder estimates, which we shall quote later on. For the Schauder estimates we use a different version of "non-dimensional" norms, similar to $|\cdot|'_{k,\alpha}$, in which the "non-dimensionality" is expressed by the distance from the boundary rather than by the diameter. In order to accommodate the situation of Poisson's equation, we allow "altering" the exponent of the distance from the boundary. More formally, for an integer $k \geq 0$ and $\sigma \geq 0$ we define

$$[f]^{(\sigma)}_{k,0;\Omega} = [f]^{(\sigma)}_{k;\Omega} = \sup_{x \in \Omega, |\beta| = k} d_x^{k+\sigma} |D^\beta f(x)| \qquad \text{and} \qquad |f|^{(\sigma)}_{k;\Omega} = \sum_{j=0}^{k} [f]^{(\sigma)}_{j;\Omega}.$$

For $0 < \alpha < 1$, denoting $d_{x,y} := \min(d_x, d_y)$, we also define

$$[f]^{(\sigma)}_{k,\alpha;\Omega} = \sup_{x \neq y \in \Omega, |\beta| = k} d_{x,y}^{k+\alpha+\sigma} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha} \qquad \text{and} \qquad |f|^{(\sigma)}_{k,\alpha;\Omega} = |f|^{(\sigma)}_{k;\Omega} + [f]^{(\sigma)}_{k,\alpha;\Omega}.$$

In the case where $\sigma = 0$, we denote these quantities by $[\,\cdot\,]^* := [\,\cdot\,]^{(0)}$ and $|\cdot|^* := |\cdot|^{(0)}$. Note that $|\cdot|^{(\sigma)}_{k;\Omega}$ and $|\cdot|^{(\sigma)}_{k,\alpha;\Omega}$ are norms on the subspaces of $C^k(\Omega)$ and $C^{k,\alpha}(\Omega)$ respectively for which they are finite. It is easy to verify that

$$|fg|^{(\sigma+\tau)}_{0,\alpha;\Omega} \leq |f|^{(\sigma)}_{0,\alpha;\Omega} \cdot |g|^{(\tau)}_{0,\alpha;\Omega}. \tag{58}$$

For other properties, we will use the following interpolation inequalities.

Proposition 4.11. *Suppose $j + \beta < k + \alpha$, where $j, k \in \mathbb{Z}_{\geq 0}$ and $0 \leq \alpha, \beta \leq 1$. Let Ω be an open subset of $\mathbb{R}^n$ and assume $u \in C^{k,\alpha}(\Omega)$. Then for any $\varepsilon > 0$ and some constant $C = C(\varepsilon, k, j)$ we have*

$$[u]^*_{j,\beta;\Omega} \leq C|u|_{0;\Omega} + \varepsilon [u]^*_{k,\alpha;\Omega} \qquad \textit{and} \qquad |u|^*_{j,\beta;\Omega} \leq C|u|_{0;\Omega} + \varepsilon [u]^*_{k,\alpha;\Omega}.$$

We again use the notation $D^j f$ for the derivatives' norms, meaning that whenever we write $|D^j f|_{k,\alpha}^{(\sigma)}$ for some integer $j \geq 0$, it should be understood as $\sup_{|\beta|=j} |D^\beta f|_{k,\alpha}^{(\sigma)}$, where β is a 2-dimensional multi-index. As before, when the domain is fixed, we sometimes omit it from our notation. Fix some $0 < \alpha < 1$ and an open set Ω in $\mathbb{R}^n$. By definition, for any $f \in C^{2,\alpha}(\Omega)$ we have that

$$|D^1 f|_{0,\alpha}^{(1)} \leq |f|_{1,\alpha}^* \qquad \text{and} \qquad |D^2 f|_{0,\alpha}^{(2)} \leq |f|_{2,\alpha}^*. \tag{59}$$

Together with Proposition 4.11, we see that for some universal constant $C > 0$,

$$\max\left\{|f|_{0,\alpha}^*\,,\ |D^1 f|_{0,\alpha}^{(1)}\right\} \leq C \cdot |f|_{2,\alpha}^*. \tag{60}$$

For the reciprocal of a non-vanishing $f \in C^\alpha(\Omega)$ we again have the inequality

$$|1/f|_{0,\alpha}^* \leq |1/f|_0 + [f]_{0,\alpha}^* \cdot \left(|1/f|_0\right)^2. \tag{61}$$

The Schauder estimate we will use is as follows.

Proposition 4.12. *Let $u \in C^2(\Omega)$, $f \in C^\alpha(\Omega)$ satisfy $\Delta u = f$ in an open set Ω of $\mathbb{R}^n$. Then*

$$|u|_{2,\alpha;\Omega}^* \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)})$$

where $C = C(n,\alpha)$ is a constant depending on n and α.

Recall that we write $x \lesssim y$ if there exists a constant $C \geq 1$ depending only on α, such that $x \leq C \cdot y$. In case $x \lesssim y$ and $y \lesssim x$, we write $x \eqsim y$.

Lemma 4.13. *Let $0 < \alpha < 1$, let M be a C^2-smooth Riemannian surface, fix a point $p_0 \in M$ and let $\delta > 0$ be such that the injectivity radius of all points in the Riemannian disc $B := B_M(p_0, \delta)$ is at least 2δ. Suppose that $\delta^2 |K|_{0,\alpha;B}' \leq 1/100$, and let φ be the conformal factor corresponding to an isothermal coordinate chart $z : \overline{B} \to \delta\overline{\mathbb{D}}$ with $z(p_0) = 0$ and $z(\partial B) = \delta\mathbb{S}^1$. Then*

$$|\varphi - 1|_{2,\alpha;\delta\mathbb{D}}^* \lesssim \delta^2 |K|_{0,\alpha;B}'.$$

Proof. Since $\delta^2 |K|_{0;B} \leq 1/100$, by Corollary 1.4 we see that for any $p \neq q \in B$ we have

$$e^{-\alpha/25} \frac{|K(p) - K(q)|}{|z(p) - z(q)|^\alpha} \leq \frac{|K(p) - K(q)|}{d_g^\alpha(p,q)} \leq e^{\alpha/25} \frac{|K(p) - K(q)|}{|z(p) - z(q)|^\alpha}.$$

Furthermore, the Riemannian diameter of B and the Euclidean diameter of $\delta\mathbb{D}$ are both 2δ. Thus we have that

$$|K|_{0,\alpha;B}' \eqsim |K \circ z^{-1}|_{0,\alpha;\delta\mathbb{D}}'.$$

Therefore throughout this proof, all of the norms are implicitly with respect to $\delta\mathbb{D}$, where the distances are Euclidean. We refer to K as a function on $\delta\mathbb{D}$ as an abuse of notation instead of $K \circ z^{-1}$. Hence $\delta^2 |K|_{0,\alpha}' \lesssim 1$, and it suffices to show that

$$|\varphi - 1|_{2,\alpha}^* \lesssim \delta^2 |K|_{0,\alpha}'. \tag{62}$$

By virtue of Corollary 1.3, together with the fact that $|e^x - 1| \leq 2|x|$ for $|x| < 1$, we see that

$$|\varphi - 1|_0 \leq 2|\log\varphi|_0 \leq 16 \cdot \delta^2 |K|_0. \tag{63}$$

By Proposition 4.11 and (63), we see that in order to show (62), it suffices to show that

$$[\varphi]^*_{2,\alpha} \lesssim \delta^2 |K|'_{0,\alpha}. \tag{64}$$

By (63) we clearly have $|\varphi|_0 \lesssim 1$. By Liouville's equation (5) together with Proposition 4.10, we thus obtain that

$$\begin{aligned} \sup_{x\in\delta\mathbb{D},|\beta|=1} d_x|D^\beta\varphi(x)| &\lesssim \sup_{x\in\delta\mathbb{D},|\beta|=1} d_x|D^\beta\log\varphi(x)| \lesssim \sup_{\delta\mathbb{D}}|\log\varphi| + \sup_{\delta\mathbb{D}} d_x^2|K\varphi| \\ &\lesssim \sup_{\delta\mathbb{D}}|\log\varphi| + \delta^2\sup_{\delta\mathbb{D}}|K|\sup_{\delta\mathbb{D}}|\varphi| \lesssim 1. \end{aligned}$$

Hence $|\varphi|^*_1 \lesssim 1$, and from Proposition 4.11 it follows that

$$|\varphi|^*_{0,\alpha} \lesssim 1. \tag{65}$$

For the following inequalities, we implicitly use the triangle inequality and (58). Using Liouville's equation (5), Proposition 4.12, Corollary 1.3 and (65), we see that

$$\begin{aligned} |\log\varphi|^*_{2,\alpha} &\lesssim |\log\varphi|_0 + |K\varphi|^{(2)}_{0,\alpha} \leq |\log\varphi|_0 + |K|^{(2)}_{0,\alpha}|\varphi|^*_{0,\alpha} \\ &\lesssim \delta^2|K|_0 + \delta^2|K|'_{0,\alpha}|\varphi|^*_{0,\alpha} \lesssim \delta^2|K|'_{0,\alpha}. \end{aligned} \tag{66}$$

For any $i, j \in \{1, 2\}$ we have that $\partial_i\partial_j\varphi = \varphi \cdot \partial_i\partial_j\log\varphi + \partial_i\log\varphi \cdot \partial_j\log\varphi$. Thus by (65), together with (59), (60) and (66), we obtain that

$$[\varphi]^*_{2,\alpha} \leq |D^2\varphi|^{(2)}_{0,\alpha} \lesssim |D^2\log\varphi|^{(2)}_{0,\alpha} + \left(|D^1\log\varphi|^{(1)}_{0,\alpha}\right)^2 \lesssim \delta^2|K|'_{0,\alpha}.$$

Hence (64) holds and the proof is completed. □

Proof of Claim (2) from Theorem 1.8. Suppose that M is α-almost-flat via curvature at a point $p_0 \in M$ to distance $\delta > 0$, i.e. the injectivity radius of all points of $B := B_M(p_0, \delta)$ is at least 2δ, and

$$\delta^2 \cdot |K|'_{0,\alpha;B} \leq 1.$$

Consider $\delta_0 := \frac{\delta}{10\sqrt{C_0}}$, where $C_0 \geq 1$ is the implied constant depending on α from Lemma 4.13. Writing $B_0 := B_M(p_0, \delta_0)$, we see that

$$\delta_0^2|K|'_{0,\alpha;B_0} \leq C_0\delta_0^2|K|'_{0,\alpha;B_0} = (\delta^2/100)|K|'_{0,\alpha;B_0} \leq 1/100.$$

Let $z_0 : \overline{B_0} \to \delta_0\overline{\mathbb{D}}$ be an isothermal coordinate chart such that $z_0(p_0) = 0$ and $z_0(\partial B_0) = \delta_0\mathbb{S}^1$, with conformal factor φ_0. By Lemma 4.13 we obtain that

$$|\varphi_0 - 1|^*_{2,\alpha;\delta_0\mathbb{D}} \leq C_0\delta_0^2|K|'_{0,\alpha;B_0} \leq 1/100. \tag{67}$$

Now consider $\delta_1 = \delta_0/3$, the Riemannian disc $B_1 := B_M(p_0, \delta_1)$ and the coordinate map $z_0\big|_{\overline{B_1}} : \overline{B_1} \to U$ for some open set $U \subset \mathbb{C}$. Clearly these are also isothermal coordinates, with the conformal factor being $\varphi_0\big|_U$. For any $q \in \partial B_1$ we have that $d_g(p_0, q) = \delta_1$, and by Corollary 1.4, we see that

$$|z_0(q)| \le e^{1/25}\delta_1 < \delta_0/2$$

Thus we have that $\mathrm{dist}(U, \delta_0\mathbb{S}^1) > \delta_0/2$ and $d := \mathrm{diam}(U) \le \delta_0$. Writing $d_x = \mathrm{dist}(x, \delta_0\mathbb{S}^1)$, we see that for any $x \in U$ we have $d \le 2d_x$. Hence for any $k \in \{1, 2\}$ we have

$$d^k |D^k\varphi_0|_{0;U} \le 4 \cdot \sup_{x\in U, |\beta|=k} d_x^k \cdot |D^\beta \varphi_0(x)| \le 4 \cdot [\varphi_0]^*_{k,0;\delta_0\mathbb{D}}.$$

Similarly, writing $d_{x,y} = \min(d_x, d_y)$, we have that

$$d^{2+\alpha}[D^2\varphi_0]_{\alpha;U} \le 2^{2+\alpha} \cdot \sup_{x\neq y\in U, |\beta|=2} d_{x,y}^{2+\alpha} \frac{|D^\beta\varphi_0(x) - D^\beta\varphi_0(y)|}{|x-y|^\alpha} \le 2^{2+\alpha} \cdot [\varphi_0]^*_{2,\alpha;\delta_0\mathbb{D}}.$$

Using (67) we thus obtain that

$$|\varphi_0 - 1|'_{2,\alpha;U} \le 8 \cdot |\varphi_0 - 1|^*_{2,\alpha;\delta_0\mathbb{D}} \le 2/25.$$

As mentioned in Remark 4.3, we may now "fix the origin" by applying a linear transformation. Write $\xi = \sqrt{\varphi_0(0)}$, let $V := \xi U$ and set $z_1 := \xi z_0\big|_{\overline{B_1}} : \overline{B_1} \to V$. Recall that z_1 are also isothermal coordinates with the conformal factor $\varphi_1(z) = \xi^{-2}\varphi_0(z/\xi)$. Since the norms $|\cdot|'_{k,\alpha}$ are "non-dimensional", we have that

$$|\varphi_1 - \xi^{-2}|'_{2,\alpha;V} = \xi^{-2} \cdot |\varphi_0(z/\xi) - 1|'_{2,\alpha;V} = \xi^{-2} \cdot |\varphi_0 - 1|'_{2,\alpha;U} \le 2\xi^{-2}/25.$$

Thus

$$|\varphi_1 - 1|'_{2,\alpha;V} \le |\varphi_1 - \xi^{-2}|'_{2,\alpha;V} + |1 - \xi^{-2}| \le 2\xi^{-2}/25 + |1 - \xi^{-2}|.$$

Since $100/101 \le \xi^{-2} \le 100/99$, we obtain that

$$\varphi_1(0) = 1 \qquad \text{and} \qquad |\varphi_1 - 1|'_{2,\alpha;V} \le 1. \tag{68}$$

To summarize, taking $C = \frac{1}{30\sqrt{C_0}}$, which is a constant depending on α, we have that $\delta_1 = C\delta$, and we showed that the Riemannian disc $B_M(p_0, \delta_1)$ is isometric to a bounded domain $V \subset \mathbb{C}$ endowed with the metric $g = \varphi_1|dz_1|^2$, where φ_1 satisfies (68). Hence we obtained that $\rho_{\text{chart}} \ge C \cdot \rho_{\text{curvature}}$. □

5 Appendix

The following lemma follows directly from the definition of a convex (resp. concave) function.

Lemma 5.1. *Suppose that $g : [0, a] \to \mathbb{R}$ is a continuous strictly convex (resp. concave) function such that $f(x) = g(x)/x$ is a well-defined continuous function. Then f is strictly increasing (resp. decreasing).*

Proof. Let $x_0 \in (0, a)$, and consider the function

$$h(x) := x(f(x) - f(x_0)) = g(x) - f(x_0) \cdot x.$$

The function h is strictly convex (resp. concave) and satisfies $h(0) = h(x_0) = 0$. Thus $h(x) < 0$ (resp. $h(x) > 0$) on $(0, x_0)$, which implies that $f(x) < f(x_0)$ (resp. $f(x) > f(x_0)$) for any $x \in (0, x_0)$. □

We use this characterization of convex (resp. concave) functions to show the following results we used.

Lemma 5.2. *We have that*

$$\frac{\sinh(x)}{x} < \exp(x^2/4) \qquad \textit{for any } x > 0, \tag{69}$$

and

$$\frac{\sin(x)}{x} > \exp(-x^2/4) \qquad \textit{for any } 0 < x < \pi/2. \tag{70}$$

Proof. The first inequality follows directly from the asymptotic expansion of the two functions. Indeed, we have that

$$\frac{\sinh(x)}{x} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} \qquad \text{and} \qquad \exp(x^2/4) = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n \cdot n!}.$$

Since $(2n+1)! > 4^n \cdot n!$ for any $n \geq 1$, we see that (69) holds. In order to show (70), define $g(x) = \sin(x)\exp(x^2/4)$. Differentiating twice we obtain that

$$g''(x) = \frac{1}{4} \cdot \exp(x^2/4) \left[(x^2 - 2)\sin(x) + 4x\cos(x)\right].$$

Let $h(x) = (x^2 - 2)\sin(x) + 4x\cos(x)$. Differentiating the function h twice we obtain that $h''(x) = -(x^2 + 4)\sin(x)$, so that $h'' < 0$ on $(0, \pi/2)$, which implies that h is strictly concave on $(0, \pi/2)$. Since $h(0) = 0$ and $h(\pi/2) > 0$, we have that $h > 0$ on $(0, \pi/2)$. Thus $g'' > 0$ on $(0, \pi/2)$, and by Lemma 5.1 we see that $f(x) = g(x)/x$ is strictly increasing. Since $f(0) = 1$, we have that $f(x) > 1$ for any $x \in (0, \pi/2)$, thus completing the proof. □

Lemma 5.3. *For any $a > 0$, The function $x/\tanh(ax)$ is increasing on $(0, \infty)$, and the function $x/\tan(ax)$ is decreasing on $(0, \frac{\pi}{2a})$.*

Proof. Let $a > 0$. The function $\tanh(ax)$ satisfies $\lim_{x\to 0} \tanh(ax)/x = a$, and it is strictly concave on $(0, \infty)$. Thus by Lemma 5.1, we see that $\tanh(ax)/x$ is strictly decreasing on $(0, \infty)$, and since $\tanh(ax) > 0$ on $(0, \infty)$, we obtain that $x/\tanh(ax)$ is increasing. Similarly, the function $\tan(ax)$ satisfies $\lim_{x\to 0} \tan(ax)/x = a$, and it is strictly convex on $(0, \frac{\pi}{2a})$. Thus by Lemma 5.1, we see that $\tan(ax)/x$ is strictly increasing on $(0, \frac{\pi}{2a})$, and since $\tan(ax) > 0$ on $(0, \frac{\pi}{2a})$, we obtain that $x/\tan(x)$ is decreasing. □

Lemma 5.4. *Let $0 < a < \pi/2$. Then the function $f : (0, \pi] \to \mathbb{R}$ given by*

$$f(x) = \frac{\cos^{-1}(\cos^2(a) + \sin^2(a)\cos(x))}{x}$$

is strictly decreasing on $(0, \pi)$, and in particular attains its infimum at π.

Proof. Define $g : [0, \pi] \to \mathbb{R}$ by

$$g(x) := \cos^{-1}(\cos^2(a) + \sin^2(a)\cos(x)).$$

Then the function g satisfies

$$\frac{\partial^2 g}{\partial x^2}(x) \cdot (1 - (\sin^2(a)\cos(x) + \cos^2(a))^2)^{3/2} = -4\sin^4(a)\cos^2(a)\sin^4(x/2) < 0,$$

which implies that it is strictly concave on $(0, \pi)$. Moreover, the function $f(x) = g(x)/x$ is well-defined and continuous on $[0, \pi]$. Hence by Lemma 5.1, we see that f is strictly decreasing. □

References

[1] Deturck, D., Kazdan, J., *Some regularity theorems in Riemannian geometry.* Ann. Sci. École Norm. Sup. 14, 249-260 (1981).

[2] Calabi, E., Hartman, P., *On the Smoothness of Isometries.* Duke Math. J., Vol. 37, 741–750 (1970).

[3] Killing, W., *Ueber die Clifford-Klein'schen Raumformen.* Math. Ann. 39, 257–278 (1891).

[4] Hopf, H., *Zum Clifford-Kleinschen Raumproblem.* Math. Ann. 95, 313–339 (1926).

[5] Struik, Dirk J., *Lectures on Classical Differential Geometry.* 2nd. ed. Dover Publications (1961).

[6] Whitehead, J. H. C., *Convex regions in the geometry of paths.* Quart. J. Math., Vol. 3, Issue 1, 33–42 (1932).

[7] Cheeger, J., Ebin, David G., *Comparison theorems in Riemannian geometry.* North-Holland Publishing Company (1975).

[8] Ahlfors, L. V., *Conformal Invariants: Topics in geometric function theory.* McGraw-Hill, New York (1973).

[9] Gamelin, T. W., *Complex Analysis.* Springer-Verlag (2001).

[10] Painlevé, P., *Sure les lignes singulares des functions analytiques.* Thesis, Gauthier-Villars (1887).

[11] Kellogg, O. D., *Harmonic functions and Greens integral.* Trans. Amer. Math. Soc. 13, 109–132 (1912).

[12] Peterson, P., *Riemannian Geometry.* Springer New York (2006).

[13] Milman, E., *Isoperimetric Bounds on Convex Manifolds.* Concentration, Functional Inequalities and Isoperimetry, Contemporary Mathematics 545, Amer. Math. Soc., 195–208 (2011).

[14] Bridson, Martin R., Haefliger, A., *Metric Spaces of Non-Positive Curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin (1999).

[15] Anderson, M., *Convergence and rigidity of manifolds under Ricci curvature bounds.* Invent. Math. 102, 429–445 (1990).

[16] Hebey, E., Herzlich, M., *Harmonic coordinates, harmonic radius and convergence of Riemannian manifolds.* Rend. Mat. Appl. (7) 17 (1997), no. 4, 569–605 (1998).

[17] Zhang, Q. S., Zhu, M., *Bounds on harmonic radius and limits of manifolds with bounded Bakry-Emery Ricci curvature.*, J. Geom. Anal. (2018).

[18] Hayman, W.K., Kennedy, P., *Subharmonic functions.* Vol. 1, London Mathematical Society Monographs, No. 9, Academic Press (1976).

[19] Gilbarg, D., Trudinger, N., *Elliptic Partial Differential Equations of Second Order.* Springer Berlin Heidelberg (2001).