Volumes in High Dimensions - Home Work 1

It is highly recommended to do all the questions. You need to submit the solution to question 1 **or** 3.

Question 1. *Let $n \geq 100$. Let $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}$ be a unit vector such that*

$$\forall i, \qquad |\theta_i| \leq \frac{5}{\sqrt{n}}.$$

Let X be a random vector in $\mathbb{R}^n$, distributed uniformly in $[-\sqrt{3}, \sqrt{3}]^n$. Denote by $f_\theta(t)$ the continuous density of $\langle X, \theta \rangle$. Prove that

$$\left| f_\theta(t) - \frac{\exp(-t^2/2)}{\sqrt{2\pi}} \right| \leq \frac{C}{n} \qquad (t \in \mathbb{R})$$

for a universal constant $C > 0$.

Question 2. *For $\theta \in S^{n-1}, t \in \mathbb{R}$ we set $H_{\theta,t} = \{x \in \mathbb{R}^n \,;\, x \cdot \theta = t\}$. Set,*

$$f_\theta(t) = \frac{\mathrm{Vol}_{n-1}\left\{\sqrt{n}B_2^n \cap H_{\theta,t}\right\}}{\mathrm{Vol}_n\left\{\sqrt{n}B_2^n\right\}} = \frac{\mathrm{Vol}_{n-1}\left(B_2^{n-1}\right)}{\mathrm{Vol}_n\left(\sqrt{n}B_2^n\right)} \cdot \left(n - t^2\right)^{(n-1)/2}.$$

Prove that

$$f_\theta(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}} + O\left(\frac{1}{n}\right).$$

Question 3. *Let $p \geq 1$ be an integer, and consider the unit ball of ℓ_p^n, namely,*

$$B_p^n = \left\{ x \in \mathbb{R}^n \,;\, \|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \leq 1 \right\}.$$

The cone measure μ on its boundary is defined, for a Borel set $A \subseteq \partial B_p^n$ via

$$\mu(A) = \frac{Vol_n(\{tx \,;\, x \in A, 0 \leq t \leq 1\})}{Vol_n(B_p^n)}.$$

1. *Let $X_1, \ldots, X_n$ be i.i.d random variables, whose density is proportional to $\exp(-|t|^p)$ $(t \in \mathbb{R})$. Prove that*

   $$(X_1, \ldots, X_n)/\|X\|_p$$

   is distributed according to the cone measure on ∂B_p^n.

2. *Prove that $(X_1, \ldots, X_n)/\|X\|_p$ and $\|X\|_p$ are independent. What is the density of $\|X\|_p^p$?*

3. *Prove that the density of $\sum_{i=1}^p |X_i|^p$ is an exponential random variable of parameter one (i.e., density e^{-t} on $[0, \infty)$).*

4. *Let E be an exponential random variable of parameter one, independent of the X_i's. Prove that $\|X\|_p/(\|X\|_p^p + E)^{1/p}$ has density nt^{n-1} in $[0, 1]$.*

5. *Prove that*

$$(X_1, \ldots, X_n)/(\|X\|_p^p + E)^{1/p}$$

is distributed uniformly in B_p^n.

6. *Conclude the generalized Archimedes principle: If* $(X_1, \ldots, X_n)$ *is distributed according to the cone measure on* ∂B_p^n*, then* $(X_1, \ldots, X_{n-p})$ *is distributed uniformly in* B_p^{n-p}.