Volumes in High Dimensions - Home Work 2

It is highly recommended to do all the questions. You need to submit the solution to question 3.

Question 1 *Let X be a random vector in $\mathbb{R}^n$, with $\mathbb{E}|X|^2 < \infty$ and X is not supported on an hyperplane. Prove that $\exists b \in \mathbb{R}^n$ and a matrix $A > 0$ such that $Ax + b$ is isotropic.*

Question 2 *For a centrally symmetric convex body $K \subseteq \mathbb{R}^n$ we define its polar body by*

$$K^\circ = \{x \in \mathbb{R}^n;\ x \cdot y \leq 1,\ \forall y \in K\}.$$

Prove

1. *For any K we have*
$$(K^\circ)^\circ = K.$$
(Hint: If $x \notin K$, then there is a separating hyperplane)
2. *We have $K^\circ = K$ if and only if K is the unit Euclidean ball. (Prove that $K \subseteq B_2^n$ and use the order-reversing property of polarity).*
3. *For any $p \geq 1$ we denote by B_p^n the unit ball in the space ℓ_p^n. Show that*
$$\left(B_p^n\right)^\circ = B_q^n,$$
where $1/p + 1/q = 1$. (Hint: Hölder)

Question 3 *In this question we prove the Santaló inequality. Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex body. We define*

$$\rho(K) = Vol(K)\ Vol(K^\circ)\,.$$

Let $u \in S^{n-1}$ and define

$$T = S_u K,$$

where S_u is the Steiner symmetrization with respect to $u^\perp$.

1. *Let A be an invertible matrix. Show that $\rho(AK) = \rho(K)$.*
2. *For any $A \subseteq \mathbb{R}^n$ and $y \in \mathbb{R}$ denote $A_y = \{x \in A;\ x_n = y\}$. Assuming $u = e_n$, show that*
$$(K^\circ)_y + (K^\circ)_{-y} \subseteq 2T_y, \quad \forall y \in \mathbb{R}.$$
3. *Prove that $Vol(K^\circ) \leq Vol(T^\circ)$ and that $\rho(K) \leq \rho(T)$.*
(Hint: Brunn-Minkowski)
4. *Prove that for any centrally symmetric convex body K we have*
$$\rho(K) \leq \rho(B_2^n).$$

Question 4 *Let $f : S^{n-1} \to (0, \infty)$ be a 1-Lipschitz function and set $M = \sqrt{\int f^2 d\sigma}$. Prove that for all $t > 0$,*

$$\sigma\left(\left\{x \in S^{n-1}\,;\ |f(x) - M| \geq t\right\}\right) \leq Ce^{-ct^2 n}$$

where $c, C > 0$ are universal constants.