Volumes in High Dimensions - Home Work 3

It is highly recommended to do all the questions. You need to submit the solution to question 1 or 3.

Question 1 *Let $f_1, \dots, f_n : [0,1] \to \mathbb{R}$ be measurable functions with $\left|\sum_{j=1}^n f_j^2 - n\right| \leq 1$. Assume that $\forall \theta \in S^{n-1}$ we have*

$$\int_0^1 \left(\sum_{j=1}^n \theta_j f_j \right)^2 \leq n^{1/10}.$$

Prove that $\exists \mathcal{F} \subseteq S^{n-1}$ with $\sigma_{n-1}(\mathcal{F}) \geq 1 - 1/n$ such that for all $\theta \in \mathcal{F}$ we have

$$\left| m \left\{ x \in [0,1];\ \sum_{j=1}^n \delta_i f_i \geq t \right\} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-s^2/2} \right| \leq \frac{C}{n^\alpha}, \quad \forall t \in \mathbb{R},$$

where $C, \alpha > 0$.

Question 2

1. *Prove that $(\varepsilon/\sqrt{n})\mathbb{Z}^n \cap B_2^n$ is an $\varepsilon-$net for B_2^n of size $(c/\varepsilon)^n$.*
   (Hint: $B_2^n \subseteq \sqrt{n}B_1^n$, and estimate the size of $(\varepsilon/\sqrt{n})\mathbb{Z}^n \cap \sqrt{n}B_1^n$ by elementary combinatorics)
2. *Let $0 < \varepsilon < 1$. Let $\mathcal{N} = \{X_1, \dots, X_m\}$ be independent uniform points on S^{n-1}, where $m > (1/\varepsilon)^n$. Prove that*

   $$\mathbb{P}(N \text{ is an } \varepsilon - \text{net}) \geq 1 - C\exp\left(-c(1/\varepsilon)^n\right),$$

   for some $C, c > 0$. (Hint: $\exists \widetilde{\mathcal{N}}$ which is an $\varepsilon-$net on S^{n-1})

Question 3 *Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex body, such that $B_2^n \subseteq K$. Assume that there exists an isotropic measure μ that is supported on the contact points of K and B_2^n.*

Here we use the following definition of isotropic measures on the sphere: for all x we have

$$\int_{S^{n-1}} \langle x, u \rangle^2 \, d\mu(u) = \frac{1}{n}|x|^2.$$

1. *Let*

   $$L = \{y \in \mathbb{R}^n;\ \langle y, u \rangle \leq 1,\ \forall u \in supp\,(\mu)\}.$$

   Show that $K \subseteq L$.
2. *Let $E \subseteq L$ be an ellipsoid defined by*

   $$E = \left\{ x \in \mathbb{R}^n;\ \sum_{i=1}^n \frac{\langle x, v_i \rangle^2}{\alpha_i^2} \leq 1 \right\},$$

   for some orthonormal set $v_1, \dots, v_n$, and positive $\alpha_1, \dots, \alpha_n$. For any $u \in supp\,(\mu)$ we define

   $$y(u) = \sum_{i=1}^n \alpha_i \langle u, v_i \rangle v_i.$$

   Show that $y(u) \in E$ and conclude that $\sum_{i=1}^n \alpha_i \langle u, v_i \rangle^2 \leq 1$.

3. *Prove that*

$$\frac{1}{n}\sum_{i=1}^{n}\alpha_i \leq 1.$$

(Hint: use part 2 of the question)

4. *Show that* $Vol(E) \leq Vol(B_2^n)$, *and conclude that* B_2^n *is maximal at* K.
(Hint: arithmetic-geometric inequality.)