Volumes in High Dimensions - Home Work 4

It is highly recommended to do all the questions. You need to submit the solution to question 3.

Question 1 *In this question we prove Khintchine's inequality: Let $p \geq 1$ and let $\varepsilon_1, \ldots, \varepsilon_n$ be independent Bernoulli random variables with $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = 1/2$. Prove that*

$$c_p \|x\|_2 \leq \left(\mathbb{E} \left| \sum_{i=1}^{n} \varepsilon_i x_i \right|^p \right)^{1/p} \leq C_p \|x\|_2 , \quad \forall x \in \mathbb{R}^n, \tag{1}$$

Where $C_p \approx \sqrt{p}$. A possible proof:

1. *Show that $\cosh t \leq e^{t^2/2}$ for all $t \in \mathbb{R}$.*
2. *Show that $t^p \leq p^p e^{-p} e^t$ for all $t \geq 0$ and all $p \geq 1$.*
3. *Show that $\mathbb{E} \exp\left(\left|\sum_{i=1}^{n} \varepsilon_i x_i\right|\right) \leq 2\sqrt{e}$ for all $x \in S^{n-1}$.*
4. *Prove inequality (1) for $p \geq 2$.*
5. *Prove inequality (1) for $1 \leq p \leq 2$. (Hint: Hölder)*

Question 2 *Let $2 < p < \infty$. Our goal is to prove*

$$c_1 n^{2/p} \leq k(\ell_p^n) \leq c_2 p n^{2/p}. \tag{2}$$

1. *Prove that $b = 1$ and $M \geq n^{1/q-1/2}$, where b and M are defined as in the Dvoretzky-Milman theorem.*
2. *Prove the left inequality in (2).*
3. *Assuming a subspace of ℓ_p^n that is $4-$isomorphic to ℓ_2^k we have $u_1, \ldots, u_k \in \mathbb{R}^n$ such that*

$$\|a\|_2 \leq \left\| \sum_{i=1}^{k} a_i u_i \right\|_p \leq 4 \|a\|_2 , \quad \forall a \in \mathbb{R}^k. \tag{3}$$

   Show that

$$k^{p/2} \leq c_p^p \sum_{i=1}^{n} \left(\sum_{j=1}^{k} u_{j,i}^2 \right)^{p/2} ,$$

   where $u_j = (u_{j,1}, \ldots, u_{j,n})$ and $c_p \approx \sqrt{q}$. (Hint: Khintchine - inequality (1))

4. *Prove that*

$$\sum_{j=1}^{k} u_{j,i}^2 \leq 4^2.$$

   (Hint: use (3))

5. Prove the right inequality in (2).

Question 3 *Prove that*

$$c_1\sqrt{\frac{\log k}{n}} \leq \int_{S^{n-1}} \max_{i=1,\ldots,k} |x_i| d\sigma_{n-1}(x) \leq c_2\sqrt{\frac{\log k}{n}},$$

where $c_1, c_2 > 0$ *are universal constants.*