Volumes in High Dimensions - Home Work 5

It is highly recommended to do all the questions. You need to submit the solution to question 1.

Question 1 *Let $\varepsilon > 0$ and $0 < k < n$.*

1. *Let $F \subseteq \mathbb{R}^n$ be a subspace of dimension k. Let X be a random uniform vector in S^{n-1}. Show that*

$$\mathbb{P}\left(\left||Proj_F(x)|^2 - \frac{k}{n}\right| \geq \varepsilon\frac{k}{n}\right) \leq \sqrt{\pi/2}e^{-\varepsilon^2 k/8}.$$

2. *Let $y_1, \ldots, y_N \in S^{n-1}$ where $N \leq e^{c\varepsilon^2 k}$. Let F be a random subspace of dimension k. Show that*

$$\mathbb{P}\left(\left||Proj_F(y_j)|^2 - \frac{k}{n}\right| \geq \varepsilon\frac{k}{n},\ \forall j = 1, \ldots, N\right) \leq e^{-c\varepsilon^2 k}.$$

Question 2 *Prove that for Gaussian variables, the density is proportional to $\exp(-\langle Ax, x\rangle/2)$ where A^{-1} is the covariance matrix. What happens when the covariance matrix is singular?*

Question 3 *Denote by $R(A)$ the minimal radius of a ball circumscribing A. Let $F \subseteq \mathbb{R}^n$ be a random subspace of dimension k. Prove that with probability greater than $1 - e^{-ck}$ we have*

1.

$$R(Proj_F K) \geq \sqrt{\frac{k}{n}}R(K).$$

(Hint: start with $x \in K$ such that $|x| = R(K)$)

2.

$$R(Proj_F K) \geq c\max\left\{M^*, \sqrt{\frac{k}{n}}R(K)\right\}.$$

(Hint: Dvoretzky-Milman theorem)