Volumes in High Dimensions - Home Work 6

It is highly recommended to do all the questions. You need to submit the solution to question 1.

Question 1 *Let $\beta > 0$ and define*

$$f(x) = f_\beta(x) = \frac{1}{\beta} \log \sum_{i=1}^{n} e^{\beta x_i}.$$

1. *Prove that*
$$f_\beta(x) \xrightarrow[\beta\to\infty]{} \max_{i\le n} x_i.$$

2. *Let X, Y be two independent Gaussian vectors with covariance matrices C^X, C^Y. Let*
$$Z(t) = \sqrt{t}X + \sqrt{1-t}Y, \quad t \in [0,1].$$
Prove that,
$$\frac{d}{dt}\mathbb{E}f(Z(t)) = \frac{1}{2}\sum_{i,j=1}^{n} \left(C_{ij}^X - C_{ij}^Y\right)\mathbb{E}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(Z(t))\right)$$
(Hint: use the chain rule and the integration by parts formula $\mathbb{E}Xf(X) = C^X \mathbb{E}\nabla f(X)$)

3. *Set $V_{ij}^X = -2C_{ij}^X + C_{ii}^X + C_{jj}^X$ and similarly define V_{ij}^Y. Assuming that $V_{ij}^X \le V_{ij}^Y$ for all i, j, prove that $\frac{d}{dt}\mathbb{E}f((Z(u)) \le 0$. (Hint: Writing $\partial^i f = p_i(x)$, observe that $\partial^{ij} f = \beta(p_i\delta_{ij} - p_i p_j)$ and $\sum p_i = 1$.)*

4. *Deduce the sharp Slepian's lemma: Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be centered Gaussian processes, such that*
$$\mathbb{E}|X_i - X_j|^2 \le \mathbb{E}|Y_i - Y_j|^2, \quad \forall i, j.$$
Then,
$$\mathbb{E}\max X_i \le \mathbb{E}\max Y_i.$$

Question 2 *Formulate and prove the conclusion of Slepian's lemma for an infinite collection of Gaussian random variables, given the one for a finite collection of Gaussian random variables.*

Question 3 *A centered random variable Z is called* sub-Gaussian *if for all $t > 0$.*

$$\mathbb{P}\left(|Z| > t\right) \le 5e^{-t^2/2}.$$

(The specific choice of constnats 5 and 1/2 is rather arbitrary). Let $Z_1, \ldots, Z_N$ be centered, sub-gaussian random variables, possibly dependent. Prove that

$$\mathbb{E}\max_i Z_i \le C\sqrt{\log N},$$

where $C > 0$ is a universal constant.