Volumes in High Dimensions - Home Work 7

It is highly recommended to do all the questions. You need to submit the solution to questions 2 and 3.

Question 1 *Let $h = h_\tau : \mathbb{R} \to [0,1]$ be a smooth approximation of $1_{(-\infty,\tau]}$ as in the proof of Slepian's lemma (see Figure 1). Let*

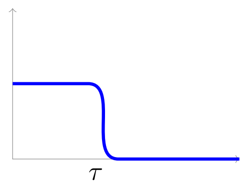

Figure 1: The function h_τ.

$$f(x) = \prod_i \left(1 - \prod_j h(x_{ij})\right).$$

Using the function f prove Gordon's version of Slepian's lemma: Let $(X_{ut})_{u\in U, t\in T}$ and $(Y_{ut})_{u\in U, t\in T}$ be centered Gaussian processes. Assume that

$$\begin{cases} \mathbb{E}X_{ut}^2 = \mathbb{E}Y_{ut}^2, & \forall u, t, \\ \mathbb{E}|X_{ut} - X_{ut'}|^2 \leq \mathbb{E}|Y_{ut} - Y_{ut'}|^2, & \forall u, t, t', \\ \mathbb{E}|X_{ut} - X_{u't'}|^2 \geq \mathbb{E}|Y_{ut} - Y_{u't'}|^2, & \forall t, t' \text{ and } u \neq u'. \end{cases}$$

Then

$$\mathbb{P}\left(\inf_u \sup_t X_{ut} \geq \tau\right) \leq \mathbb{P}\left(\inf_u \sup_t Y_{ut} \geq \tau\right).$$

(Hint: see question 1 in home assignment 6.)

Question 2 *Define a sub-exponential process $(X_t)_{t\in T}$ (not sub-Gaussian!) with respect to the metric space (T,d).*

1. *For such a process prove*
$$\mathbb{E}\sup_{t\in T} X_t \leq C\int_0^\infty \log \mathcal{N}(T,d,\varepsilon)d\varepsilon,$$
where $\mathcal{N}(T,d,\varepsilon)$ is the convering number.

2. *Is it tight in the example where $X_t = \langle \Gamma, t\rangle$ where $t \in S^{n-1}$ and Γ is a standard Gaussian vector in $\mathbb{R}^n$?*

Question 3 *Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Set*

$$T = \left\{\frac{e_k}{\sqrt{\log k}};\ k = 1, \ldots, n\right\}.$$

1. *Prove that the Dudley sum tends to infinity with the dimension:*

$$\inf_{(T_k)} \sum_{k=0}^{\infty} 2^{k/2} \sup_{t \in T} d(t, T_k) \to \infty.$$

2. *Prove that the generic chaining bound is bounded by a universal constant:*

$$\gamma_2(T) = \inf_{(T_k)} \sup_{t \in T} \sum_{k=0}^{\infty} 2^{k/2} d(t, T_k) \leq C.$$

The infimum is taken over all admissible sequences (T_k).

Question 4 *We saw concentration of Gaussian random vector for Lipschitz functions with respect to expectation.*

1. *Prove that in the Gaussian concentration inequality, we may work with median (or $2/3$ quantile) in place of expectation.*

2. *Let K be a symmetric convex body with non empty interior. Let $f(x) = \|x\|_K$. Prove that the Gaussian median of f and its Gaussian expectation differ at most by a universal constant.*

Question 5 *In class we proved that for a 1-Lipschitz function f on the sphere S^{n-1}, and a random $k-$dimensional subspace E,*

$$\mathbb{E} \sup_{x \in E} |f(x) - M| \leq C\sqrt{\frac{k}{n}},$$

where $M = \int f d\sigma_{n-1}$. Prove that

$$\mathbb{P}\left(\sup_{x \in E} |f(x) - M| \leq C\sqrt{\frac{k}{n}} \right) \geq 1 - Ce^{-ck},$$

where $c, C > 0$ are universal constants.