Volumes in High Dimensions - Home Work 8

It is highly recommended to do all the questions. You need to submit the solution to question 2.

Question 1 *This question shows different conditions that are equivalent to the sub-Gaussian condition.*

1. *Let X be a random variable. Prove that,*

$$\mathbb{P}(X \geq u) \leq 2\exp\left(-\frac{u^2}{K^2}\right),$$

if and only if

$$\mathbb{E}e^{\lambda X} \leq e^{\lambda K_2^2}, \quad \forall \lambda.$$

Where K and K_2 are comparable ($K \leq K_2 \leq 3K$). (Hint: moment generating map.)

2. *Deduce that if $(Z_i)_{i=1,\ldots,n}$ are independent identically distributed, such that*

$$\mathbb{P}(Z_1 \geq u) \leq 2\exp\left(-\frac{u^2}{K^2}\right),$$

then

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i \geq u\right) \leq 2\exp\left(-\frac{nu^2}{cK^2}\right).$$

3. *Observe that if Z_1 is a bounded random variable such that $|Z_1| \leq L$, then obviously for all u,*

$$\mathbb{P}(|Z_1| \geq u) \leq 2\exp\left(-\frac{u^2}{2L^2}\right).$$

Question 2 *In this question we prove a uniform Monte-Carlo bound for Lipschitz functions on $[0,1]$. Let*

$$\mathcal{F} = \{f : [0,1] \to [0,1];\ \|f\|_{Lip} \leq 1\}.$$

Let μ be a probability measure on $[0,1]$, and let Y be a random variable distributed according to μ, and let $Y_1, Y_2, \ldots$ be independent copies of Y.

1. *Prove that for any $f \in \mathcal{F}$ we have*

$$\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^{n} f(Y_i) - \mathbb{E}f(Y)\right| \leq \frac{C}{\sqrt{n}},$$

where $C > 0$ is a universal constant. (Note that $\sum f(Y_i)/n \to \mathbb{E}f(Y)$ a.s by the law of large numbers.)

2. *For any $f \in \mathcal{F}$ define*

$$X_f = \frac{1}{n}\sum f(Y_i) - \mathbb{E}f(Y).$$

Prove that $(X_f)_{f\in\mathcal{F}}$ has sub-Gaussian increments, with respect to the metric d, where

$$\frac{\sqrt{n}}{C}d(f,g) = \|f-g\|_\infty = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

(Hint: define $Z_i = f(Y_i) - g(Y_i) - (\mathbb{E}f(Y) - \mathbb{E}g(Y))$ and use Question 1.)

3. *Prove that* $\mathcal{N}(\mathcal{F}, d, \varepsilon) \leq (1/\varepsilon)^{1/\varepsilon}$.

   Hint: see figure 3.

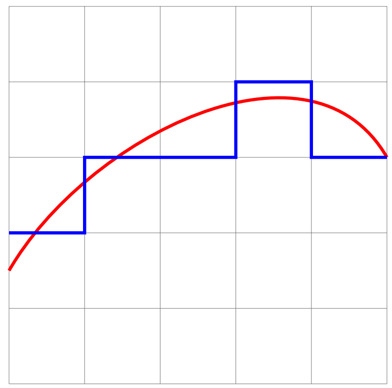

Figure 1: Approximate the red with the blue.

4. *Prove that*

$$\mathbb{E} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} f(Y_i) - \mathbb{E} f(Y) \right| \leq \frac{C}{\sqrt{n}},$$

   where $C > 0$ *is a universal constant. (Hint: Dudley's inequality.)*

5. *Can we get the same result for continuous functions without the Lipschitz condition?*

Remark 3 *There are some improvements and generalizations to Question 2:*

- *We can generalize this to*

$$\mathcal{F} = \{f : [0,1] \to \mathbb{R};\ \|f\|_{Lip} \leq L\}.$$

  and get

$$\mathbb{E} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} f(Y_i) - \mathbb{E} f(Y) \right| \leq \frac{CL}{\sqrt{n}}.$$

- *If we work in* $[0,1]^n$ *we can get covering number bounds of* e^{C/ε^n}.