Several Complex Variables – Exercise 1

Due by November 4th, 2009. Please let me know immediately when you find a mistake or a misprint.

1. **Differentiaion under the integral sign**. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain, μ is a finite measure on Ω, and let $u(t,x)$ be a complex function defined in the domain $\tilde{\Omega} = (-\varepsilon, \varepsilon) \times \Omega$, such that $x \mapsto u(t,x)$ is $\mu-$ integrable for $|t| < \varepsilon$. Define
$$f(t) = \int_\Omega u(t,x) d\mu(x) \qquad (-\varepsilon < t < \varepsilon).$$
Assume that the derivative $\partial u / \partial t$ exists and is uniformly continuous in $\tilde{\Omega}$. Prove that
$$f'(0) = \int_\Omega \frac{\partial u}{\partial t}(0,x) d\mu(x).$$

2. Generalize the previous question to line integrals over smooth curves, and explain why we can differentiate Cauchy integral formula under the integral sign any finite number of times.

3. Verify that $\partial \bar{f}/\partial z = \overline{\partial f/\partial \bar{z}}$ and $\partial \bar{f}/\partial \bar{z} = \overline{\partial f/\partial z}$, that $\partial^2 f/\partial z \partial \bar{z} = \frac{1}{4}\triangle f$, that the composition of two anti-holomorphic functions (i.e., function with $\partial f/\partial z = 0$) is actually holomorphic, and prove also the chain rule:
$$\frac{\partial (f \circ g)}{\partial z} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{g}}{\partial z}, \qquad \frac{\partial (f \circ g)}{\partial \bar{z}} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{g}}{\partial \bar{z}}.$$

4. **Approximation by rational functions**. Let $\Omega \subset \mathbb{C}$ be a bounded domain with a smooth boundary γ, and let $K \subset\subset \Omega$ (i.e., $\overline{K} \subset \Omega$). Suppose that f is a continuous function on γ. Prove that for any $\varepsilon > 0$, there exists a rational function R, all of its poles lie in γ, for which
$$\sup_{\xi \in K} \left| \int_\gamma \frac{f(z)}{z - \xi} dz - R(\xi) \right| < \varepsilon.$$

5. Suppose $U, V \subset \mathbb{C}$ are open sets, with $V \subset U$ and $\partial V \cap U = \emptyset$. Let H be a connected component of U with $H \cap V \neq \emptyset$. Prove that $H \subset V$.

6. Suppose $K \subset \mathbb{R}^n$ is a compact, convex set. Prove that for any $x \notin K$ there exists an affine functional φ on $\mathbb{R}^n$ with
$$|\varphi(x)| > \sup_{y \in K} |\varphi(y)|.$$